16.

ON THE CHANGE OF SYSTEMS OF INDEPENDENT VARIABLES.

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(1) THE theorem contained in the subjoined pages having been printed *, with many typographical and other errors†, in the *Proceedings of the Royal Society*, Vol. VII. No. 8, I think, on account of its importance to the direct march of the differential calculus, of which, as an instrument of expansion, it may be said to complete the processes, that the reissue of it in a more correct form may be acceptable and useful to the readers of this journal.

The purpose of the theorem is to effect for any number of variables the same end which has been accomplished by Burmann and others for a single variable; that is to say, ϑ being supposed to be a function of the variables, $x, y, \ldots z$, each of which is a given function of $u, v, \ldots w$, and $\alpha, \beta, \ldots \gamma$, being any positive integers, the theorem gives the complete development of $\left(\frac{d}{dx}\right)^\alpha \left(\frac{d}{dy}\right)^\beta \ldots \left(\frac{d}{dz}\right)^\gamma \vartheta$ in terms of $\frac{d}{du}, \frac{d}{dv}, \ldots \frac{d}{dw}, x, y, \ldots z, \vartheta$. Such, I say, is the primary form of the theorem; but it enables us, as will hereafter be shown in this paper, in fact, and as a consequence, to do much more than this, namely, to solve the question of differential transformation, under its most general aspect. The question so proposed may be stated as follows:

Given $\phi_1 = 0, \phi_2 = 0, \ldots \phi_n = 0$, where each ϕ is a function of $x_1, x_2, \ldots x_{n+i}$, it is required to pass from an expression in which the differentiations have respect to $x_1, x_2, \ldots x_i$ to an equivalent expression, in each of the terms of which the differentiations have respect to $x_{\theta_1}, x_{\theta_2}, \ldots x_{\theta_i}$, these last-written quantities being any i arbitrarily chosen terms out of the given set of $n+i$ variables, $x_1, x_2, \ldots x_{n+i}$. Through the medium of the reversion of series, the solution of this problem for the case contemplated in the theorem about to be enunciated (where $x_1, x_2, \ldots x_i$ are given *explicitly* in terms of

* Want of leisure prevented me then, and still prevents me, from producing the proof of the theorem, or the investigation by which I arrived at it. It must, however, be understood, that the theorem was not obtained tentatively, but that the proof of it is in my possession.

[† p. 50 above.]

$u_1, u_2, \ldots u_i$), enables us to write down the solution for the case where these two systems of variables are connected by equations in the more general manner just above supposed. It may then be asked whether it is meant to affirm that Burmann's law for passing from one independent variable x to another y, of which the first is a known function, conducts immediately to the law for effecting such change, when x and y are connected through the intervention of one equation between x and y, or several equations between x, y, and other connecting variables. The answer to this question is in the negative; for even if we take the simpler case where x and y are connected by a single equation, it will be found that to solve the problem for this case in the manner indicated, we shall need to know the solution of the problem, how to pass to *two* variables, u, v, from two others, x, y, given explicitly as functions of the former two; and so in general it is the fact, that the theorem applicable to the case of implicit connections between any number of variables, is always a corollary to the theorem applicable to the case of explicit connection between a *greater* number of variables. Thus it comes to pass, that Burmann's law for one variable explicitly connected with another, does not contain within itself the law for one variable implicitly connected with another; but the general law which I have discovered for a system of *any number* of variables explicitly connected with another such system, does contain within itself the general law for systems implicitly so connected*.

As the theorem is one of considerable complexity, it will be rendered most easily intelligible by taking separately and successively the cases of two and of three variables; the reader will then not experience any difficulty in seeing how it is to be extended to any greater number.

Problem for Two Variables.

(2) Let x, y be given functions of u, v, it is required to express $\left(\frac{d}{dx}\right)^f \left(\frac{d}{dy}\right)^g \vartheta$ in terms of the partial differential coefficients of x, y, ϑ in respect of u and v.

Solution.

Form the Jacobian determinant

$$\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{dv} \\ \frac{dy}{du}, & \frac{dy}{dv} \end{vmatrix},$$

* Any linear function of infinity is still infinity, and all infinity is one, but not so of a finite integer; thus it is that the particular does not carry with it the particular, although the general does the general.

which call J; the required expression will be made up of terms, each of which will have for its components; 1°, a power of (−); 2°, a positive integer numerical multiplier; 3°, a negative power of J; 4° and 5° (subject to a subsequent distinction into two sets), factors of the form

$$\left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q x,\quad \left(\frac{d}{du}\right)^{p'} \left(\frac{d}{dv}\right)^{q'} y;$$

and 6°, a factor of the form $\left(\frac{d}{du}\right)^A \left(\frac{d}{dv}\right)^B \vartheta$.

The distinction of the factors under the combined headings 4 and 5 into two sets, referring to these headings separately taken, is dependent upon the values of p, q; p', q'. The 4th heading is intended to comprise the factors, for which $p=1$ and $q=0$ or $p=0$, $q=1$, and similarly for p' and q', that is, factors for which $p+q$ or $p'+q'$ is unity. The 5th heading comprises those factors in which $p+q$ or $p'+q'$ (as the case may be), exceeds unity. These two sets require to be carefully distinguished and considered apart: those values of p, q; p', q' belonging to the second set will be distinguished by the letters a, b; a', b', so that it is to be understood that $a+b>1$, $a'+b'>1$.

The general term may thus be put under the form

$$(-)^i N \frac{1}{J^\omega} \left(\frac{dy}{dv}\right)^\alpha \times \left(-\frac{dy}{du}\right)^\beta \times \left(-\frac{dx}{dv}\right)^{\alpha'} \times \left(\frac{dx}{du}\right)^{\beta'}$$

$$\times \left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{dv}\right)^b x\right\}^l \times \text{\&c.}$$

$$\times \left\{\left(\frac{d}{du}\right)^{a'} \left(\frac{d}{dv}\right)^{b'} y\right\}^{l'} \times \text{\&c.}$$

$$\times \left(\frac{d}{du}\right)^A \left(\frac{d}{dv}\right)^B \vartheta.$$

The negative signs are employed with $\frac{dy}{du}$, $\frac{dx}{dv}$ in the first line of factors, because, as will be seen when we pass to the case of more than two variables, it is the first minors of J which give rise to these factors, and these first minors are respectively

$$\frac{dy}{dv};\quad -\frac{dy}{du};\quad -\frac{dx}{dv};\quad \frac{dx}{du}.$$

The &c. in the second line of factors refers to a, b, l becoming changed into a_1, b_1, l_1; a_2, b_2, l_2 &c.; and indicates that the product is to be taken of all the factors thus formed upon the type of

$$\left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{dv}\right)^b x\right\}^l.$$

Similarly, the &c. in the third line of factors refers to a', b', l' becoming changed into a_1', b_1', l_1' &c.; and the product taken of all such factors so formed upon the type of

$$\left\{\left(\frac{d}{du}\right)^{a'}\left(\frac{d}{dv}\right)^{b'} y\right\}^{l'}.$$

[We may of course, if we please, write the first line under the form

$$(-)^{i'}\frac{N}{J^{\omega}}\left(\frac{dy}{dv}\right)^{\alpha}\left(\frac{dy}{du}\right)^{\beta}\left(\frac{dx}{dv}\right)^{\alpha'}\left(\frac{dx}{du}\right)^{\beta'}$$

by making $i' = i + \beta + \alpha'$.]

