4.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

[*Continued from* p. 422 *of Volume I. of this Reprint.*]

[*Cambridge and Dublin Mathematical Journal*, IX. (1854), pp. 85—103.]

SECTION VII. (*Continued.*)

BEFORE proceeding further I must guard against a misconception as to my meaning to which the modification of the title of this memoir might give birth; it is not to be understood that I regard the Theory of Invariants as coextensive with the Calculus of Forms, but only with a certain portion of that Calculus which is here exclusively treated of; the Calculus of Forms itself has for its subject-matter the whole theory of the Composition, Decomposition, and Comparison of Forms. In the theory of invariants the composition of single forms with sets of linear forms is alone considered, and the idea of invariance must be regarded as a transient idea arising out of an artificial mode of viewing the effects of composition, so as to ignore the presence in the result of factors which depend on the resultants of the linear forms employed, which resultants, although in this portion of the subject treated as mere moduli and as such generally supposed to be reduced to unity, yet in regard to the general theory are as important as the factors which are retained as the sole objects of contemplation; so that in fact the idea of invariance is but a special and it may be said accidental notion which merges in the more general notion of permanency of character in the resultant of forms compounded in a given manner out of given forms. Again, as to combinants, the idea contained in this word may, by a change in the mode of statement of the definition, be extended to functions of unlike degrees. A combinant of U, V, W, ..., all functions of the same system or systems of variables, is in fact only another name for invariants of the function $\lambda U + \mu V + \nu W +$ &c., where, over and above the sets of variables contained in U, V, W, ... there is a new correlated set of variables λ, μ, ν, &c. So now, more generally, if U, V, W, ... are of p, q, r, ... dimensions in one set of variables, of which the highest number is I, if λ is taken of $I-p$, μ of $I-q$, ν of $I-r$, &c. dimensions in the same, the functions λ, μ, ν, &c. being each the most general of their kind, any invariant of $\lambda U+\mu V+\nu W+\ldots$ which is such as well in respect to the coefficients in λ, μ, ν, ... which must be considered as forming a set among themselves, as also in respect to the set of variables in U, V, W, ... will be a combinant to the system U, V, W, ... ; and so, more generally, if U, V, W, ... contain several (say i) unrelated sets or systems of sets of variables, we must form in an analogous manner

$$\lambda_1\lambda_2\ldots\lambda_i U+\mu_1\mu_2\ldots\mu_i V+\nu_1\nu_2\ldots\nu_i W+\&c.,$$

and then an invariant in respect to the i given sets in U, V, W, ... and the i new sets contained in $(\lambda_1, \mu_1, \nu_1, \ldots)$, $(\lambda_2, \mu_2, \ldots \nu_2)$, &c. $(\lambda_i, \mu_i, \nu_i, \ldots)$ will be a combinant to the system U, V, W, ... Perhaps, however, a more immediate extension of the idea of combinants to the case supposed of i unrelated sets or systems of sets would be to take, instead of $\lambda_1\lambda_2\ldots\lambda_i$, $\mu_1\mu_2\ldots\mu_i$, &c., the perfectly general forms of the same degrees in each set of the variables as these quantities are respectively of the same; to use these general forms, the coefficients of which will constitute not i new sets but a single new set of variables, as the syzygetic multipliers to U, V, W, ..., and then the invariant of the corresponding conjunctive in respect to the i original sets or systems of sets, and the one new set of variables thus obtained will be a combinant to the given system of functions*. As a matter of punctilio I may here take the opportunity of observing that the process for obtaining the relation between שׁ, ט (inadvertently written טּ), and R, would have been more perfectly symmetrical to the eye had the equation for W [p. 416 of Vol. I.] been written $\tau(z^2-y^2)=W$ in lieu of $\sigma(y^2-z^2)=W$. I now return to take up the subject from the point where it was brought to a close in the last number of the *Journal*.

Let us consider what the equation (A)† becomes when U, V, W become the first partial derivatives (quâ x, y, z) of a single homogeneous cubic function ψ, so that

$$U=\frac{d\psi}{dx},\quad V=\frac{d\psi}{dy},\quad W=\frac{d\psi}{dz};$$

ϑ then becomes the Hessian of ψ, and the S of this (like every other invariant of ψ)‡ may be expressed, as is well known, as a rational integral function of

* I propose to append at the end of the next or some subsequent Section what ought to have been given in this or previous place, viz. the general differential equations for any concomitant to any congeries of forms, comprising amongst them any number of various distinct (that is unrelated) classes of systems of sets of variables, the relations between the sets belonging to any one system being supposed to be either simple or compound, and after the manner of either cogredience or contragredience; in fact, to do this only requires a slight extension of the formulæ given by me with that object in the fifth section of my paper in the *Philosophical Transactions* for the year 1853, Part III., which see. [Vol. I., p. 551.]

† *Vide* last number of this *Journal*, near the end of Author's paper therein. [Vol. I., p. 420.]

