

ON DIFFERENTIAL TRANSFORMATION AND THE REVERSION OF SERIESSES*.

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WITH a view to its publication in the *Proceedings* of the Society, I take occasion to communicate the result of my investigations, as far as they have yet extended, into the general theory of differential transformations, containing a complete and general solution of the important problem of expanding a given partial differential coefficient of a function in respect of one system of independent variables in terms of the partial differential coefficients thereof, in respect to a second system of independent variables, each respectively given as explicit functions of the first set.

This question may be shown to be exactly coincident with that of the reversion of simultaneous serieses proposed by Jacobi, which may be thus stated: given $(n + 1)$ quantities, each expressed by rational infinite serieses as functions of n others; required to express any one of the first set in a rational infinite series in terms of the other n of the same set. This question has only been resolved by Jacobi for a particular case; the result hereunder given for the transformation of differential coefficients contains the solution of the general question. My method of investigation is entirely different from that adopted by the great Jacobi, and I hope in a short time to be able to lay it in a complete form before the Society, and probably to add a solution of the still more general question comprising the reversion of serieses as a particular case, namely, the question of expressing any one of n quantities connected by m equations in terms of any $(n - m)$ others of the same.

Let there be any number of variables, say u, v, w , of which x, y, z, \mathfrak{S} are given functions, it is required to expand

$$\left(\frac{d}{dx}\right)^f \left(\frac{d}{dy}\right)^g \left(\frac{d}{dz}\right)^h \mathfrak{S}$$

in terms of the partial differential coefficients of \mathfrak{S}, x, y, z in respect of u, v, w .

[* See p. 65, below.]

Form the determinant

$$\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix},$$

which call J .

The required expansion will contain in each term an integer numerical coefficient, a power of $\frac{1}{J}$, one factor of the form

$$\left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q \left(\frac{d}{dw}\right)^r \mathfrak{D},$$

and other factors of the form

$$\begin{aligned} &\left(\frac{d}{du}\right)^l \left(\frac{d}{dv}\right)^m \left(\frac{d}{dw}\right)^n x, \\ &\left(\frac{d}{du}\right)^{l'} \left(\frac{d}{dv}\right)^{m'} \left(\frac{d}{dw}\right)^{n'} y, \\ &\left(\frac{d}{du}\right)^{l''} \left(\frac{d}{dv}\right)^{m''} \left(\frac{d}{dw}\right)^{n''} z. \end{aligned}$$

Let the latter class of factors be distinguished into two sets, those where $l + m + n = 1$,

$$\begin{pmatrix} l=1 & m=0 & n=0 \\ \text{or } l=0 & m=1 & n=0 \\ \text{or } l=0 & m=0 & n=1 \end{pmatrix},$$

which I shall call uni-differential factors, and those in which $l + m + n > 1$, which I shall call pluri-differential factors.

First, then, as to the form of the general term abstracting from the numerical coefficient and the uni-differential factors (except of course so far as they enter into J). This will be as follows:

$$\begin{aligned} &\left(\frac{d}{du}\right)^{l_1} \left(\frac{d}{dv}\right)^{m_1} \left(\frac{d}{dw}\right)^{n_1} x \times \left(\frac{d}{du}\right)^{2l_1} \left(\frac{d}{dv}\right)^{2m_1} \left(\frac{d}{dw}\right)^{2n_1} x \times \dots \times \left(\frac{d}{du}\right)^{e_1 l_1} \left(\frac{d}{dv}\right)^{e_1 m_1} \left(\frac{d}{dw}\right)^{e_1 n_1} x \\ &\quad \times \left(\frac{d}{du}\right)^{l_2} \left(\frac{d}{dv}\right)^{m_2} \left(\frac{d}{dw}\right)^{n_2} y \times \dots \times \left(\frac{d}{du}\right)^{e_2 l_2} \left(\frac{d}{dv}\right)^{e_2 m_2} \left(\frac{d}{dw}\right)^{e_2 n_2} y \\ &\quad \times \left(\frac{d}{du}\right)^{l_3} \left(\frac{d}{dv}\right)^{m_3} \left(\frac{d}{dw}\right)^{n_3} z \times \dots \times \left(\frac{d}{du}\right)^{e_3 l_3} \left(\frac{d}{dv}\right)^{e_3 m_3} \left(\frac{d}{dw}\right)^{e_3 n_3} z \\ &\quad \times \left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q \left(\frac{d}{dw}\right)^r \mathfrak{D} \times \frac{1}{J}, \end{aligned}$$

subject to the limitations about to be expressed.

