13.

NOTE ON AN INTUITIVE PROOF OF THE EXISTENCE OF TWENTY-SEVEN CONICS OF CLOSEST CONTACT WITH A CURVE OF THE THIRD DEGREE.

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IN general a conic can only be made to have five coincident points with a curve, and if the curve be of the third degree, the conic will of course cut it in a remaining sixth point; but at certain points of the cubic all these six points may come together. How many of these are there, and where are they? This question, which originated with Steiner, who stated the number, and was subsequently treated by Plücker, who assigned the position of the points, may be resolved by very simple considerations and without calculation. For if we can succeed in putting the characteristic of the curve (I mean what is commonly, but not altogether commodiously, called "the-left-hand-side-of-the-equation-to-the-curve-when-the-right-hand-side-of-it-is-made-equal-to-zero") under the form $u^3 + v(uw + \omega^2)$, it is obvious that the conic $uw + \omega^2$ will intersect the cubic curve in the six coincident points $u^3 = 0$, $\omega^2 = 0$.

If now we take for our cubic the reduced form $x^3 + y^3 + z^3 - 6mxyz$, and make $x + y + 2mz = p$, $\rho x + \rho^2 y + 2mz = q$, $\rho^2 x + \rho y + 2mz = r$ [where ρ is an imaginary cube root of unity], it may be written under the form

$$(1 - 8m^3) z^3 + pqr, \text{ say } -\mu z^3 + pqr;$$

or, if we please, under the form

$$-\mu(z + kp)^3 + p(qr + \mu k^3 p^2 + 3\mu k^2 pz + 3\mu k z^2).$$

And if we assume k properly, $z + kp$ may be made to touch the multiplier of p, that is, the cubic may be made to take the form

$$-\mu(z + kp)^3 + p\{(z + kp)v + \omega^2\}.$$

From the symmetry which reigns between x and y, it is obvious *à priori* that any value of k which is rightly assumed for the object in view will make

ω (when z is eliminated from it by means of the equation $z+kp=0$) a multiple either of $x-y$ or $x+y$; the latter obviously cannot be true, since such values would make the given cubic a function of $x+y$ and z; the proper values of k will therefore make $x-y=0$, from which, combined with the equation $2x^3+z^3+6mx^2z=0$, the values of $x:y:z$ may be determined. These will be three in number; and as we may write, instead of x and y, ρx, $\rho^2 y$, or ρy, $\rho^2 x$, we obtain three sets of three points, corresponding to p being taken $x+y+2mz$; and consequently, by interchanging z with x and with y successively, we obtain altogether three systems of three sets of three points each; any such factor as $x+y+2mz$ is a tangent to a point of inflexion, and it is clear *à priori* that if the cubic is put under the form $u^3+v(uw+\omega^2)$, since $v=0$ makes $u^3=0$, *v can only be* a tangent at an inflexion. Hence the nine sets of three points just assigned are *all* that can be found enjoying the property in question, and it is readily seen that $x-y$ is the straight line containing the three points of intersection in which the second emanant,

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)^2(x^3+y^3+z^3-6mxyz),$$

at the point of inflexion ($x+y=0$, $z=0$) cuts the given cubic over and above the three coincident points $x+y=0$, $z=0$. In other words, each ternary group of the twenty-seven points in question consists of the three points in which the curve is met by the tangents drawn from a point of inflexion, which agrees with the geometrical construction given by Plücker in *Crelle's Journal*.