‡ I have given a perfectly rigid demonstration in the *Philosophical Magazine*, in the early part of 1853, that every invariant to a cubic function of three variables is a rational integral function of the two Aronholdian invariants S and T. [Vol. I., p. 599.]

the S and T of ψ. The relation between the S of the H and the S and T may readily be obtained from the canonical form

$$(\psi) = x^3 + y^3 + z^3 + 6mxyz.$$

The Hessian of this is

$$(1 + 2m^3)\, xyz - m^2 (x^3 + y^3 + z^3),$$

and making $-\dfrac{1 + 2m^3}{6m^2} = \mu$, the S of this Hessian will be

$$6^4 . m^8 \times (\mu - \mu^4),$$

which is $\qquad (1 + 2m^3) \{(1 + 2m^3)^3 + 216m^6\}.$

That is $\qquad 1 + 8m^3 + 240m^6 + 464m^9 + 16m^{12},$

$$= (1 - 20m^3 - 8m^6)^2 + 48 (m - m^4)^3,$$

$$= (S)^3 + 48 (T)^3,$$

where (S) and (T) are respectively the S and T of (ψ). Hence we have in general

$$S . H . \psi = (S\psi)^3 + 48 (T\psi)^3.$$

So that ש becomes $T^2 + 48S^3$, and ט evidently from Calculus of Forms [cf. Vol. I., p. 311] becomes

$$\tfrac{16}{8} (1 - 20m^3 - 8m^6),$$

that is $2T$, so that $\qquad 4\text{ש} - \frac{1}{4}\text{ט}^2 = 3T^2 + 192S^3;$

so that equation (A) becomes

$$R = T^2 + 64S^3,$$

the Aronholdian representation of the Discriminant of ψ.

We see from this numerical calculation that it is not $\Sigma\Omega$ but $\frac{1}{2}\Sigma\Omega$, which ought to receive the appellation of ט, making which modification the general equation, written (A), becomes

$$\tfrac{1}{3}R = 4\text{ש} - \text{ט}^2.$$

The ש it will be observed is a compound combinant, being a biquadratic function of quantities all of which are invariants of the system U, V, W; the ט on the other hand is a simple combinant of the sixth degree.

The general dodecadic combinant ש may also in another manner be exhibited as a biquadratic function of cubic functions of the coefficients of the three given quadratics; but these cubic functions will no longer be invariants of the given quadratics. Thus, form the Jacobian of U, V, W, that is, the determinant

$$\begin{vmatrix} \dfrac{dU}{dx}, & \dfrac{dU}{dy}, & \dfrac{dU}{dz} \\ \dfrac{dV}{dx}, & \dfrac{dV}{dy}, & \dfrac{dV}{dz} \\ \dfrac{dW}{dx}, & \dfrac{dW}{dy}, & \dfrac{dW}{dz} \end{vmatrix},$$

which will be a cubic covariant to the system. The S of this will be another form of שׁ. So too, again, if we border the matrix to the Jacobian determinant above written vertically and horizontally with ξ, η, ζ, and call the determinant of the matrix thus formed I', I' will be quadratic in the system x, y, z, in the system ξ, η, ζ, and in the system formed by the coefficients of U, V, W, and the result of affecting this with the operator Σ will be the same as the result of the operation upon Ω with the same symbol; that is to say, $\frac{1}{24}E \,.\, I'$ will be equal to ט, this latter symbol being so taken (as last explained) in such a way that $3R$ shall equal $4\,\text{שׁ} - \text{ט}^2$, and each of the four lines in the operator Σ being supposed to go through their complete number (6) of permutations.

The terms sextic and dodecadic combinants will not be sufficient *per se* to characterize שׁ or ט (to a numerical factor *près*), supposing that there exist combinants of the 3rd and 9th degrees respectively in the coefficients, in which case the general sextic would contain two and the general dodecadic five arbitrary numerical parameters.

This makes so much the more remarkable and satisfactory the method above developed for finding שׁ and ט as undecompounded forms; the general dodecadic combinant at all events being rendered indeterminate by virtue of the existence of a sextic combinant above demonstrated.

It is interesting to evince the identity of the S of the Jacobian with that of the discriminant to the conjunctive of U, V, W, which latter has been called שׁ.

Starting with the canonical forms of the system U, V, W, and neglecting the ρ and σ, which cannot influence the result of the intended comparison, we have

$$J(U, V, W) = \begin{vmatrix} x, & -y, & 0 \\ 0, & -y, & z \\ gz+hy, & hx+y+fz, & gz+hy \end{vmatrix}$$
$$= fx(y^2+z^2) + gy(x^2+z^2) + hz(x^2+y^2) + xyz.$$

And multiplying by 6 and adopting the same notation as before (from the *Higher Plane Curves*, p. 182), we have

$$b_1 = 2f, \quad b_2 = 0, \quad b_3 = 2h,$$
$$a_1 = 0, \quad a_2 = 2g, \quad a_3 = 2h,$$
$$c_1 = 2f, \quad c_2 = 2y, \quad c_3 = 0,$$
$$d = 1.$$

And the expression for S in *Higher Plane Curves*, p. 184, becomes, omitting every term containing a_1, b_2, or c_3,

$$d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2)$$
$$- (b_3c_2a_2a_3 + a_3c_1b_1b_3 + a_2b_1c_1c_2) + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2),$$

that is

$$1 - 8(f^2 + g^2 + h^2) + 48fgh - 16(h^2g^2 + g^2f^2 + f^2h^2) + 16(f^4 + g^4 + h^4),$$

so that

$$\underset{x,\, y,\, z}{S}\; J(U, V, W) = \text{שׁ} = \underset{\lambda,\, \mu,\, \nu}{S}\;\; \underset{x,\, y,\, z}{\square}\, (\lambda U + \mu V + \nu W),$$

as was to be shown. As observed above, the form first found has the advantage over the one just obtained in disclosing the elements (cubic invariants to U, V, W) of which the שׁ is a biquadratic function. So, analogously, the resultant of two quadratic functions (P, Q) of x and y may be exhibited either under the form of the discriminant in respect to the coefficients of conjunction of the discriminant in respect to the original variables of the conjunctive of P, Q, or under the form of the discriminant of the Jacobian of P, Q. The former discloses the invariantive composition of the resultant which remains latent in the latter. As regards the ט, the proof of its being capable of the second mode of generation above indicated must, on account of the tediousness of the calculation, be for the present reserved; nor can I assert the fact with entire confidence until I have made a more complete investigation into the combinants of the system U, V, W, the remarks concerning which, in p. [416, Vol. I.], I wish to be considered as provisionally withdrawn.