In the first place,

$$i = l + \&c. + l' + \&c.$$

In the second place,

$$\omega = \alpha + \beta + \alpha' + \beta'.$$

In the third place,

$$\alpha + \alpha' = la + \&c. + l'a' + \&c. + A, \qquad (1)$$

$$\beta + \beta' = lb + \&c. + l'b' + \&c. + B, \qquad (2)$$

and

$$\alpha + \beta = f + \Sigma l, \qquad (3)$$

$$\alpha' + \beta' = g + \Sigma l', \qquad (4)$$

which two systems of equations of course imply the existence of the equation

$$\Sigma l\,(a + b - 1) + \Sigma l'\,(a' + b' - 1) = (f + g) - (A + B). \qquad (5)$$

And finally:

$$N = D \times \frac{\Pi\,(\alpha + \beta - 1)\,\Pi\,(\alpha' + \beta' - 1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$

$$\times \frac{\Pi\,(\alpha + \alpha' - 1)\,\Pi\,(\beta + \beta' - 1)}{\{\Pi l\,(\Pi a\,\Pi b)^{l}\} \times \&c. \times \{\Pi l'\,(\Pi a'\,\Pi b')^{l'}\} \times \&c. \times \Pi A\,\Pi B},$$

Πn for any value of the integer n indicating the factorial $1 \,.\, 2 \,.\, 3 \ldots n$, and D denoting the determinant hereunder written, namely:

$$\begin{vmatrix} \alpha + \beta, & 0, & la + \&c., & lb + \&c. \\ 0, & \alpha' + \beta', & l'a' + \&c., & l'b' + \&c. \\ \alpha, & \alpha', & \alpha + \alpha', & 0 \\ \beta, & \beta', & 0, & \beta + \beta' \end{vmatrix},$$

which writing $la + \&c. = \Sigma la$, $l'a' + \&c. = \Sigma l'a'$, and substituting for $\alpha + \alpha'$, $\Sigma la + \Sigma l'a' + A$, and for $\beta + \beta'$, $\Sigma lb + \Sigma l'b' + B$, becomes when developed

$$\begin{aligned}&(\alpha + \beta)(\alpha' + \beta') AB\\ &+ \{\beta(\alpha' + \beta') \Sigma la + \beta'(\alpha + \beta) \Sigma l'a'\} B\\ &+ \{\alpha(\alpha' + \beta') \Sigma lb + \alpha'(\alpha + \beta) \Sigma l'b'\} A.\end{aligned}$$

D being essentially positive, N can only vanish when the following equations (or the analogues to them obtained by the interchange of a, α, A with b, β, B) are fulfilled, namely:

$$A = 0, \quad B = 0,$$

or $A = 0, \quad \beta = 0, \quad \beta' = 0,$

or $A = 0, \quad \beta = 0, \quad \alpha = 0,$

or $A = 0, \quad \alpha' = 0, \quad \beta' = 0,$

or $A = 0, \quad \beta = 0, \quad \Sigma l'a' = 0,$

or $A = 0, \quad \beta' = 0, \quad \Sigma la = 0,$

or $A = 0, \quad \Sigma la = 0, \quad \Sigma l'a' = 0.$

(3) By way of illustration, let us suppose $f = 2$, $g = 0$, so that the expression to be developed is $\frac{d^2\vartheta}{dx^2}$, which is to be expressed in terms of $\frac{d}{du}$, $\frac{d}{dv}$, x, y, ϑ. It will be the simpler mode of proceeding to find this development by actual expansion, and compare the result with that given by the theorem in the text.

We shall find without difficulty by the ordinary process

$$\begin{aligned}\frac{d^2\vartheta}{dx^2} = \frac{1}{J^2}\left(\frac{dy}{dv}\right)^2 \frac{d^2\vartheta}{du^2} - \frac{2}{J^2}\frac{dy}{du}\frac{dy}{dv}\frac{d^2\vartheta}{du\,dv} + \frac{1}{J^2}\left(\frac{dy}{du}\right)^2\frac{d^2\vartheta}{dv^2}\\
+ \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{du} - \frac{2}{J^3}\frac{dx}{dv}\frac{dy}{du}\frac{dy}{dv}\frac{d^2y}{du\,dv}\frac{d\vartheta}{du}\\
+ \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{du}\\
- \frac{1}{J^3}\frac{dy}{du}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{du} + \frac{2}{J^3}\left(\frac{dy}{dv}\right)^2\frac{dy}{du}\frac{d^2x}{du\,dv}\frac{d\vartheta}{du}\\
- \frac{1}{J^3}\frac{dy}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2x}{dv^2}\frac{d\vartheta}{du}\\
- \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{dv} + \frac{2}{J^3}\frac{dx}{du}\frac{dy}{du}\frac{dy}{dv}\frac{d^2y}{du\,dv}\frac{d\vartheta}{dv}\\
- \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{dv}\\
+ \frac{1}{J^3}\frac{dy}{du}\left(\frac{dy}{dv}\right)^2\frac{d^2x}{du^2}\frac{d\vartheta}{dv} - \frac{2}{J^3}\frac{dy}{du}\frac{dy}{du}\frac{dy}{dv}\frac{d^2x}{du\,dv}\frac{d\vartheta}{dv}\\
+ \frac{1}{J^3}\frac{dx}{du}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{dv}.\end{aligned}$$

(4) In the first term

$$\alpha = 2, \quad \beta = 0, \quad \alpha' = 0, \quad \beta' = 0,$$
$$a = 0, \quad b = 0, \quad \&c. = 0, \quad \Sigma l = 0,$$
$$a' = 0, \quad b' = 0, \quad \&c. = 0, \quad \Sigma l' = 0,$$
$$A = 2, \quad B = 0,$$

and we have, as indicated by the theorem,

$$i = \Sigma l + \Sigma l' = 0, \quad \omega = \alpha + \beta + \alpha' + \beta' = 2,$$
$$\alpha + \alpha' = \Sigma la + \Sigma l'a' + A = 2,$$
$$\beta + \beta' = \Sigma lb + \Sigma l'b' + B = 0,$$
$$\alpha + \beta = f = 2,$$
$$\alpha' + \beta' = g = 0.$$

N becomes

$$\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'} \times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi A} \times (\alpha+\beta)(\alpha'+\beta')\,AB;$$

it is easily seen that

$$\Pi(\beta+\beta'-1) \times B$$
$$= \Pi(\beta+\beta'-1) \times (\beta+\beta')$$
$$= \Pi 0 = 1,$$
$$(\alpha'+\beta')\,\Pi(\alpha'+\beta'-1) = \Pi(\alpha'+\beta') = \Pi 0 = 1,$$

so that the value of the fraction above written is in fact

$$\frac{\Pi(\alpha+\beta)\,A}{\Pi\alpha\,\Pi A} = \frac{(\Pi 2)^2}{(\Pi 2)^2} = 1.$$

In the second term,

$$\alpha = 1, \quad \beta = 1, \quad \alpha' = 0, \quad \beta' = 0;$$

everything else remains as before, except that the numerical factor is $(-)^{\beta} N$, that is, $-N$, where $N = 2$.

