9.

ON SOME NEW THEOREMS IN ARITHMETIC.

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LET $S_i\,(a, b, c, \ldots k, l)$ denote, as is not unusual, the complete sum of the products of the elements (n in number) $a, b, c, \ldots k, l$, combined in every possible way i together. Let $\ddot{S}_i\,(a, b, c, \ldots k, l)$ denote the sum of the products of the same elements combined i together, but so that all combinations are excluded in which any two consecutive elements as a and b, or b and c, ... or k and l, appear simultaneously. S_i may be termed a complete sum of ith products, and $\ddot{S}_i$ a sum of products of *anakolouthic* elements, or briefly an anakolouthic sum of ith products. If we expand the continued fraction

$$\frac{1}{\rho +}\,\frac{a}{\rho +}\,\frac{b}{\rho +}\,\ldots\,\frac{k}{\rho +}\,\frac{l}{\rho},$$

it will be easily found to take the form

$$\frac{\rho^{n-1} + \ddot{S}_1{}'\rho^{n-3} + \ddot{S}_2{}'\rho^{n-5} + \&c.}{\rho^n + \ddot{S}_1\rho^{n-2} + \ddot{S}_2\rho^{n-4} + \&c.};$$

where $\ddot{S}_i{}'$ is intended to denote the anakolouthic sum of the ith products of $b, c, \ldots l$, and $\ddot{S}_i$ the anakolouthic sum of the ith products of $a, b, c, \ldots l$.

It is this fact, and the close relation of reciprocity in which the generating continued fraction for anakolouthic sums stands to ordinary continued fractions (a reciprocity which becomes more apparent when ρ is made unity), which gives a peculiar importance to the theory of anakolouthic sums of the kind denoted by $\ddot{S}$; otherwise we might be tempted to embark upon a premature generalization, extending the force of the term anakolouthic so as to denote by $\dddot{S}$ a sum of products in which no *three* consecutive elements came together, $\ddddot{S}$ a sum of products in which no four consecutive elements came together, and so on; these more general forms of anakolouthic sums may hereafter merit and reward attention, but my present business will be exclusively with a statement of some remarkable properties which have accidentally fallen under my observation, of anakolouthic sums of the kind first mentioned, and referring to elements formed in a manner presently to be explained, from the natural progression of numbers. In order to familiarize the reader with the construction of anakolouthic series, I subjoin the following examples:

$$\ddot{S}_1(abcde) = a + b + c + d + e,$$
$$\ddot{S}_2(abcde) = ac + ad + ae + bd + be + ce,$$
$$\ddot{S}_3(abcde) = ace,$$
$$\ddot{S}_4(abcde) = 0,$$
$$\ddot{S}_4(abcdef) = 0,$$
$$\ddot{S}_4(abcdefg) = aceg,$$
$$\ddot{S}_4(abcdefgh) = aceg + aceh + bdfh.$$

First Theorem. Let n be any odd number; form the $\frac{1}{2}(n-1)$ elements

$$n,\ 2(n-1),\ 3(n-2)\ldots\frac{n-1}{2}\cdot\frac{n+3}{2};$$

the anakolouthic sum of the ith products of these elements is equal to the ith power of negative unity into the complete sum of the $2i$th products of the elements n, $-(n-2)$, $(n-4)$, ... ± 1. Thus suppose $n = 7$, the elements for the anakolouthic sums will be

$$7,\ 12,\ 15;$$

and for the complete sums,

$$7,\ -5,\ 3,\ -1;$$

and we find

$$\ddot{S}_1(7, 12, 15) = 7 + 12 + 15 = 34,\quad S_2(7, -5, 3, -1) = -7.3 - 5.2 - 3 = -34,$$

$$\ddot{S}_2(7, 12, 15) = 7.15 = 105,\quad S_4(7, -5, 3, -1) = 1.3.5.7 = 105.$$

Or, again, if $n = 9$, the one set of elements will be

$$9,\ 16,\ 21,\ 24,$$

and the other set $9,\ -7,\ 5,\ -3,\ 1;$

and we have

$$-(9 + 16 + 21 + 24) = -70 = 9\times(-4) + 7(-3) + 5(-2) + 3(-1),$$

$$9.21 + 9.24 + 16.24$$
$$= 789 = 9.7.5.3 + 9.7.3.1 - 9.7.5.1 - 9.5.3.1 + 7.5.3.1.$$