The analogy between the invariants of a cubic form of three variables and a biquadratic of two has been frequently insisted upon in the foregoing pages; but we shall now see that this analogy has its foundation in the deeper-seated analogy which connects a ternary system of quadratics of three variables with a binary system of cubics of two variables.

We may suppose the two given functions so combined that the linear conjunctive $lP + mQ$ shall contain two equal roots, and so take the form x^2y; this may then be combined with either of the given functions so as to give a conjunctive of the form

$$ax^3 + 3xy^2 + dy^3,$$

and writing for x and y, $\dfrac{x}{\sqrt[3]{a}}$, $\dfrac{y}{\sqrt[3]{d}}$, respectively, and multiplying $lP + mQ$ by $\sqrt[3]{a^2d}$, we obtain for our standard form

$$P = 3x^2y,$$

$$Q = x^3 + 3exy^2 + y^3.$$

The resultant of this system rejecting an universally-irrelevant numerical factor is 1.

Again, write

$$\lambda P + \mu Q = 3\lambda x^2y + \mu x^3 + 3\mu exy^2 + \mu y^3,$$

and operate upon this with the commutator (say ω) [see Vol. I., p. 306],

$$\begin{vmatrix} \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \end{vmatrix}.$$

Keeping one of the lines (for example, the first) stationary, and, for greater brevity, writing δ_λ, δ_μ, δ_x, δ_y in place of $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{dx}, \frac{d}{dy}$, we obtain 8 positions, which, remembering that the order *in* the lines of these positions (and not the order *of* the lines) is the only thing to be attended to, are equivalent to

$$\begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \end{vmatrix} - 3 \times \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \\ \delta_y, & \delta_x \end{vmatrix} + 3 \times \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \end{vmatrix} - \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \end{vmatrix}.$$

Hence we have $\frac{1}{36}\omega(\lambda P + \mu Q) = -e$.

I need hardly observe, that in general for any two odd-degreed functions of the same degree in x, y, as

$$a_0 x^m + m a_1 x^{m-1} y + \tfrac{1}{2} m(m-1) a_2 x^{m-2} y^2 + \ldots + m(a_1) x y^{m-1} + (a_0) y^m,$$
$$b_0 x^m + m b_1 x^{m-1} y + \tfrac{1}{2} m(m-1) b_2 x^{m-2} y^2 + \ldots + m(b_1) x y^{m-1} + (b_0) y^m,$$

we may obtain, in an analogous manner, the combinant

$$a_0(b_0) - m a_1(b_1) + \tfrac{1}{2} m(m-1) a_2(b_2) + \&c.$$

Moreover it is easily shown that when m is an even integer the above expression will remain invariant, although of course it is no longer a combinant.

Again, the Hessian to $\lambda P + \mu Q$ will be

$$\begin{vmatrix} \mu x + \lambda y, & \lambda x + \mu e y \\ \lambda x + \mu e y, & \mu e x + \mu y \end{vmatrix},$$

which is equal to

$$e\mu^2 x^2 + \mu^2 xy - e^2 \mu^2 y^2 + \lambda\mu y^2 - e\lambda\mu xy - \lambda^2 x^2,$$

which call $H.C$ (C meaning the conjunctive of P, Q). Let this be operated upon with the commutator

$$\begin{matrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2, \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2, \end{matrix}$$

which call Ω.

Since neither $y^2\lambda^2$ nor $xy\lambda^2$ enters in $H.C$, we have only to consider out of the full number 6 of positions the two effective positions

$$\begin{vmatrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2 \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2 \end{vmatrix} - \begin{vmatrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2 \\ \delta_\lambda^2, & \delta_\mu^2, & \delta_\lambda\delta_\mu \end{vmatrix}.$$

Hence $$\tfrac{1}{16}EHC(P, Q) = 1 - e^3.$$

So that $$\{-\tfrac{1}{36}\omega C(P, Q)\}^3 + \{\tfrac{1}{16}EHC(P, Q)\} = R(P, Q).$$

Thus R is expressed in terms of the cube of a simple quadratic combinant and a sextic compound combinant, which is made up of quadratic invariants. When P and Q become of the form $\frac{d\psi}{dx}, \frac{d\psi}{dy}$, respectively ($\psi$ being a quartic in x and y), these become respectively (to numerical factors *près*) the quadrinvariant of the given function and the cube invariant of its Hessian, which latter is a linear function of the cube of the quadrinvariant and the square of the cubinvariant of the given function, as we know *à priori* from the fact of the fundamental scale of the quartic consisting of the quadrinvariant and cubinvariant (for a rigid demonstration of which fact see the *Philosophical Magazine* in the early part of 1853 [Vol. I., p. 599]), and the expression for the resultant thus resolves itself into the known composite form of the sum of a square and cube.