(5) If we take the eighth term (the second one of the fourth line) we have

$$\alpha = 2, \quad \beta = 1, \quad \alpha' = 0, \quad \beta' = 0,$$
$$a = 1, \quad b = 1, \quad a' = 0, \quad b' = 0, \quad \Sigma l = l = 1,$$
$$A = 1, \quad B = 0,$$

and we have

$$i = l + \beta + \alpha' = \Sigma l + \beta + \alpha' = 2,$$
$$\omega = \alpha + \beta + \alpha' + \beta' = 3,$$
$$\alpha + \alpha' = la + A = 2,$$
$$\beta + \beta' = lb + B = 1,$$
$$\alpha + \beta = f + l = 3,$$
$$\alpha' + \beta' = g = 0,$$

and N becomes $$\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$
$$\times\frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi a\,\Pi b\,\Pi A\,\Pi B}\times\{\alpha(\alpha'+\beta')\}\,bA,$$
which, since $$\Pi(\alpha'+\beta'-1)\times(\alpha'+\beta')=\Pi 0=1,$$
$$=\frac{\Pi(\alpha+\beta-1)\,A}{\Pi A}=\frac{\Pi 2\,.\,2}{\Pi 2}=2.$$

(6) The above examples, although taken from the simplest terms, are in a certain sense exceptional cases, inasmuch as N for these cases involves one or more fractions of the form $\frac{0}{0}$; but this is a mere accident, resulting from the peculiar form of representation which I choose to employ, as being in general the most convenient to operate with.

If we take the fifth term (that is, the second term of the second line), this exception does not apply. We have for this term
$$\alpha=1,\quad \beta=1,\quad \alpha'=1,\quad \beta'=0,$$
$$a=0,\quad b=0,\quad a'=1,\quad b'=1,\quad \Sigma l'=l'=1,$$
$$A=1,\quad B=0,$$
and we find $$N=\frac{\Pi 1\times\Pi 0}{\Pi 1\times\Pi 1}\times\frac{\Pi 1\,\Pi 0}{\Pi 1\times\Pi 1\times\Pi 1}\times D=D$$
$$=\left\{\begin{array}{c}2\times 0\times 1\times 0\\ +(1\times 1+0+0\times 1\times 1)\,0\\ +\;1\times 1\times 0+1\times 2\times 1\end{array}\right\}=2.$$

(7) In general, to form all the terms in $\left(\frac{d}{dx}\right)^f\left(\frac{d}{dy}\right)^g\vartheta$, that is, to find all the systems of indices, we may begin by taking $A+B=\mu$, and giving to μ in succession, every value from 1 to $f+g$, and calling $f+g=n$, and writing
$$\Sigma l\,(a+b-1)=L,$$
$$\Sigma l'\,(a'+b'-1)=L',$$
we have to combine each solution of the equation $A+B=\mu$ with each of the equation $L+L'=n-\mu$, that is, we may assume for A in succession each value from 1 to μ, and for L, from 1 to $n-\mu$.

It will be convenient to denote in general an integer which may be anything from 1 to p by $[p]$. We have then
$$\mu=[n],$$
$$A=\Big[[n]\Big],\quad B=[n]-\Big[n-[n]\Big],$$
$$L=\Big[n-[n]\Big],\quad L'=n-[n]-\Big[n-[n]\Big].$$

We have then to break up L in every possible way into parts which will give by combining equal parts into groups all the values of l, $(a+b-1)$. In like manner, the partitionment of L' will give all the values of l', $(a'+b'-1)$.

Any of the values of $a+b-1$ and of $a'+b'-1$ respectively being called c and c', we have

$$a=[c+1],\quad b=c+1-[c+1],\quad a'=[c'+1],\quad b'=c'+1-[c'+1].$$

Hence every system of $\quad l_1,\ a_1,\ b_1;\ \ l_2,\ a_2,\ b_2;\ \ldots$

and of $\quad l_1',\ a_1',\ b_1';\ \ l_2',\ a_2',\ b_2';\ \ldots$

satisfying the equations of condition may be found. To find the corresponding values of α, β; α', β' we must observe that one combination of the equations (1), (2), (3), (4), having been employed to obtain the quantities already found, only three of these equations are independent; we shall accordingly have

$$\alpha=[f+\Sigma l],\quad \beta=f+\Sigma l-[f+\Sigma l],$$
$$\alpha'=\Sigma la+\Sigma l'a'+A-\alpha,$$
$$\beta'=\Sigma lb+\Sigma l'b'+B-\beta,$$

and the problem is completely resolved.

(8) If now we pass to the case of three variables x, x', x'', given explicitly as functions of u, u', u'', we must take

$$J=\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{du'}, & \frac{dx}{du''} \\ \frac{dx'}{du}, & \frac{dx'}{du'}, & \frac{dx'}{du''} \\ \frac{dx''}{du}, & \frac{dx''}{du'}, & \frac{dx''}{du''} \end{vmatrix},$$

which, for greater brevity, using $\bar{u}, \bar{u}', \bar{u}''$, to denote $\frac{d}{du}, \frac{d}{du'}, \frac{d}{du''}$, may be written

$$\begin{vmatrix} \bar{u}x, & \bar{u}'x, & \bar{u}''x \\ \bar{u}x', & \bar{u}'x', & \bar{u}''x' \\ \bar{u}x'', & \bar{u}'x'', & \bar{u}''x'' \end{vmatrix}.$$

The nine first minor determinants may then be expressed under the respective forms

$$\begin{matrix} \frac{dJ}{d\bar{u}x}, & \frac{dJ}{d\bar{u}'x}, & \frac{dJ}{d\bar{u}''x}, \\ \frac{dJ}{d\bar{u}x'}, & \frac{dJ}{d\bar{u}'x'}, & \frac{dJ}{d\bar{u}''x'}, \\ \frac{dJ}{d\bar{u}x''}, & \frac{dJ}{d\bar{u}'x''}, & \frac{dJ}{d\bar{u}''x''}. \end{matrix}$$

The general term in $\left(\frac{d}{dx}\right)^f \left(\frac{d}{dx'}\right)^{f'} \left(\frac{d}{dx''}\right)^{f''} \vartheta$ will then be

$$(-)^i \frac{N}{J^\omega}\left(\frac{dJ}{d\bar{u}x}\right)^\alpha \left(\frac{dJ}{d\bar{u}'x}\right)^\beta \left(\frac{dJ}{d\bar{u}''x}\right)^\gamma$$
$$\times \left(\frac{dJ}{d\bar{u}x'}\right)^{\alpha'} \left(\frac{dJ}{d\bar{u}'x'}\right)^{\beta'} \left(\frac{dJ}{d\bar{u}''x'}\right)^{\gamma'}$$
$$\times \left(\frac{dJ}{d\bar{u}x''}\right)^{\alpha''} \left(\frac{dJ}{d\bar{u}'x''}\right)^{\beta''} \left(\frac{dJ}{d\bar{u}''x''}\right)^{\gamma''}$$
$$\times \left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{du'}\right)^b \left(\frac{d}{du''}\right)^c x\right\}^l \times \text{\&c.}$$
$$\times \left\{\left(\frac{d}{du}\right)^{a'} \left(\frac{d}{du'}\right)^{b'} \left(\frac{d}{du''}\right)^{c'} x'\right\}^{l'} \times \text{\&c.}$$
$$\times \left\{\left(\frac{d}{du}\right)^{a''} \left(\frac{d}{du'}\right)^{b''} \left(\frac{d}{du''}\right)^{c''} x''\right\}^{l''} \times \text{\&c.}$$
$$\times \left(\frac{d}{du}\right)^A \left(\frac{d}{du'}\right)^B \left(\frac{d}{du''}\right)^C \vartheta;$$

