

8.

NOTE ON A FORMULA BY AID OF WHICH AND OF A TABLE OF SINGLE ENTRY THE CONTINUED PRODUCT OF ANY SET OF NUMBERS (OR AT LEAST A GIVEN CONSTANT MULTIPLE THEREOF) MAY BE EFFECTED BY ADDITIONS AND SUBTRACTIONS ONLY WITHOUT THE USE OF LOGARITHMS.

[*Philosophical Magazine*, VII. (1854), pp. 430—436.]

INTRODUCTION.

THE remark to which this note refers is not new; it has been well observed somewhere in Gergonne's *Annales* (Mr Cayley being my informant), that by aid of the formula $4ab = (a + b)^2 - (a - b)^2$ the question of finding the product of two numbers is virtually reduced to a process of addition and subtraction, and of finding the values of two squares out of a table of squares. If the two factors a and b are both even or both odd, the formula ought to be changed into

$$ab = \left(\frac{a + b}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2;$$

if one of them is odd and the other even, we may employ the formula

$$ab = \left(\frac{a + b - 1}{2}\right)^2 - \left(\frac{a - b + 1}{2}\right)^2 + a.$$

So, again, for the product of three numbers, there exists the analogous formula

$$24abc = (a + b + c)^3 - (a + b - c)^3 - (b + c - a)^3 - (c + a - b)^3.$$

OBJECT OF THE PAPER.

The object of this brief note is to exhibit and demonstrate the generalization of the above formulæ, that is, to express the product of any n quantities $a_1, a_2, a_3, \dots, a_n$ under the form of the sum of powers of simple linear functions of a_1, a_2, \dots, a_n . This may be done as follows:

GENERAL FORMULA.

Let $\theta_1, \theta_2, \theta_3, \dots, \theta_n$
 be disjunctively equal to $1, 2, 3, \dots, n$,
 then $2 \cdot 4 \cdot 6 \dots (2n) a_1 a_2 \dots a_n$
 $= (a_{\theta_1} + a_{\theta_2} + a_{\theta_3} + \dots + a_{\theta_n})^n - \Sigma (-a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n$
 $+ \Sigma (-a_{\theta_1} - a_{\theta_2} + a_{\theta_3} + \dots + a_{\theta_n})^n + \&c.$
 $+ (-)^n (-a_{\theta_1} - a_{\theta_2} - \dots - a_{\theta_n})^n,$

which I call the principal equation.

DEMONSTRATION OF THE PRINCIPAL EQUATION.

Let $\phi_1, \phi_2, \phi_3, \dots, \phi_{n-1}$
 be disjunctively equal to $1, 2, 3, \dots, (n-1)$,
 then it is easily seen that

$$\begin{aligned} (a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n &= (a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} + a_n)^n \\ \Sigma (-a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n &= (a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} - a_n)^n \\ &\quad + \Sigma (-a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} + a_n)^n \\ \Sigma (-a_{\theta_1} - a_{\theta_2} + \dots + a_{\theta_n})^n &= \Sigma (-a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} - a_n)^n \\ &\quad + \Sigma (-a_{\phi_1} - a_{\phi_2} + \dots + a_{\phi_{n-1}} + a_n)^n \\ \&c. &= \&c. \\ \Sigma (-a_{\theta_1} - a_{\theta_2} \dots - a_{\theta_{n-1}} + a_{\theta_n})^n &= \Sigma (-a_{\phi_1} - a_{\phi_2} \dots - a_{\phi_{n-2}} + a_{\phi_{n-1}} - a_n)^n \\ &\quad + (-a_{\phi_1} - a_{\phi_2} \dots - a_{\phi_{n-1}} + a_n)^n \\ (-a_{\theta_1} - a_{\theta_2} \dots - a_{\theta_{n-1}} - a_{\theta_n})^n &= (-a_{\phi_1} - a_{\phi_2} \dots - a_{\phi_{n-1}} - a_n)^n. \end{aligned}$$