The simple sextic combinant represented by $E.H.C(P, Q)$ may also, analogous to what has been observed concerning the ט, be expressed as a commutant (in fact the cubinvariant) of the Jacobian to P and Q, but then the form will no longer disclose its invariantive sub-composition. So too, if it were thought worth while to push the analogies to an extreme, the quadricombinant to P, Q might have been found, first by bordering the Hessian to the conjunctive to P, Q with ξ, η horizontally and vertically, and operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix},$$

or by bordering the Jacobian to P, Q with ξ, η, as before, and then operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx'}, & \frac{d}{dy'} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi'}, & \frac{d}{d\eta'} \end{vmatrix}.$$

I propose hereafter to return to the consideration of the fundamental scale of combinants to the two systems, namely of three quadratics in x, y, z, and of two cubics in x, y, which have been treated of in this section.

Section VIII.

On the Reduction of a Sextic Function of Two Variables to its Canonical Form.

In the *London and Edinburgh Philosophical Magazine* for Nov. 1851, after giving a simple method for representing any function of two variables of an odd degree $(x, y)^{2m+1}$ under the form of

$$u_1^{2m+1} + u_2^{2m+1} + \ldots + u_{m+1}^{2m+1},$$

where $u_1, u_2 \ldots u_{m+1}$ are linear functions of x, y (which form, as appears from the method of obtaining it, is unique), I proceeded [Vol. I., p. 269] to show how by a certain method therein explained the biquadratic and octavic functions of x, y, $(x, y)^4$, $(x, y)^8$ could be thrown under the respective forms

$$u_1^4 + u_2^4 + mu_1^2u_2^2,$$

$$u_1^8 + u_2^8 + u_3^8 + u_4^8 + mu_1^2u_2^2u_3^2u_4^2,$$

the number of values of m in the first form being three and in the second form five, the quantity m in the one case depending on the solution of the equation

$$\begin{vmatrix} a_0, & a_1, & a_2+\lambda \\ a_1, & a_2-\frac{1}{2}\lambda, & a_3 \\ a_2+\lambda, & a_3, & a_4 \end{vmatrix} = 0,$$

where a_0, a_1, a_2, a_3, a_4 are the coefficients of $(x, y)^4$ multiplied respectively by $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1$; and in the other case, on the solution of the equation

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & a_4+\lambda \\ a_1, & a_2, & a_3, & a_4-\frac{1}{4}\lambda, & a_5 \\ a_2, & a_3, & a_4+\frac{1}{6}\lambda, & a_5, & a_6 \\ a_3, & a_4-\frac{1}{4}\lambda, & a_5, & a_6, & a_7 \\ a_4+\lambda, & a_5, & a_6, & a_7, & a_8 \end{vmatrix} = 0,$$

$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8,$

being the coefficients of $(x, y)^8$ multiplied respectively by

$$1, \tfrac{1}{8}, \tfrac{1}{28}, \tfrac{1}{56}, \tfrac{1}{70}, \tfrac{1}{56}, \tfrac{1}{28}, \tfrac{1}{8}, 1.$$

Before proceeding to investigate the theory of these methods of reduction under any more general point of view, it will be convenient to seek to obtain the representation of $(x, y)^6$ under some analogous form.

It might at first be supposed that the corresponding form should be

$$u_1^6 + u_2^6 + u_3^6 + mu_1^2 u_2^2 u_3^2;$$

if, however, the method which succeeds for the quartic and octavic functions be attempted to be applied to this it will be found entirely to fail. Here, however, considerations of a purely morphological character step in to our aid and immediately lead to the true canonical representation of the sextic function. Algebraically speaking, the only connexion between two identical forms F and F is through the equation $F = \psi^{-1}\psi F$; but, morphologically considered, a *form* F may admit of being derived by a series of entirely heterogeneous operations from *itself*. In general, supposing

$$F(x, y) = ax^n + nbx^{n-1}y + \&\text{c.} \ldots + n(b)\, xy^{n-1} + (a)\, y^n,$$

the form
$$\xi^n \frac{d}{da} + \xi^{n-1}\eta \frac{d}{db} + \ldots + \xi\eta^{n-1}\frac{d}{d(b)} + \eta^n \frac{d}{d(a)},$$

operating upon any concomitant to F will, we know (from the law of reciprocity in Section IV.), produce another concomitant. The operative form above written is termed the evector, and the result of operating therewith upon a concomitant is termed the evectant of the latter, which is said, when so operated upon, to be evected*. The polar reciprocal of the evector may be termed the contravector, and for two variables is of course of the form

$$y^n \frac{d}{da} - y^{n-1}x \frac{d}{db} \pm \&\text{c.}$$

If we suppose n to be even, $F(x, y)$ will have the well-known quadrinvariant

$$a(a) - nb(b) + \tfrac{1}{2}n(n-1)\, c(c) \mp \&\text{c.},$$

and if this be operated upon with the contravector, or if we like so to say, be contravected, we recover the original function F, so that any function of two variables of an even degree is the contravect of its quadrinvariant.