and similarly to the last case

$$i = \Sigma l + \Sigma l' + \Sigma l'',$$
$$\omega = \alpha + \beta + \gamma + \alpha' + \beta' + \gamma' + \alpha'' + \beta'' + \gamma'',$$
$$\alpha + \alpha' + \alpha'' = \Sigma la + \Sigma l'a' + \Sigma l''a'' + A$$
$$\beta + \beta' + \beta'' = \Sigma lb + \Sigma l'b' + \Sigma l''b'' + B$$
$$\gamma + \gamma' + \gamma'' = \Sigma lc + \Sigma l'c' + \Sigma l''c'' + C,$$
$$\alpha + \beta + \gamma = f + \Sigma l$$
$$\alpha' + \beta' + \gamma' = f' + \Sigma l'$$
$$\alpha'' + \beta'' + \gamma'' = f'' + \Sigma l'',$$

from which six equations we may deduce

$$\Sigma l\,(a+b+c-1) + \Sigma l'\,(a'+b'+c'-1) + \Sigma l''\,(a''+b''+c''-1) = f+g+h-(A+B+C).$$

(9) And the six equations first written may be solved in a manner analogous to the four equations in the preceding case.

We have finally

$$N = \frac{\Pi\,(\alpha+\beta+\gamma-1)\,\Pi\,(\alpha'+\beta'+\gamma'-1)\,\Pi\,(\alpha''+\beta''+\gamma''-1)}{\Pi\alpha\,\Pi\beta\,\Pi\gamma\,\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'\,\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}$$
$$\times \frac{\Pi\,(\alpha+\alpha'+\alpha''-1)\,\Pi\,(\beta+\beta'+\beta''-1)\,\Pi\,(\gamma+\gamma'+\gamma''-1)}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^l \times \text{\&c.} \times \Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'} \times \text{\&c.} \times \Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''} \times \text{\&c.}}$$
$$\times D \div (\Pi A\,\Pi B\,\Pi C).$$

where D = the determinant following, namely,

$$\begin{vmatrix} \alpha+\beta+\gamma, & 0, & 0, & \Sigma la, & \Sigma lb, & \Sigma lc \\ 0, & \alpha'+\beta'+\gamma', & 0, & \Sigma l'a', & \Sigma l'b', & \Sigma l'c' \\ 0, & 0, & \alpha''+\beta''+\gamma'', & \Sigma l''a'', & \Sigma l''b'', & \Sigma l''c'' \\ \alpha, & \alpha', & \alpha'', & \alpha+\alpha'+\alpha'', & 0, & 0 \\ \beta, & \beta', & \beta'', & 0, & \beta+\beta'+\beta'', & 0 \\ \gamma, & \gamma', & \gamma'', & 0, & 0, & \gamma+\gamma'+\gamma'' \end{vmatrix},$$

which, employing the equations

$$\alpha + \alpha' + \alpha'' = \Sigma la + \Sigma l'a' + \Sigma l''a'' + A$$
$$\beta + \beta' + \beta'' = \Sigma lb + \Sigma l'b' + \Sigma l''b'' + B$$
$$\gamma + \gamma' + \gamma'' = \Sigma lc + \Sigma l'c' + \Sigma l''c'' + C,$$

may be expressed under the forms

$$\lambda ABC + \mu BC + \mu' CA + \mu'' AB + \nu A + \nu' B + \nu'' C,$$

where all the coefficients λ, μ, ν, are essentially positive functions of α, β, γ, &c., Σla, Σlb, Σlc, &c.

The general form of D is apparent, as is also the reason why there is no term in which one of the indices, A, B, C ... does not appear, namely, that the sum of the lines in the lower half of the square, minus the sum of the lines in its upper half, gives rise to the line of terms following, which may be substituted in place of any one of the existing lines

$$0,\ 0,\ 0\ \dots\ A,\ B,\ C\ \dots$$

so that one of the letters A, B, C ... must appear in every actual term of the development.

(10) Let us return for a moment to show what the theorem becomes for the case of a single variable x, from which the transition is to be made to u.

For this case $$J = \frac{dx}{du},$$

and the 1st minor which is a determinant of zero places, as is well known to those conversant with determinants, must be taken $+1$. The formula then becomes

$$(-)^i (1)^a \frac{N}{J^\omega} \left\{ \left(\frac{d}{du}\right)^{a_1} x \right\}^{l_1} \left\{ \left(\frac{d}{du}\right)^{a_2} x \right\}^{l_2} \dots \left\{ \left(\frac{d}{du}\right)^{a_e} x \right\}^{l_e} \cdot \left(\frac{d}{du}\right)^{A} \vartheta,$$

where $$i = l_1 + l_2 + \dots + l_e,\ \ \omega = \alpha = l_1 a_1 + l_2 a_2 + \dots + l_e a_e + A,$$

and $$N = \frac{\Pi(\alpha - 1)}{\Pi \alpha} \frac{\Pi(\alpha - 1)}{\Pi l_1 (\Pi a_1)^{l_1} \Pi l_2 (\Pi a_2)^{l_2} \dots \Pi l_e (\Pi a_e)^{l_e} \Pi A} \cdot D,$$

where
$$D = \begin{vmatrix} \alpha, & \alpha - A \\ \alpha, & \alpha \end{vmatrix} = \alpha A.$$

Hence
$$N = \frac{\Pi(\alpha - 1)}{\Pi l_1 (\Pi a_1)^{l_1} \times \&c. \times \Pi l_e (\Pi a_e)^{l_e} \times \Pi(A-1)},$$
agreeing, as it ought, with Burmann's Law.

(11) For particular classes of terms N admits of a reduction to a simpler form.

Thus, in the case of three variables, suppose that the matrix

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix} \quad \text{assumes the form} \quad \begin{matrix} \alpha, & 0, & 0, \\ 0, & \beta', & 0, \\ 0, & 0, & \gamma'', \end{matrix}$$

by which I mean that
$$\beta = 0, \quad \gamma = 0,$$
$$\alpha' = 0, \quad \gamma' = 0,$$
$$\alpha'' = 0, \quad \beta'' = 0.$$

Then by substituting for the 4th, 5th, and 6th lines in D the differences between the 4th and 1st, the 5th and 2nd, the 6th and 3rd, respectively, D assumes the form

$$\begin{vmatrix} \alpha, & 0, & 0, & \Sigma la, & \Sigma lb, & \Sigma lc \\ 0, & \beta', & 0, & \Sigma l'a', & \Sigma l'b', & \Sigma l'c' \\ 0, & 0, & \gamma'', & \Sigma l''a'', & \Sigma l''b'', & \Sigma l''c'' \\ 0, & 0, & 0, & \Sigma l'a' + \Sigma l''a'' + A, & -\Sigma lb, & -\Sigma lc \\ 0, & 0, & 0, & -\Sigma l'a', & \Sigma lb + \Sigma l''b'' + B, & -\Sigma l'c' \\ 0, & 0, & 0, & -\Sigma l''a'', & -\Sigma l''b'', & \Sigma lc + \Sigma l'c' + C \end{vmatrix},$$

which

$$= \alpha\beta'\gamma'' \times \begin{vmatrix} \Sigma l'a' + \Sigma l''a'' + A, & -\Sigma l'a', & -\Sigma l''a'' \\ -\Sigma lb, & \Sigma lb + \Sigma l''b'' + B, & -\Sigma l''b'' \\ -\Sigma lc, & -\Sigma l'c', & \Sigma lc + \Sigma l'c' + C \end{vmatrix},$$

which we may call $\alpha\beta'\gamma'' D'$.