Second Theorem. Take away the last element belonging to the anakolouthic group above written, so as to reduce the elements to the following sequence:

$$n,\ 2(n-1),\ 3(n-2)\ldots\frac{n-3}{2}\cdot\frac{n+5}{2};$$

$\frac{1}{2}(n+1)$ times the anakolouthic sum of ith products of this sequence will be equal to $(-1)^i$ multiplied by the complete sum of the $(2i+1)$th products of the series $n, -(n-2), (n-4), \ldots \pm 1$. Thus if $n=9$, the two series of elements are respectively

$$9,\ 16,\ 21; \qquad 9,\ -7,\ 5,\ -3,\ 1;$$

and we find

$$5 \cdot 1 = 9 - 7 + 5 - 3 + 1,$$

$$5 \cdot (9 + 16 + 21) = 230 = 9 \cdot 7 \cdot 5 - 9 \cdot 7 \cdot 3 + 9 \cdot 7 \cdot 1 + 9 \cdot 5 \cdot 3 - 9 \cdot 5 \cdot 1 + 9 \cdot 3 \cdot 1 - 7 \cdot 5 \cdot 3 + 7 \cdot 5 \cdot 1 - 7 \cdot 3 \cdot 1 + 5 \cdot 3 \cdot 1,$$

$$5 \cdot (9 \cdot 21) = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1.$$

I now pass on to the cases where n is an even number.

Third Theorem. Let n be of the form $4m+k$, where k is zero or 2; construct the sequence

$$1 \cdot n,\ 2(n-1),\ 3(n-2) \ldots \left(\frac{n}{2}-1\right)\left(\frac{n}{2}+2\right);$$

the ith anakolouthic series of products formed out of these elements is equal to the ith complete series of products formed out of the elements $(n-2)^2, (n-6)^2, \ldots (k+2)^2$.

Ex. Let $n=10$, the two sequences will be

$$10,\ 18,\ 24,\ 28,$$

$$64,\ 16,$$

and we have

$$10 + 18 + 24 + 28 = 80 = 64 + 16,$$

$$10 \cdot 24 + 10 \cdot 28 + 18 \cdot 28 = 1024 = 64 \cdot 16.$$

So, if $n=12$, the two sequences will be

$$12,\ 22,\ 30,\ 36,\ 40,$$

$$100,\ 36,\ 4;$$

and we have

$$12 + 22 + 30 + 36 + 40 = 140 = 100 + 36 + 4,$$

$$12 \cdot (30 + 36 + 40) + 22 \cdot (36 + 40) + 30 \cdot 40 = 4144 = 100 \cdot 36 + 100 \cdot 4 + 36 \cdot 4,$$

$$12 \cdot 30 \cdot 40 = 4 \cdot 36 \cdot 100.$$

Fourth Theorem. If n be any even number, and we form the three sequences

$$1 \cdot n,\quad 2(n-1),\quad 3(n-2) \ldots \frac{n}{2}\left(\frac{n}{2}+1\right),$$

$$1 \cdot (n+2),\quad 2(n+1),\quad 3(n) \ldots \frac{n}{2}\left(\frac{n}{2}+3\right),$$

$$1 \cdot n,\quad 2(n-1),\quad 3(n-2) \ldots \left(\frac{n}{2}-2\right)\left(\frac{n}{2}+3\right),$$

the ith anakolouthic sum in respect to the second sequence less the ith anakolouthic sum in respect to the first sequence is equal to $\frac{n}{2}\left(\frac{n}{2}+1\right)$ into the $(i-1)$th anakolouthic sum in respect to the third sequence.

Ex. Take the three sequences

$$1.10,\quad 2.9,\quad 3.8,\quad 4.7,\quad 5.6,$$
$$1.12,\quad 2.11,\quad 3.10,\quad 4.9,\quad 5.8,$$
$$1.10,\quad 2.9,\quad 3.8.$$

These, written out with simple elements, are as follows:

$$10\quad 18\quad 24\quad 28\quad 30,$$
$$12\quad 22\quad 30\quad 36\quad 40,$$
$$10\quad 18\quad 24;$$

and we have

$$(12+22+30+36+40)-(10+18+24+28+30)=30\,.\,1,$$

$$\{12\,.\,(30+36+40)+22\,.\,(36+40)+30\,.\,40\}$$
$$-\{10\,.\,(24+28+30)+18\,.\,(28+30)+24\,.\,30\}$$
$$=4144-2584=1560=30\,.\,(10+18+24),$$

$$12\,.\,30\,.\,40-10\,.\,24\,.\,30=14400-7200=7200=30\,.\,(10\,.\,24).$$

These four theorems are only particular cases of one much more general relating to a determinant, to which I was led by my method of integrating the system of two partial differential equations to the general invariant of a function or system of functions of two variables. In like manner the integration of the system of t partial differential equations to the general invariant of a function or system of functions of t variables conducts to a determinant*, of which *à priori* we know the constitution, and which will (save as to the periodic occurrence of a single factor λ) resolve itself into factors of the form $\lambda^t \pm m^t$, m being an integer; and thus promises to lay open a road to the discovery of a new genus of theorems relating to the powers of the natural progression of integer numbers, destined apparently to occupy a sort of neutral ground between the formal and quantitative arithmetics.

* The integration of this system of equations always depends essentially upon the integration of one homogeneous equation which is doubly linear, that is of the first degree in the variables, and also of the first degree in respect to the order of the differentiations; such an equation can always be integrated, and the integral will depend upon the solution of an algebraical equation expressed by equating a certain determinant to zero.