* These terms "evector, evectant, contravectant, to evect and contravect," will of course admit of an immediate extension to functions of any number of variables. Evection gives rise to contravariants, contravection to covariants; but on this account to interchange the meanings respectively attached to the terms evector and contravector, and their respective allied terms, would be a simplification too dearly purchased at the expense of contravening the principle that the *word for the base should be the base for the word.*

If now we return to the representation of $(x, y)^4$ under the form

$$u_1^4 + u_2^4 + m(u_1 u_2)^2,$$

and make $$u_1 u_2 = F_2(x, y);$$

or to that of $(x, y)^8$ under the form

$$u_1^4 + u_2^4 + u_3^4 + u_4^4 + m(u_1 u_2 u_3 u_4)^2,$$

and make $$u_1 u_2 u_3 u_4 = F_4(x, y),$$

the outstanding terms multiplied by the parameter m may be regarded in each of these two cases as the squared contravects of the quadrinvariants F_2 and F_4 respectively. Under this point of view we at once see a ground for the proved fact of $(x, y)^6$ not being capable of being thrown under the form

$$u_1^6 + u_2^6 + u_3^6 + m\{F_3(x, y)\}^2,$$

where $$u_1 u_2 u_3 = F_3(x, y),$$

because there exists no quadrinvariant to $F_3(x, y)$, the only invariant which it possesses being the discriminant which is of the fourth degree; if however instead of $m\{F_3(x, y)\}^2$ we write $mF_3(x, y)\, G_3(x, y)$, where $G_3(x, y)$ is the contravect of the discriminant of F_3, we shall find that the method applied to the reduction of $(x, y)^4$ and to $(x, y)^8$ will perfectly well succeed for $(x, y)^6$, as I proceed to demonstrate.

Let this function be written under the form

$$a_0x^6 + 6a_1x^5y + 15a_2x^4y^2 + 20a_3x^3y^3 + 15a_4x^2y^4 + 6a_5xy^5 + a_6y^6,$$

which suppose made equal to

$$(p_1x + q_1y)^6 + (p_2x + q_2y)^6 + (p_3x + q_3y)^6$$
$$+ (Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2y + Nxy^2 + Py^3);$$

where

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y) = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3;$$

the discriminant of this will be, as is well known,

$$A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD,$$

and contravecting this with the operator

$$-y^3\frac{d}{dA} + y^2x\frac{d}{dB} - yx^2\frac{d}{dC} + x^3\frac{d}{dD},$$

and identifying the result with $Lx^3 + Mx^2y + Nxy^2 + Py^3$, we have

$$L = -6ABC + 2A^2D + 4B^3,$$
$$M = 6ABD - 12AC^2 + 6B^2C,$$
$$N = -6ACD + 12DB^2 - 6BC^2,$$
$$P = 6BCD - 2AD^2 - 4C^3.$$

A, B, C, D are known functions of p_1, p_2, p_3; q_1, q_2, q_3, and we shall have seven equations for determining these six unknown quantities and the unknown parameter m.

Let
$$q_1 = p_1\lambda_1, \quad q_2 = p_2\lambda_2^4, \quad q_3 = p_3\lambda_3,$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 3s_1, \quad \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = 3s_2, \quad \lambda_1\lambda_2\lambda_3 = s_3,$$
$$p_1 p_2 p_3 = m.$$

Then
$$A = m, \quad 3B = 3ms_1, \quad 3C = 3ms_2, \quad D = s_3.$$
$$L = m^3(4s_1^3 - 6s_1s_2 + 2s_3),$$
$$M = m^3(6s_1^2s_2 + 6s_1s_3 - 12s_2^2),$$
$$N = m^3(12s_1^2s_3 - 6s_1^2s_2 - 6s_2s_3),$$
$$P = m^3(+6s_1s_2s_3 - 4s_2^3 - 2s_3^2).$$

Let
$$(Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2y + Nxy^2 + Py^3)$$
$$= K_0x^6 + K_1x^5y + K_2x^4y^2 + K_3x^3y^3 + K_4x^2y^4 + K_5xy^5 + K_6y^6 = T.$$

Then, equating this term for term with
$$\lambda_1^6(x + p_1y)^6 + \lambda_2^6(x + p_2y)^6 + \lambda_3^6(x + p_3y)^6 + \mu T,$$

we obtain the seven equations following:

$$p_1^6 + p_2^6 + p_3^6 + \mu K_0 = a_0, \tag{1}$$

$$p_1^6\lambda_1 + p_2^6\lambda_2 + p_3^6\lambda_3 + \frac{\mu}{6}K_1 = a_1, \tag{2}$$

$$p_1^6\lambda_1^2 + p_2^6\lambda_2^2 + p_3^6\lambda_3^2 + \frac{\mu}{15}K_2 = a_2, \tag{3}$$

$$p_1^6\lambda_1^3 + p_2^6\lambda_2^3 + p_3^6\lambda_3^3 + \frac{\mu}{20}K_3 = a_3, \tag{4}$$

$$p_1^6\lambda_1^4 + p_2^6\lambda_2^4 + p_3^6\lambda_3^4 + \frac{\mu}{15}K_4 = a_4, \tag{5}$$

$$p_1^6\lambda_1^5 + p_2^6\lambda_2^5 + p_3^6\lambda_3^5 + \frac{\mu}{6}K_5 = a_5, \tag{6}$$

$$p_1^6\lambda_1^6 + p_2^6\lambda_2^6 + p_3^6\lambda_3^6 + \mu K_6 = a_6. \tag{7}$$