Hence it is apparent that when $a_n = 0$, the right-hand side of the so-called principal equation spontaneously vanishes; it will therefore always contain a_n as a factor, and by parity of reasoning it will contain every one of the quantities a_1, a_2, \dots, a_n as a factor, and will consequently be equal to the product $a_1 a_2 \dots a_n$ multiplied by a numerical factor, which, by making a_1, a_2, \dots, a_n each equal to unity, is readily seen to be

$$2^n \times (1 \cdot 2 \cdot 3 \dots n),$$

or if we please so to say, $2 \cdot 4 \cdot 6 \dots (2n)$. Q. E. D.

CONCLUSION.

If n is odd and be called $2m + 1$, we have

$$\begin{aligned} & 4 \cdot 6 \cdot 8 \dots (2n) a_1 a_2 \dots a_n \\ &= (a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n - \Sigma (-a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n \\ &+ \Sigma (-a_{\theta_1} - a_{\theta_2} + a_{\theta_3} + \dots + a_{\theta_n})^n \mp \&c. \\ &+ (-)^m (-a_{\theta_1} - a_{\theta_2} \dots - a_{\theta_m} + a_{\theta_{m+1}} + a_{\theta_{m+2}} + \dots + a_{\theta_n})^n; \end{aligned}$$

and if n be even and be called $2m$, we have

$$\begin{aligned} & 4 \cdot 6 \cdot 8 \dots (2n) a_1 a_2 \dots a_n \\ &= (a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n - \Sigma (-a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n \\ &+ \Sigma (-a_{\theta_1} - a_{\theta_2} + a_{\theta_3} + \dots + a_{\theta_n})^n \mp \&c. \\ &+ \frac{1}{2} (-)^m \Sigma (-a_{\theta_1} - a_{\theta_2} \dots - a_{\theta_m} + a_{\theta_{m+1}} + a_{\theta_{m+2}} + \dots + a_{\theta_n})^n; \end{aligned}$$

where, it should be observed, that the last term is made up of integer parts, notwithstanding the presence of the factor $\frac{1}{2}$, which factor may be construed as only serving to denote that, of any pair of complementary linear functions of those which enter into this term, such as

$$-a_{q_1} - a_{q_2} \dots - a_{q_m} + a_{q_{m+1}} + a_{q_{m+2}} + \dots + a_{q_n}$$

and

$$-a_{q_{m+1}} - a_{q_{m+2}} \dots - a_{q_n} + a_{q_1} + a_{q_2} + \dots + a_{q_m},$$

one only is to be retained. The entire term is of course made up exclusively of such pairs.

COROLLARY.

If $R(a_1, a_2, \dots, a_n)$ denote any symmetrical algebraic function whatever of a_1, a_2, \dots, a_n ,

$$\sum_n^0 \sum_{\nu_i}^0 (-)^i R(-a_{\theta_1}, -a_{\theta_2}, \dots - a_{\theta_i}, a_{\theta_{i+1}}, a_{\theta_{i+2}}, \dots, a_{\theta_n})$$

will contain $a_1 a_2 a_3 \dots a_n$ as a factor. In this formula ν_i denotes the number of combinations of n things taken i together.

POSTSCRIPT.

In constructing a table of single entry for applying the formula

$$4ab = (a+b)^2 - (a-b)^2,$$

that is,

$$ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2,$$

it is only necessary to retain the integer part of the quarters of the squares of all the numbers from 2 to the sum of the highest of the values of a and b to which the application of the table is proposed to be restricted, because the

fractional parts of $\left(\frac{a+b}{2}\right)^2$ and $\left(\frac{a-b}{2}\right)^2$ will always destroy one another. A table for the multiplication of a ternary set of factors by means of the formula

$$abc = \frac{1}{24}(a+b+c)^3 - \frac{1}{24}(a+b-c)^3 - \frac{1}{24}(a-b+c)^3 - \frac{1}{24}(-a+b+c)^3$$

will imply the registration of the values of the 24th parts of all numbers up to the highest value of $(a+b+c)$, and it becomes a question of some practical interest to determine in what way the fractional remainders of these 24th parts are to be dealt with.