The entire value of N is consequently

$$\alpha\beta'\gamma''\frac{\Pi(\alpha-1)\,\Pi(\beta'-1)\,\Pi(\gamma''-1)}{\Pi\alpha\,\Pi\beta'\,\Pi\gamma''}$$

$$\times\frac{\Pi(\alpha-1)\,\Pi(\beta'-1)\,\Pi(\gamma''-1)}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^{l}\times\&c.\times\Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'}\times\&c.\times\Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''}\times\&c.}$$

$$\times\frac{D}{\Pi A\,\Pi B\,\Pi C}$$

$$=\frac{\Pi\alpha\,\Pi\beta'\,\Pi\gamma''}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^{l}\times\&c.\times\Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'}\times\&c.\times\Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''}\times\&c.}$$

$$\times\frac{D'}{\Pi A\,\Pi B\,\Pi C}.$$

(12) The form of D' is deserving of consideration on its own account. Call

$$\Sigma l'a' = A_b,\quad \Sigma l''a'' = A_c,$$
$$\Sigma lb = B_a,\quad \Sigma l''b'' = B_c,$$
$$\Sigma lc = C_a,\quad \Sigma l'c' = C_b.$$

Then
$$D' = ABC + (A_b + A_c)\,BC + (B_c + B_a)\,CA + (C_a + C_b)\,AB$$
$$+ (B_cC_a + B_aC_b + B_aC_a)\,A + (C_aA_b + C_bA_c + C_bA_b)\,B$$
$$+ (A_bB_c + A_cB_a + A_cB_c)\,C.$$

The entire number of terms is 16. In general, for m variables the corresponding number will be $(m+1)^{m-1}$, as may easily be shown*.

* The number of terms in D', since each of them has positive unity for its numerical coefficient, is evidently the value of a determinant, which, for three variables, is

$$\begin{vmatrix} 3, & -1, & -1 \\ -1, & 3, & -1 \\ -1, & -1, & 3 \end{vmatrix}.$$

To find in general the value of such a determinant in its more general form

$$\begin{vmatrix} a, & -1, & -1 \\ -1, & a, & -1 \\ -1, & -1, & a \end{vmatrix},$$

which is the discriminant of $a(x^2+y^2+z^2)-2yz-2zx-2xy$, we may observe that this latter formula becomes a perfect square, that is, loses two orders when $a=-1$. Hence $(a+1)^2$ is a factor of the determinant. Again, when $a=2$ the sum of all the terms in each column is zero. Hence $(a-2)$ is also contained in it as a factor; the complete value of the determinant is therefore $(a-2)(a+1)^2$, that is, 4^2, when $a=3$; and so for a determinant of the mth order we obtain $\{a-(m-1)\}(a+1)^{m-1}$, which becomes $(m+1)^{m-1}$ when $a=m$.

The same result may also be obtained directly by the integration of a linear equation of differences of the second order of the form given in the example at the foot of page 14, in Mr Cohen's paper in this *Journal*.

If we take D, which also, like D', consists exclusively of positive terms, only with unit coefficients, the number of these terms for the case of 1, 2, 3 variables I find to be 1, 12, 432; and for the general case of m variables I presume that the law is $m^m(m+1)^{m-1}$.

The terms themselves may be found without calculation by means of a simple rule.

Suppose that there are four variables, we may then find D' for the case corresponding to the one just treated of for three variables by taking the product of

$$A_b + A_c + A_d + A,$$
$$B_a + B_c + B_d + B,$$
$$C_a + C_b + C_d + C,$$
$$D_a + D_b + D_c + D,$$

and *rejecting* every term in such product in which any group of the letters forms a cycle.

Thus, for example, every term in which $A_b \times B_a$ enters must be rejected, because AB, BA is a cycle.

So, again, every term in which $A_b \times B_c \times C_a$ enters must be rejected, because AB, BC, CA forms a cycle.

We might take the product of $A_a + A_b + A_c + A_d + A$, and the quantities similarly formed, and proceed as above; for since AA is a cycle, as is also BB, CC, DD, therefore $A_a B_b C_c D_d$ will not appear in the final result.

Applying the method of rejection, we find without difficulty D', which represents the determinant

$$\begin{vmatrix} A + A_b + A_c + A_d, & -A_b, & -A_c, & -A_d \\ -B_a, & B + B_a + B_c + B_d, & -B_c, & -B_d \\ -C_a, & -C_b, & C + C_a + C_b + C_d, & -C_d \\ -D_a, & -D_b, & -D_c, & D + D_a + D_b + D_c \end{vmatrix},$$

$$= ABCD + \Sigma (A_b + A_c + A_d) BCD + \Sigma (A_bB_c + A_bB_d + A_cB_a + A_cB_c + A_cB_d + A_dB_a + A_dB_c + A_dB_d) CD$$
$$+ \Sigma \begin{pmatrix} A_dB_dC_d + (A_b + A_c) B_dC_d + (B_c + B_a) C_dA_d + (C_a + C_b) A_dB_d \\ \{B_a (C_a + C_b) + B_cC_a\} C_d + \{C_b (A_b + A_c) + C_aA_b\} A_d \\ + \{A_c (B_c + B_a) + A_bB_c\} B_d \end{pmatrix} D.$$

The total number of terms being

$$1 + 4 \times 3 + 8 \times 6 + 4 \times 16 = 125 = 5^3,$$

as it ought to be.

Other cases of simplification will readily suggest themselves; and, of course, when $\gamma = 0$, $\gamma' = 0$, $\gamma'' = 0$, which equations imply also $\Sigma lc = 0$, $\Sigma l'c' = 0$, $\Sigma l''c'' = 0$, and $C = 0$, the value of N will reduce as it ought to the form corresponding to the case of only two variables, and so in general (the value of the coefficient of any term in the development of the transformed value of any differential coefficient of a function of several variables depending only upon such of them as appear in the term itself, and in no way upon the other variables not so appearing).

(13) To indicate the method of passing from the theory of transformation of systems explicitly to that of systems of variables implicitly connected, let us suppose $\phi(x, y) = 0$ and that $\frac{d^f \vartheta}{dx^f}$ is to be expressed in terms of $\frac{d}{dy}$, ϕ, ϑ.

We may make this transformation depend upon our being able to solve the following question in the reversion of series, namely:

Given $$\xi = a\rho + b\sigma + \frac{1}{1\,.\,2}(c\rho^2 + 2d\rho\sigma + e\sigma^2) + \&c.,$$

$$\eta = a'\rho + b'\sigma + \frac{1}{1\,.\,2}(c'\rho^2 + 2d'\rho\sigma + e'\sigma^2) + \&c.,$$

to express $\rho^h\sigma^k$ in terms of ξ, η. The solution of this question, when $b = 0$, $a' = 0$, has been given by Jacobi, *Crelle*, t. VI. 1830; and as is obvious and pointed out by Jacobi, the general case, by either of two methods, namely, combination of the equations or linear transformations effected in the variables ρ, σ contained in them, may be made to depend on the particular case for which $b = 0$, $a' = 0$; but Jacobi has not followed out the effects of these processes, and apparently was not aware of the results being (as we may now see is the case) capable of an explicit representation, which mode of representation is essential for the purpose we have in view.