Eliminating linearly

p_1^6,	p_2^6,	p_3^6	between	equations	1, 2, 3, 4,
$\lambda_1p_1^6$,	$\lambda_2p_2^6$,	$\lambda_3p_3^6$	,,	,,	2, 3, 4, 5,
$\lambda_1^2p_1^6$,	$\lambda_2^2p_2^6$,	$\lambda_3^2p_3^6$	,,	,,	3, 4, 5, 6,
$\lambda_1^3p_1^6$,	$\lambda_2^3p_2^6$,	$\lambda_3^3p_3^6$	,,	,,	4, 5, 6, 7,

we obtain the four equations following, namely,

$$a_0 s_3 - 3a_1 s_2 + 3a_2 s_1 - a_3 = \mu\vartheta_0,$$

$$a_1 s_3 - 3a_2 s_2 + 3a_3 s_1 - a_4 = \mu\vartheta_1,$$

$$a_2 s_3 - 3a_3 s_2 + 3a_4 s_1 - a_5 = \mu\vartheta_2,$$

$$a_3 s_3 - 3a_4 s_2 + 3a_5 s_1 - a_6 = \mu\vartheta_3,$$

where
$$\vartheta_0 = K_0 s_3 - \tfrac{3}{6} K_1 s_2 + \tfrac{3}{15} K_2 s_1 - \tfrac{1}{20} K_3$$
$$= \tfrac{1}{60}(60K_0 s_3 - 30K_1 s_2 + 12K_2 s_1 - 3K_3),$$

$$\vartheta_1 = \tfrac{1}{6} K_1 s_3 - \tfrac{3}{15} K_2 s_2 + \tfrac{3}{20} K_3 s_1 - \tfrac{1}{15} K_4$$
$$= \tfrac{1}{60}(10K_1 s_3 - 12K_2 s_2 + 9K_3 s_1 - 4K_4),$$

$$\vartheta_2 = \tfrac{1}{15} K_2 s_3 - \tfrac{3}{20} K_3 s_2 + \tfrac{3}{15} K_4 s_1 - \tfrac{1}{6} K_5$$
$$= \tfrac{1}{60}(4K_2 s_3 - 9K_3 s_2 + 12K_4 s_1 - 10K_5),$$

$$\vartheta_3 = \tfrac{1}{20} K_3 s_3 - \tfrac{3}{15} K_4 s_2 + \tfrac{3}{6} K_5 s_1 - K_6$$
$$= \tfrac{1}{60}(3K_3 s_3 - 12K_4 s_2 + 30K_5 s_1 - 60K_6),$$

and

$$\frac{1}{m^4} K_0 = \frac{A}{m} \cdot \frac{L}{m^3} = 4s_1^3 - 6s_1 s_2 + 2s_3,$$

$$\frac{1}{m^4} K_1 = \frac{AM + 3BL}{m^4} = (6s_1^2 s_2 + 6s_1 s_3 - 12s_2^2) + 3s_1(4s_1^3 - 6s_1 s_2 + 2s_3)$$
$$= 12s_1^4 - 12s_1^2 s_2 + 12s_1 s_3 - 12s_2^2,$$

$$\frac{1}{m^4} K_2 = \frac{AN + 3BM + 3CL}{m^4} = (12s_1^2 s_3 - 6s_1 s_2^2 - 6s_2 s_3)$$
$$+ (18s_1^3 s_2 + 18s_1^2 s_3 - 36s_1 s_2^2)$$
$$+ (12s_1^3 s_2 - 18s_1 s_2^2 + 6s_2 s_3)$$
$$= 30s_1^3 s_2 + 30s_1^2 s_3 - 60s_1 s_2^2,$$

$$\frac{1}{m^4} K_3 = \frac{AP + 3BN + 3CM + DL}{m^4} = (6s_1 s_2 s_3 - 4s_2^3 - 2s_3^2)$$
$$+ (36s_1^3 s_3 - 18s_1^2 s_2^2 - 18s_1 s_2 s_3)$$
$$+ (18s_1^2 s_2^2 + 18s_1 s_2 s_3 - 36s_2^3)$$
$$+ (4s_1^3 s_3 - 6s_1 s_2 s_3 + 2s_3^2)$$
$$= 40s_1^3 s_3 - 40s_2^3,$$

$$\frac{1}{m^4} K_4 = \frac{3BP + 3CN + DM}{m^4} = 18s_1^2 s_2 s_3 - 12s_1 s_2^3 - 6s_1 s_3^2$$
$$+ 36s_1^2 s_2 s_3 - 18s_1 s_2^3 - 18s_2^2 s_3$$
$$+ 6s_1^2 s_2 s_3 + 6s_1 s_3^2 - 12s_2^2 s_3$$
$$= 60s_1^2 s_2 s_3 - 30s_1 s_2^3 - 30s_2^2 s_3,$$

$$\frac{1}{m^4}K_5 = \frac{3CP + DN}{m^4} = 18s_1s_2^2s_3^2 - 12s_2^4 - 6s_2s_3^2$$
$$+ 12s_1^2s_3^2 - 6s_1s_2^2s_3 - 6s_2s_3^2$$
$$= 12s_1^2s_3^2 + 12s_1s_2^2s_3 - 12s_2^4 - 12s_2s_3^2,$$