$$\begin{aligned} \text{Call} \quad & {}^1l_1 + {}^2l_1 + \dots + {}^{e_1}l_1 = L_1, \\ & {}^1l_2 + {}^2l_2 + \dots + {}^{e_2}l_2 = L_2, \\ & {}^1l_3 + {}^2l_3 + \dots + {}^{e_3}l_3 = L_3, \end{aligned}$$

and form the analogous quantities $M_1, M_2, M_3; N_1, N_2, N_3$. Then we must have

$$L_1 + L_2 + L_3 + M_1 + M_2 + M_3 + N_1 + N_2 + N_3 + p + q + r = f + g + h + e_1 + e_2 + e_3;$$

and as the sum of any group of indices l, m, n must not be less than 2, we have

$$f + g + h + e_1 + e_2 + e_3 + p + q + r, \text{ not less than } 2e_1 + 2e_2 + 2e_3,$$

so that $e_1 + e_2 + e_3$ must not exceed $f + g + h + p + q + r$; furthermore, $p + q + r$ must not exceed $f + g + h$; and finally,

$$\omega = f + g + h + e_1 + e_2 + e_3.$$

1. We may first take $e_1 + e_2 + e_3 = E$, giving to E in succession all integer values from $f + g + h$ to $2f + 2g + 2h$, and find all possible solutions of this equation with permutations between the values of e_1, e_2, e_3 .

2. We may then take $p + q + r = s$, giving s in succession all integer values from 1 to $f + g + h$, and find all possible solutions of this equation with permutations between f, g, h .

3. We may then take $L + M + N = f + g + h + E - s$, and find all the values of L, M, N , with permutations allowable between the values of L, M, N .

4. We may then take

$$\begin{aligned} L_1 + L_2 + L_3 &= L, \\ M_1 + M_2 + M_3 &= M, \\ N_1 + N_2 + N_3 &= N, \end{aligned}$$

and solve these several equations in every way possible, with permutations as before.

5. We must take

$$\begin{aligned} {}^1l_1 + {}^2l_1 + \dots + {}^{e_1}l_1 &= L_1, & {}^1m_1 + {}^2m_1 + \dots + {}^{e_1}m_1 &= M_1, & {}^1n_1 + {}^2n_1 + \dots + {}^{e_1}n_1 &= N_1, \\ {}^1l_2 + {}^2l_2 \dots & {}^{e_2}l_2 = L_2, & {}^1m_2 + {}^2m_2 \dots & {}^{e_2}m_2 = M_2, & {}^1n_2 + {}^2n_2 \dots & {}^{e_2}n_2 = N_2, \\ {}^1l_3 + {}^2l_3 \dots & {}^{e_3}l_3 = L_3, & {}^1m_3 + {}^2m_3 \dots & {}^{e_3}m_3 = M_3, & {}^1n_3 + {}^2n_3 \dots & {}^{e_3}n_3 = N_3, \end{aligned}$$

and solve in every possible manner these equations, but without admitting permutations between the values of ${}^1l_1, {}^2l_1 \dots {}^{e_1}l_1$, or between the values of the members of the other of the third sets taken each *per se*, and subject to the

condition that every such sum as ${}^r l_i + {}^r m_i + {}^r n_i$ must be greater than unity. Every possible system of values of these nine sets will furnish a corresponding pluri-differential part to the general term.

Next, as to the uni-differential part, we may form the quantity

$$\begin{aligned} & \left(\frac{dy}{dv} \frac{dz}{dw} - \frac{dy}{dw} \frac{dz}{dv}\right)^{\lambda_1} \left(\frac{dy}{dw} \frac{dz}{du} - \frac{dy}{du} \frac{dz}{dw}\right)^{\mu_1} \left(\frac{dy}{du} \frac{dz}{dv} - \frac{dy}{dv} \frac{dz}{du}\right)^{\nu_1}, \\ & \left(\frac{dz}{dv} \frac{dx}{dw} - \frac{dz}{dw} \frac{dx}{dv}\right)^{\lambda_2} \left(\frac{dz}{dw} \frac{dx}{du} - \frac{dz}{du} \frac{dx}{dw}\right)^{\mu_2} \left(\frac{dz}{du} \frac{dx}{dv} - \frac{dz}{dv} \frac{dx}{du}\right)^{\nu_2}, \\ & \left(\frac{dx}{dv} \frac{dy}{dw} - \frac{dx}{dw} \frac{dy}{dv}\right)^{\lambda_3} \left(\frac{dx}{dw} \frac{dy}{du} - \frac{dx}{du} \frac{dy}{dw}\right)^{\mu_3} \left(\frac{dx}{du} \frac{dy}{dv} - \frac{dx}{dv} \frac{dy}{du}\right)^{\nu_3}, \end{aligned}$$