Let x, y be functions of u, v; and suppose x, y, ϑ to become $x + \xi$, $y + \eta$, $\vartheta + \tau$, when u and v become $u + h$ and $v + k$ respectively; then we shall have

$$\xi = \frac{dx}{du}h + \frac{dx}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}x}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.,$$

$$\eta = \frac{dy}{du}h + \frac{dy}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}y}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.,$$

$$\tau = \frac{d\vartheta}{du}h + \frac{d\vartheta}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}\vartheta}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.;$$

but treating τ as a function of ξ, η, we have also

$$\tau = \&c. + \left\{\&c. + \frac{1}{\Pi f\,\Pi g}\cdot\frac{d^{f+g}\vartheta}{dx^f dy^g}\xi^f\eta^g + \&c.\right\} + \&c.$$

Hence $\frac{1}{\Pi f\,\Pi g}\cdot\frac{d^{f+g}\vartheta}{dx^f dy^g}$ being expanded by means of our theorem in terms of $\frac{d}{du}$, $\frac{d}{dv}$, x, y, ϑ, the coefficient in such expansion of $\frac{d^{A+B}\vartheta}{du^A dv^B}$ will exhibit the value of the coefficient of $\xi^f\eta^g$ in the expansion of $\frac{1}{\Pi A\,\Pi B}h^A k^B$ in terms of ξ and η.

(14) As there are no quantitative relations between the coefficients in the equations above written which express ξ and η, we are therefore now able to express the value of $h^A k^B$ in terms of ξ and η when ξ and η are respectively expressed as rational integral functions of (and vanishing with) h and k. Thus, let us write in general

$$\xi = \Sigma p_{r,s} h^r k^s,$$
$$\eta = \Sigma q_{r,s} h^r k^s,$$

where $p_{0,0}$ and $q_{0,0}$ are each zero, but all the other values of p and q absolutely arbitrary. We have now $p_{r,s}$, $q_{r,s}$ respectively replacing

$$\frac{1}{\Pi r \, \Pi s} \frac{d^{r+s} x}{du^r dv^s}, \quad \frac{1}{\Pi r \, \Pi s} \frac{d^{r+s} y}{du^r dv^s}$$

and consequently the general term in the expansion of $h^A k^B$ as a function of ξ and η will be $I_{f,g} \xi^f \eta^g$, where

$$I_{f,g} = \frac{\Pi A \, \Pi B}{\Pi f \, \Pi g} \Sigma (-)^{i+\beta+\alpha'} \frac{N}{J^{\alpha+\beta+\alpha'+\beta'}} q^{\alpha}{}_{0,1} p^{\alpha'}{}_{0,1} q^{\beta}{}_{1,0} p^{\beta'}{}_{1,0}$$
$$\times (\Pi a \Pi b \, p_{a,b})^l \times \&c. \times (\Pi a' \Pi b' \, q_{a',b'})^{l'} \times \&c.,$$

where

$$N = \frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$
$$\times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi l \,(\Pi a \, \Pi b)^l \times \&c. \times \Pi l' \,(\Pi a' \, \Pi b')^{l'} \times \&c.} \times \frac{D}{\Pi A \, \Pi B},$$

and

$$J = \begin{vmatrix} p_{1,0}, & p_{0,1} \\ q_{1,0}, & q_{0,1} \end{vmatrix}.$$

Hence

$$I_{f,g} = \Sigma \left\{ (-)^{\Sigma l + \Sigma l' + \beta + \alpha'} \frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'} \right.$$
$$\times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi f \, \Pi g} \times \frac{q^{\alpha}{}_{0,1} q^{\beta}{}_{1,0} p^{\alpha'}{}_{0,1} p^{\beta'}{}_{1,0}}{(p_{1,0} q_{0,1} - p_{0,1} q_{1,0})^{\alpha+\beta+\alpha'+\beta'}}$$
$$\times \begin{vmatrix} \alpha+\beta, & & \Sigma la, & \Sigma lb \\ & \alpha'+\beta', & \Sigma l'a', & \Sigma l'b' \\ \alpha, & \alpha', & \alpha+\alpha', & \\ \beta, & \beta', & & \beta+\beta' \end{vmatrix}$$
$$\left. \times \left(\frac{p^l{}_{a,b}}{\Pi l} \times \&c. \times \frac{q^{l'}{}_{a',b'}}{\Pi l'} \times \&c. \right) \right\},$$

α, β; α', β'; l, l', &c., being any system of positive integers which are capable of satisfying the equations

$$\alpha+\beta=\Sigma l+f,$$
$$\alpha'+\beta'=\Sigma l'+f',$$
$$\alpha+\alpha'=\Sigma la+\Sigma l'a'+A,$$
$$\beta+\beta'=\Sigma lb+\Sigma l'b'+B.$$

Hence the value of $h^A k^B$, which $=\Sigma I_{f,g}\xi^f\eta^g$, is completely determined as an explicit function of ξ, η, and the coefficients p, q, of the equations by which ξ, η, are given in terms of h and k.

(15) So for three variables, supposing

$$\xi=\Sigma m_{r,s,t}\, h^r k^s l^t,$$
$$\eta=\Sigma n_{r,s,t}\, h^r k^s l^t,$$
$$\zeta=\Sigma p_{r,s,t}\, h^r k^s l^t,$$

where $m_{0,0,0}$, $n_{0,0,0}$ and $p_{0,0,0}$, are each zero, but all other values of m, n, p absolutely arbitrary, making

$$\begin{vmatrix} m_{1,0,0}, & m_{0,1,0}, & m_{0,0,1} \\ n_{1,0,0}, & n_{0,1,0}, & n_{0,0,1} \\ p_{1,0,0}, & p_{0,1,0}, & p_{0,0,1} \end{vmatrix}=J,$$

and writing in general

$$\frac{d\log J}{dm_{i,i',i''}}=\mu_{i,i',i''},$$
$$\frac{d\log J}{dn_{i,i',i''}}=\nu_{i,i',i''},$$
$$\frac{d\log J}{dp_{i,i',i''}}=\phi_{i,i',i''},$$

we shall find $$h^A k^B l^C=\Sigma I_{f,g,h}\xi^f\eta^g\zeta^h,$$

where

$$I_{f,g,h}=\Sigma\left\{\left((-)^{\Sigma l+\Sigma l'+\Sigma l''}\frac{\Pi(\alpha+\beta+\gamma-1)\,\Pi(\alpha'+\beta'+\gamma'-1)\,\Pi(\alpha''+\beta''+\gamma''-1)}{\Pi\alpha\,\Pi\beta\,\Pi\gamma\,\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'\,\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}\right.\right.$$
$$\left.\frac{\Pi(\alpha+\alpha'+\alpha''-1)\,\Pi(\beta+\beta'+\beta''-1)\,\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi f\,\Pi g\,\Pi h}\right)$$
$$\times D\times\mu^{\alpha}{}_{1,0,0}\,\mu^{\beta}{}_{0,1,0}\,\mu^{\gamma}{}_{0,0,1}\,\nu^{\alpha'}{}_{1,0,0}\,\nu^{\beta'}{}_{0,1,0}\,\nu^{\gamma'}{}_{0,0,1}\,\phi^{\alpha''}{}_{1,0,0}\,\phi^{\beta''}{}_{0,1,0}\,\phi^{\gamma''}{}_{0,0,1}$$
$$\left.\times\frac{m^l{}_{a,b,c}}{\Pi l}\times\&\text{c.}\times\frac{n^{l'}{}_{a',b',c'}}{\Pi l'}\times\&\text{c.}\times\frac{p^{l''}{}_{a'',b'',c''}}{\Pi l''}\times\&\text{c.}\right\},$$