$$\frac{1}{m^4}K_6 = \frac{DP}{m^4} = 6s_1s_2s_3^2 - 4s_2^3s_3 - 2s_3^3;$$

therefore
$$\frac{60\vartheta_0}{\mu m^4} = 240s_1^3s_3 - 360s_1s_2s_3 + 120s_3^2$$
$$- 360s_1^4s_2 + 360s_1^2s_2^2 - 360s_1s_2s_3 + 360s_2^3$$
$$+ 360s_1^4s_2 + 360s_1^3s_3 - 720s_1^2s_2^2$$
$$- 120s_1^3s_3 + 120s_2^3$$
$$= 120(s_3^2 + 4s_2^3 + 4s_1^3s_3 - 3s_1^2s_2^2 - 6s_1s_2s_3),$$

that is
$$\vartheta_0 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD).$$

Again,
$$\frac{\vartheta_1}{\mu m^4} = 120s_1^4s_3 - 120s_1^2s_2s_3 + 120s_1s_3^2 - 120s_2^2s_3$$
$$- 360s_1^3s_2^2 - 360s_1^2s_2s_3 + 720s_1s_2^3$$
$$+ 360s_1^4s_3 - 360s_1s_2^3$$
$$- 240s_1^2s_2s_3 + 120s_1s_2^3 + 120s_1s_2^2s_3$$
$$= 120(s_1s_3^2 + 4s_1s_2^3 + 4s_1^4s_3 - 3s_1^3s_2^2 - 6s_1^2s_2s_3);$$

therefore
$$\vartheta_1 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_1.$$

Again,
$$\frac{60\vartheta_2}{\mu m^4} = 120s_1^3s_2s_3 + 120s_1^2s_3^2 - 240s_1s_2^2s_3$$
$$- 360s_1^3s_2s_3 + 360s_2^4$$
$$+ 720s_1^3s_2s_3 - 360s_1^2s_2^3 - 360s_1s_2^2s_3$$
$$- 120s_1^2s_3^2 - 120s_1s_2^2s_3 + 120s_2^4 + 120s_2s_3^2$$
$$= 120(s_2s_3^2 + 4s_2^4 + 4s_1^3s_2s_3 - 3s_1^2s_2^3 - 6s_1s_2^2s_3),$$

therefore
$$\vartheta_2 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_2.$$

Finally,
$$\frac{60\vartheta_3}{\mu m^4} = 120s_1^3s_3^2 - 120s_2^3s_3$$
$$- 720s_1^2s_2^2s_3 + 360s_1s_2^4 + 360s_2^3s_3$$
$$+ 360s_1^3s_3^2 + 360s_1^2s_2^2s_3 - 360s_1s_2^4 - 360s_1s_2s_3^2$$
$$- 360s_1^2s_2s_3^2 + 240s_2^3s_3 + 120s_3^3$$
$$= 120(s_3^3 + 4s_2^3s_3 + 4s_1^3s_3^2 - 3s_1^2s_2^2s_3 - 6s_1s_2s_3^2),$$

therefore $\vartheta_3 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)\,s_3$.

Hence, writing

$$2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD) = \rho,$$

the four equations connecting a_0, a_1, a_2, a_3 with ϑ_0, ϑ_1, ϑ_2, ϑ_3, take the form

$$a_0s_3 - 3a_1s_2 + 3a_2s_1 - (a_3 + \rho) = 0,$$
$$a_1s_3 - 3a_2s_2 + 3\left(a_3 - \tfrac{1}{3}\rho\right)s_1 - a_4 = 0,$$
$$a_2s_3 - 3\left(a_3 + \tfrac{1}{3}\rho\right)s_2 + 3a_4s_1 - a_5 = 0,$$
$$(a_3 - \rho) - 3a_4s_2 + 3a_5s_1 - a_6 = 0.$$

Hence we derive the equation involving only the known coefficients of the given function for finding ρ, namely, the determinant

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 + \rho \\ a_1, & a_2, & a_3 - \tfrac{1}{3}\rho, & a_4 \\ a_2, & a_3 + \tfrac{1}{3}\rho, & a_4, & a_5 \\ a_3 - \rho, & a_4, & a_5, & a_6 \end{vmatrix} = 0. \qquad \text{(R)}$$

If in this matrix ρ be changed into $-\rho$, the determinant evidently remains unaltered in value; hence the odd powers of ρ disappear from the equation, and ρ may be found by the solution of a double quadratic only. In fact the above equation for finding ρ, expanded out, becomes

$$\frac{\rho^4}{9} + \left(\frac{2}{3}\begin{vmatrix} a_2, & a_3 \\ a_3, & a_4 \end{vmatrix} - \frac{2}{3}\begin{vmatrix} a_1, & a_3 \\ a_3, & a_5 \end{vmatrix} + \begin{vmatrix} a_2, & a_3 \\ a_3, & a_4 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} a_0, & a_3 \\ a_3, & a_6 \end{vmatrix}\right)\rho^2,$$

$$- \begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix} = 0;$$

that is $$\rho^4 + (15a_2a_4 - 6a_1a_5 - 10a_3^2 + a_0a_6)\rho^2$$

$$+ \begin{vmatrix} a_3, & a_2, & a_1, & a_0 \\ a_4, & a_3, & a_2, & a_1 \\ a_5, & a_4, & a_3, & a_2 \\ a_6, & a_5, & a_4, & a_3 \end{vmatrix} = 0;$$

the coefficient of ρ^2 being the well-known quadrinvariant, and the final term the meiocatalecticizant of the given function. There will consequently be four different values of ρ and four different systems of values of s_1, s_2, s_3, expressible for each system respectively in terms of ρ by means of any three out of the four equations (R), and consequently there will be four systems of values of λ_1, λ_2, λ_3, each of which may be found separately by solving the cubic equation