The formula last written may give rise to either of the two subjoined cases, according as the numbers a, b, c correspond or not to the lengths of a possible triangle, namely:

$$(1) \quad abc = \frac{1}{24}N_1^3 - \frac{1}{24}N_2^3 - \frac{1}{24}N_3^3 - \frac{1}{24}N_4^3,$$

or

$$(2) \quad abc = \frac{1}{24}N_1^3 + \frac{1}{24}N_2^3 - \frac{1}{24}N_3^3 - \frac{1}{24}N_4^3,$$

the quantities N_1, N_2, N_3, N_4 being all supposed to represent positive integers.

A very little consideration will show, that if we neglect fractions in the table there may be entailed an error of 2, 1, 0, or -1. Whether the error is, on the one hand, an error of an even order (namely, 0 or 2), or, on the other hand, of an odd order (namely, 1 or -1), would be at once obvious by looking to see whether the formula, after neglecting the fractions, gave an odd result when the result ought to be odd, and an even result when the result ought to be even, or *vice versa*. And the nature of the result as to whether it *ought* to be odd or even could be immediately inferred from observing whether a, b, c were or were not all of them odd numbers. But there would still remain an ambiguity in the correction to be applied in either case, arising from the doubt whether it should be zero or 2 in the one case, or whether it should be +1 or -1 in the other case.

This ambiguity might of course be removed by inserting in the table employed the first decimal place of $\frac{N^3}{24}$, and increasing the decimal part in the final result to unity, or lowering it to zero, according as its value might be greater or less than $\frac{1}{2}$; and it would be easy to ascertain the limits within which the decimal digit in the result must lie, and the range of values (of which 5 is one) from which it is excluded. The same end may, however, be gained much more elegantly and expeditiously, and by a method more closely analogous to that employed for the evolution of binary products, by the intervention of a very simple expedient.

The cubic residues in respect to the modulus 24 are easily verified to be as follows: 0, 1, 3, 5, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 23. Let the tabular value of $\frac{N^3}{24}$ be made $\left[\frac{N^3}{24}\right] + K_N$, where $\left[\frac{N^3}{24}\right]$ means the integer part of the quantity within the brackets, and K_N may have any one of the three values 0, $\frac{1}{2}$, 1, namely:

$K_N = 0$ when the remainder of N^3 to the divisor 24 is 0, 1, 3 or 5;

$K_N = \frac{1}{2}$ when the said remainder is 7, 8, 9, 11, 13, 15, 16 or 17;

and

$K_N = 1$ when the remainder is 19, 21 or 23;

and let $\left[\frac{N^3}{24}\right] + K_N$ be called the cubic respondent to N , and be denoted by $R(N)$;

and let the exact value of $\frac{N^3}{24}$ be called $R'(N)$.

Let

$$\begin{aligned} & R'(a+b+c) - R'(a+b-c) - R'(a-b+c) - R'(-a+b+c) \\ &= R(a+b+c) - R(a+b-c) - R(a-b+c) - R(-a+b+c) + \Delta. \end{aligned}$$

If in general we write $R'(n) - R(n) = E(n)$, Δ must be of one or the other of the two forms

$$E(n_1) - E(n_2) - E(n_3) - E(n_4),$$

or

$$E(n_1) + E(n_2) - E(n_3) - E(n_4),$$

where n_1, n_2, n_3, n_4 are supposed to be all positive integers. Now it is easily seen that $E(n)$ always lies within the limits $\pm \frac{5}{24}$; that is to say, it may reach up to $\frac{5}{24}$ or down to $-\frac{5}{24}$, but can never transgress these values in either direction. Hence it is obvious that Δ , which is made up of four terms, each of the form $E(n)$, can never be so great as +1 or so small as -1, and consequently Δ can only have one of the three values $+\frac{1}{2}, 0, -\frac{1}{2}$.