where D is a known determinant of the sixth order expressible in terms of

α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$; Σla, Σlb, Σlc; $\Sigma l'a'$, $\Sigma l'b'$, $\Sigma l'c'$; $\Sigma l''a''$, $\Sigma l''b''$, $\Sigma l''c''$, and where

$$\alpha+\beta+\gamma=\Sigma l+f,$$
$$\alpha'+\beta'+\gamma'=\Sigma l'+g,$$
$$\alpha''+\beta''+\gamma''=\Sigma l''+h,$$
$$\alpha+\alpha'+\alpha''=\Sigma la+\Sigma l'a'+\Sigma l''a''+A,$$
$$\beta+\beta'+\beta''=\Sigma lb+\Sigma l'b'+\Sigma l''b''+B,$$
$$\gamma+\gamma'+\gamma''=\Sigma lc+\Sigma l'c'+\Sigma l''c''+C.$$

(16) Suppose now that we wish from the equation $0=\Sigma p_{r,s}h^r k^s$ to deduce the value of k^β in terms of h.

We may put
$$\xi=\Sigma p_{r,s}h^r k^s,$$
$$\eta=h,$$

and then apply the formula of reversion for finding k^β in terms of ξ and η; but since $\xi=0$, we may reject all the terms out of $\Sigma I_{f,g}\xi^f\eta^g$, except those in which $f=0$; moreover, in adapting the formula applicable to this case, we must put $q_{r,s}=0$ for all values of the system r, s, except 1, 0, and for that system $q_{1,0}=1$; we have, therefore, to retain such terms only in $I_{0,g}$ for which $\alpha=0$, $\Sigma l'a'=0$, $\Sigma l'b'=0$;

D consequently becomes
$$\begin{vmatrix} \beta, & 0, & \Sigma la, & \Sigma lb \\ 0, & \alpha'+\beta', & 0, & 0 \\ 0, & \alpha', & \alpha', & 0 \\ \beta, & \beta', & 0, & \beta+\beta' \end{vmatrix}$$

$$=\alpha'(\alpha'+\beta')\begin{vmatrix} \beta, & \Sigma lb \\ \beta, & \beta+\beta' \end{vmatrix}=\alpha'(\alpha'+\beta')\beta\begin{vmatrix} 1, & \beta+\beta'-B \\ 1, & \beta+\beta' \end{vmatrix}$$
$$=\alpha'(\alpha'+\beta')\beta B;$$

hence
$$I_{0,g}=\Sigma\left\{(-)^{\Sigma l+\beta+\alpha'}\frac{\Pi\beta\,\Pi(\alpha'+\beta')}{\Pi\beta\,\Pi\alpha'\,\Pi\beta'}\times\frac{\Pi\alpha'\,\Pi(\beta+\beta'-1)B}{\Pi g}\right.$$
$$\left.\times\frac{p^{\alpha'}{}_{0,1}\,p^{\beta'}{}_{1,0}}{p_{0,1}{}^{\beta+\alpha'+\beta'}}\times\frac{p^l{}_{a,b}}{\Pi l}\times\&\text{c.}\right\},$$

with the conditions
$$\left.\begin{aligned}\alpha'=\Sigma la,\qquad\qquad \beta=\Sigma l\\ \beta+\beta'=\Sigma lb+B,\quad \alpha'+\beta'=g\end{aligned}\right\};\qquad(\omega)$$

we have, therefore, finally

$$k^B = B\Sigma\left(\left\{\Sigma(-)^{\Sigma la}\frac{\Pi\{\Sigma(lb)+B-1\}}{\Pi\,\Sigma\{l(b-1)+B\}}\cdot\frac{p_{1,0}{}^{\Sigma(l(b-1))+B}}{p_{0,1}{}^{\Sigma(lb)+B}}\cdot\frac{(p^l{}_{a_1 b_1})}{\Pi l}\cdots\frac{(p^{l_e}{}_{a_e,b_e})}{\Pi l_e}\right\}h\right)$$

with the sole condition deduced from the system (ω),

$$l_1(a_1+b_1-1)+l_2(a_2+b_2-1)+\ldots+l_e(a_e+b_e-1)=g-B.$$

Suppose, now, that $\phi(x, y)=0$, and that we wish to express $\dfrac{d^g\vartheta}{dx^g}$ (where, for greater simplicity, I consider ϑ a function only of y) in terms of x, y, without solving the equation $\phi=0$; we know that if we write

$$\frac{d\phi}{dx}h+\frac{d\phi}{dy}k+\&\text{c.}+\left\{\&\text{c.}+\frac{\left(\frac{d}{dx}\right)^\epsilon\left(\frac{d}{dy}\right)^\omega}{\Pi\epsilon\,\Pi\omega}h^\epsilon k^\omega+\&\text{c.}\right\}+\&\text{c.},$$

then $\dfrac{1}{\Pi g}\dfrac{d^g\vartheta}{dx^g}$ will be the coefficient of h^g in the expansion of

$$\frac{d\vartheta}{dy}k+\frac{d^2\vartheta}{dy^2}\frac{k^2}{1.2}+\frac{d^3\vartheta}{dy^3}\frac{k^3}{1.2.3}\ \&\text{c. in terms of } h.$$

Consequently, if we make

$$\frac{d^g\vartheta}{dx^g}=\Sigma E_B\frac{d^B\vartheta}{dy^B},$$

$$E_B=\frac{\Pi g}{\Pi(B-1)}\Sigma(-)^{\Sigma la}\frac{\Pi\{\Sigma(lb)+B-1\}}{\{\Pi\Sigma(lb)+B-\Sigma l\}}$$

$$\times\frac{\left(\frac{d\phi}{dx}\right)^{\Sigma lb+B-\Sigma l}}{\left(\frac{d\phi}{dy}\right)^{\Sigma lb+B}}\ \frac{\left\{\left(\frac{d}{dx}\right)^a\left(\frac{d}{dy}\right)^b\phi\right\}^l}{\Pi l\,(\Pi a\,\Pi b)^l}\cdots\frac{\left\{\left(\frac{d}{dx}\right)^{a_e}\left(\frac{d}{dy}\right)^{b_e}\phi\right\}^{l_e}}{\Pi l_e\,(\Pi a_e\,\Pi b_e)^{l_e}},$$

where, as before, writing in general $a_k+b_k-1=c_k$,

$$l_1c_1+l_2c_2+\ldots+l_ec_e=g-B,$$

g being now given, and B variable and subject to assume in succession every value from 1 up to B.

(17) By way of verifying the above formula, and as a protection against accidental errors of calculation, suppose $\phi=-x+\psi(y)$,

so that $$\frac{d\phi}{dx}=-1,\quad\frac{d\phi}{dy}=\frac{d\psi}{dy};$$

the only terms to be retained are those in which no (a) index appears.

We have, therefore, for this case $\Sigma l(b-1)=g-B$,

that is, $$\Sigma lb+B-\Sigma l=g,$$

and $$E_B=\frac{\Pi\{\Sigma(lb)+B-1\}}{\Pi l(\Pi b_1)^l\ldots\Pi l_e(\Pi b_e)^{l_e}}\frac{(-)^g}{\left(\frac{d\psi}{dy}\right)^{g+\Sigma l}}\left\{\left(\frac{d}{dy}\right)^b\psi\right\}^l\ldots\left\{\left(\frac{d}{dy}\right)^{b_e}\psi\right\}^{l_e}$$

agreeing, as required, with Burmann's law.