$$\lambda^3 - 3s_1\lambda^2 + 3s_2\lambda - s_3 = 0;$$

also K_0, K_1, K_2, K_3, K_4, K_5, K_6 become known multiples of m^4, and finally, the values of any λ and K system being thus determined, we may then, by means of the identity

$$p_1^6(x+\lambda_1 y)^6 + p_2^6(x+\lambda_2 y)^6 + p_3^6(x+\lambda_3 y)^6$$
$$+\mu m^4\left(\frac{K_0}{m^4}x^6 + \frac{K_1}{m^4}x^5y + \&\text{c.} + \frac{K_6}{m^4}y^6\right) = a_0x^6 + 6a_1x^5y + \&\text{c.} + a_6y^6,$$

write down at will any four equations out of the seven equations therefrom resulting, and these will serve to determine linearly the values of p_1^6, p_2^6, p_3^6, μm^4; and consequently, by means of the equations

$$q_1 = p_1\rho_1, \quad q_2 = p_2\rho_2, \quad q_3 = p_3\rho_3,$$

q_1, q_2, q_3 are known, and consequently every coefficient in

$$(p_1x+q_1y)^6 + (p_2x+q_2y)^6 + (p_3x+q_3y)^6 + \mu M$$

is completely determined. But we shall hereafter return to this theory, and seek for a direct method of finding the four values of the functions

$$(p_1x+q_1y), \quad (p_2x+q_2y), \quad (p_3x+q_3y).$$

It appears from the above investigation that there are four modes of throwing $(x, y)^6$ under the assumed form which possess the remarkable property of separating into two pairs of modes, as is obvious from the fact of the resolving equation in ρ having two pairs of roots, those of the same pair being equal but of contrary signs. As this form will be of extreme value in studying the invariants of $(x, y)^6$, it may be well to consider the simplest shape to which it admits of being reduced.

We may suppose $(p_1x+q_1y)(p_2x+q_2y)(p_3x+q_3y)$ thrown under the form of u^3+v^3, the contravectant of the discriminant to which in respect to u and v is v^3-u^3, so that we may use for the canonical form the expression

$$a(u+v)^6 + b(u+\rho v)^6 + c(u+\rho^2 v)^6 + \mu(u^6 - v^6),$$

where $\rho^3 = 1$; or if we please, more simply

$$a(u+v)^6 + b(u+\rho v)^6 + c(u+\rho^2 v)^6 + u^6 - v^6.$$

I may take this occasion to observe that there are generally two modes of a distinct kind for obtaining any simple concomitant; the difference (a most important practical one) consisting in the circumstance that in the one mode there are differentiations to be performed in respect to the coefficients, the consequence of which is that the whole of the operations must be gone through for obtaining the concomitant with the primitive in its most general form, and no advantage can be taken in the course of these operations of the simplification resulting from the absence of any terms in the primitive or of any other speciality therein; whereas in the other mode of derivation, where all the differentiations have to be performed quâ the variables only, the partial form may be operated with throughout. Thus, for instance, to find the contravectant to the discriminant of a cubic function the general form of the cubic must be employed, and then the special values of the coefficient corresponding to a specific form of the cubic substituted at the close of the operations; but this same concomitant may also be obtained by taking the resultant of the first emanant of the given cubic and of the first emanant of its Hessian in respect to the variables of emanation, and consequently the specific form may, after this mode, be retained from the first. Thus, if we start with u^3+v^3, the Hessian is uv, and the two emanants in question will be $u^2u'+v^2v'$ and $vu'+uv'$, the resultant of which in respect to u' and v' is u^3-v^3; or, again, if we commence with uvw subject to the relation that $u+v+w=0$, the Hessian will be

$$\begin{vmatrix} 0, & w, & v, & 1 \\ w, & 0, & u, & 1 \\ v, & u, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

that is to say, $$u^2+v^2+w^2-2uv-2uw-2vw.$$

The two emanants will then be

$$vwu'+wuv'+uvw'$$

$$(u-v-w)\,u'+(v-w-u)\,v'+(w-u-v)\,w',$$

subject to the relation

$$u'+v'+w'=0;$$

and taking the resultant of these three equations, or, which is the same thing, of

$$vwu'+wuv'+uvw',$$

$$uu'+vv'+ww',$$

$$u'+v'+w',$$

we obtain the determinant

$$\begin{vmatrix} vw, & wu, & uv \\ u, & v, & w \\ 1, & 1, & 1 \end{vmatrix},$$

which is equal to

$$vw\,(v-w)+wu\,(w-u)+uv\,(u-v),$$

that is to say $$(u-v)\,(v-w)\,(w-u).$$

Hence another variety of the external shape to which the canonical form for the sextic function of x, y may be reduced will be

$$au^6 + bv^6 + cw^6 + \mu uvw (u - v)(v - w)(w - u).$$

I shall presently revert to the theory of the corresponding mode of reducing to their canonical forms the biquadratic and octavic functions of x, y, the number of solutions for which will be respectively three and five, and the discovery of which, as shown by me in the Number of the *Philosophical Magazine* before adverted to, depends upon the solution of equations of the third and fifth degrees in ρ expressed by means of determinants of the third and fifth orders formed in precise correspondence with that of the fourth order, upon which, as we have found above, the reduction of the sextic function to its canonical form depends.