Hence, then, we may work with the tabular cubic respondents in lieu of the exact cubic respondents; if the result is an integer, it is good without any correction; if it is a fraction, $\frac{1}{2}$ must be added to, or taken away from it. And to ascertain which of these processes is to be applied, it is only necessary to consider whether the three factors to be multiplied are or are not all of them odd.

In practically constructing a table of cubic respondents, it would not be necessary actually to insert the fraction $\frac{1}{2}$ in any case; a dot over, or a stroke through the last integer, would serve to denote that this fraction was to be understood.

A table of quadratic respondents (that is, of the integer parts of the fourths of the square numbers) up to the base 20,000, has been actually constructed and published by a M. Antoine Voisin, under the title "Tables des Multiplications ou Logarithmes de Nombres entiers depuis 1 jusqu'à 20,000, au moyen desquelles on peut multiplier tous les nombres qui n'excèdent pas 20,000 par 20,000," &c. 12mo. à Paris, Firmin Didot, 1817. A copy of this is in Mr J. T. Graves's valuable mathematical library at Cheltenham.

By logarithms the author intends the same quantities as I term respondents, certainly a less objectionable and safer term to employ. There appears to be an error in the title in affirming that any two numbers, not separately exceeding 20,000, may be multiplied by aid of these tables, as the sum of the two factors ought not to exceed 20,000. Mr Peter Gray, so favourably known to an important section of the public as the author of many useful tables, has informed me that Major Shortredd, now in India, has computed a table of quadratic respondents extending to the argument 200,000, which he is taking measures to have published. Such tables would be very useful to computers, as they would serve for the multiplication of any two numbers whatever not containing more than five figures each. I should like to see a table of cubic respondents up to 30,000 appended to this work*.

* The best practical mode of using and arranging such a table I find, after much thought and consideration, would be as follows. It is easy to add two quantities and subtract their sum from a third by a single operation. If, then, a, b, c are the three numbers whose product it is required to find, they should be written under one another; and against (a) should be set the value of $a - b - c$; against (b), that of $b - a - c$; and against (c), that of $c - a - b$; under these three last results should be written the value of $a + b + c$; of the three former, two at least must be, all *may* be negative; their values arithmetically expressed will be of the form $K(10,000) + N$, where K is 0, 1 or 2. In order that the final process of combining the 4 cubes may be made purely additive, the tables should show the values of $(10,000)^3$ less the respondent to $K(10,000) + N$, when K is 1 or 2 for all values of N from 1 to 9999. These complements to the respondents of the simple or augmented complements of N may be termed respectively the simply and doubly affected respondents of N , but in using the tables no distinction need be drawn between the respondents and the affected respondents. The arrangement of the tables will be as follows. In each page there will be a column for the arguments, which will extend from 1 to 9999, and five other columns containing respondents and bearing respectively for their headings the numbers $\bar{2}, \bar{1}, 0, 1, 2$. The four quantities formed by addition, or by addition and subtraction, from a, b, c , will all be of the form $K\nu_1\nu_2\nu_3\nu_4$ ($\nu_1, \nu_2, \nu_3, \nu_4$ denoting respectively some one or other of the digits from 0 to 9), and K being one of the five symbols $\bar{2}, \bar{1}, 0, 1, 2$; the value corresponding to $\nu_1\nu_2\nu_3\nu_4$ will then be sought for in its proper column (according to the value of the guiding figure K), and the sum of the four values so found will be taken (the last figure to the left, which will be 2 or 3, being rejected). This result, affected, if necessary, with the proper correction of $\pm\frac{1}{2}$, will express the value of abc .