where

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= L + p, \\ \mu_1 + \mu_2 + \mu_3 &= M + q, \\ \nu_1 + \nu_2 + \nu_3 &= N + r. \end{aligned}$$

These equations are to be solved in every possible manner with permutations between the members of the λ set, the μ set, and the ν set. Finally, we have to consider the numerical coefficient. To give a perfect representation of this, we must ascertain what identities exist in the factors of the pluri-differential part. Let us suppose that one set of operators upon x is repeated θ_1 times, another θ_2 times, and so on, giving rise to the powers $\theta_1, \theta_2, \dots, \theta_a$ in the x line. Similarly, form $\phi_1, \phi_2, \dots, \phi_\beta$ from the y line, and $\psi_1, \psi_2, \dots, \psi_\gamma$ from the z line. Then the numerical part of the general term will be

$$\begin{aligned} & \frac{\Pi (\lambda_1 + \mu_1 + \nu_1) \Pi (\lambda_2 + \mu_2 + \nu_2) \Pi (\lambda_3 + \mu_3 + \nu_3)}{\Pi \lambda_1 \Pi \mu_1 \Pi \nu_1 \Pi \lambda_2 \Pi \mu_2 \Pi \nu_2 \Pi \lambda_3 \Pi \mu_3 \Pi \nu_3} \\ & \times \frac{\Pi (L + p) \Pi (M + q) \Pi (N + r)}{\left\{ \begin{array}{l} \Pi^1 l_1 \Pi^1 m_1 \Pi^1 n_1 \Pi^2 l_1 \Pi^2 m_1 \Pi^2 n_1 \dots \dots \\ \Pi^1 l_2 \Pi^1 m_2 \Pi^1 n_2 \Pi^2 l_2 \Pi^2 m_2 \Pi^2 n_2 \dots \dots \\ \Pi^1 l_3 \Pi^1 m_3 \Pi^1 n_3 \Pi^2 l_3 \Pi^2 m_3 \Pi^2 n_3 \dots \dots \end{array} \right\}} \\ & \times \frac{D}{\Pi \theta_1 \Pi \theta_2 \dots \Pi \theta_a \Pi \phi_1 \Pi \phi_2 \dots \Pi \phi_\beta \Pi \psi_1 \Pi \psi_2 \dots \Pi \psi_\gamma}, \end{aligned}$$

where in general Πm means 1. 2. 3 ... m : as regards D , it is the following determinant, namely,

$\lambda_1 + \mu_1 + \nu_1$	ν	ν	L_3	M_3	N_3
ν	$\lambda_2 + \mu_2 + \nu_2$	ν	L_2	M_2	N_2
ν	ν	$\lambda_3 + \mu_3 + \nu_3$	L_1	M_1	N_1
λ_1	λ_2	λ_3	$L_1 + L_2 + L_3 + p$	ν	ν
μ_1	μ_2	μ_3	ν	$M_1 + M_2 + M_3 + q$	ν
ν_1	ν_2	ν_3	ν	ν	$N_1 + N_2 + N_3 + r$

The result, for greater brevity, has been set out in the above pages for the case of \mathfrak{S} , a function of three variables, but the reader can have no difficulty in extending the statement to any number. In the case of a single variable, the formula can easily be identified with that given by Burman's law. It is noticeable that the determinant written is of the form

$$Apqr + Bpq + Cqr + Drp + Ep + Fq + Gr,$$

the part independent of p, q, r being easily seen to vanish. Moreover, the coefficients A, B, C, \dots are all essentially positive, so that the determinant can only vanish (except for $p=0, r=0, q=0$) by virtue of one condition at least more than the number of the variables.