(18) As another example, in illustration of the fact that our general theorem embraces the whole theory of reversion, suppose we have the equation $\Sigma m_{r,s,t}q^rk^sl^t=0$, and that it is required from this equation to deduce l^C as a function of h and k.

We may write $$\xi=\Sigma m_{r,s,t}q^rk^sl^t,$$

$$\eta=q,$$

$$\zeta=k.$$

We have then $$l^C=\Sigma I_{0,g,h}q^gk^h,$$

and in assigning the value of $I_{0,g,h}$, we need, moreover, to retain in $I_{0,g,h}$ only those terms in which the a', b', c' and a'', b'', c'' systems of indices are wanting; for

$$\Sigma l'a'=0,\quad \Sigma l'b'=0,\quad \Sigma l'c'=0,$$

$$\Sigma l''a''=0,\quad \Sigma l''b''=0,\quad \Sigma l''c''=0.$$

Moreover α, β, γ; α', β', γ'; α'', β'', γ''; being the indices respectively of the minor determinants of the matrix

$$\begin{matrix} m_{1,0,0}; & m_{0,1,0}; & m_{0,0,1}; \\ 1\ ; & 0\ ; & 0\ ; \\ 0\ ; & 1\ ; & 0\ ; \end{matrix}$$

we may consider $\alpha=0$, $\beta=0$, $\beta'=0$, $\alpha''=0$, since the minor determinants which have these indices are all zero.

Hence, for the actual terms in $I_{0,g,h}$, D becomes

$$\begin{vmatrix} \gamma, & \ldots, & \ldots, & \Sigma la, & \Sigma lb, & \Sigma lc \\ \ldots, & \alpha'+\gamma', & \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \beta''+\gamma'', & \ldots, & \ldots, & \ldots \\ \ldots, & \alpha', & \ldots, & \alpha', & \ldots, & \ldots \\ \ldots, & \ldots, & \beta'', & \ldots, & \beta'', & \ldots \\ \gamma, & \gamma', & \gamma'', & \ldots, & \ldots, & \gamma+\gamma'+\gamma'' \end{vmatrix}$$

which obviously reduces to the form

$$\alpha'\beta''(\alpha'+\gamma')(\beta''+\gamma'')\begin{vmatrix}\gamma, & \Sigma lc \\ \gamma, & \Sigma lc + C\end{vmatrix}$$

$$=\alpha'\beta''(\alpha'+\gamma')(\beta''+\gamma'')\gamma C;$$

also the equations of condition between the indices become

$$\gamma=\Sigma l, \qquad \alpha'=\Sigma la,$$

$$\alpha'+\gamma'=g, \qquad \beta''=\Sigma lb,$$

$$\beta''+\gamma''=h, \quad \gamma+\gamma'+\gamma''=\Sigma lc+C,$$

in addition to the special equations

$$\alpha=0, \quad \beta=0, \quad \alpha''=0, \quad \beta'=0.$$

Hence $l^C=\Sigma I_{g,h}q^g k^h$, where $I_{g,h}$ represents

$$(-)^{\Sigma l}\frac{\Pi g\,\Pi h}{\Pi\gamma\,\Pi\alpha'\,\Pi\gamma'\,\Pi\beta''\,\Pi\gamma''}\cdot\frac{\Pi\alpha'\,\Pi\beta''\,\Pi(\gamma+\gamma'+\gamma''-1)\,C}{\Pi g\,\Pi h}\,.\,\&c.$$

$$=(-)^{\Sigma l}\,C\,\frac{\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}\cdot\frac{m_{0,0,1}{}^{\alpha'+\beta''}(-m_{0,1,0})^{\gamma''}(-m_{1,0,0})^{\gamma'}}{m_{0,0,1}{}^{\gamma+\gamma'+\gamma''+\alpha'+\beta''}}\times\&c.$$

$$=\Sigma(-)^{\gamma+\gamma'+\gamma''}\,C\,\frac{\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}\,\frac{m^{\gamma''}{}_{0,1,0}\,m^{\gamma'}{}_{1,0,0}}{m_{0,0,1}{}^{\gamma+\gamma'+\gamma''}}$$

$$\times\frac{m^{l}{}_{abc}}{\Pi l}\times\frac{m^{l_2}{}_{a_2b_2c_2}}{\Pi l_2}\ldots\times\frac{m^{l_e}{}_{a_eb_ec_e}}{\Pi l_e},$$

where $l_1(a_1+b_1+c_1-1)+\ldots+l_e(a_e+b_e+c_e-1)=g+h-C$, g and h being assumed of any values respectively, such that their sum is not less than C: the partitionment of $g+h-C$, gives every possible system

$$l_1\ldots l_e;\quad (a+b+c-1)\ldots(a_e+b_e+c_e-1);$$

and to every such system correspond known systems of values of $a, b, c; \ldots; a_e, b_e, c_e$. We have then $\gamma=\Sigma l$, $\gamma'+\gamma''=\Sigma l(c-1)+C$, which latter equation, for each value of c, gives $\Sigma l(c-1)+C+1$ systems of values of γ' and γ''. Thus we have the complete solution of the equation $\Sigma m_{r,s,t}q^r k^s l^t=0$.

In like manner, if we suppose i variables $q_1, q_2, \ldots q_i$, and for greater simplicity, in addition to the condition always supposed of the constant term being zero, likewise conceive that the coefficient shall be unity in each linear term of the equation

$$\Sigma m_{r_1r_2\ldots r_i}q_1{}^{r_1}q_2{}^{r_2}\ldots q_i{}^{r_i}=0,$$

we shall find

$$q_i{}^{A_i}=\Sigma(-)^{\gamma_1+\gamma_2+\ldots+\gamma_i}A_i\frac{\Pi(\gamma_1+\gamma_2+\ldots+\gamma_i-1)}{\Pi\gamma_1\,\Pi\gamma_2\ldots\Pi\gamma_i}$$

$$\times\frac{(m^{l}{}_{1_a2_a\ldots i_a})^{l}}{\Pi l}\times\&c.\times\frac{(m_{1_{a_e}2_{a_e}\ldots i_{a_e}})^{l_e}}{\Pi l_e}\,q_1{}^{f_1}q_2{}^{f_2}\ldots q_{i-1}{}^{f_{i-1}},$$

with the conditions following for finding the (γ) and ${}^1a, {}^2a \dots {}^ia \dots {}^1a_e, {}^2a_e \dots {}^ia_e$ systems, namely,

$$\Sigma l\,({}^1a + {}^2a + \dots + {}^ia - 1) = f_1 + f_2 + \dots + f_{i-1} - A_i,$$
$$\gamma_1 = \Sigma l, \quad \gamma_2 + \gamma_3 + \dots + \gamma_i = \Sigma l_i\,(a_i - 1) + A_i + 1.$$

(19) In like manner we may without difficulty assign the general law for solving with like generality any number of simultaneous equations between any greater number of variables, the functions equivalent to zero being all supposed to be without a constant term, and to be expressed as rational integral functions of the variables; and we can consequently pass from one system of independent variables to any new system in whatever way, whether explicitly or implicitly, through any number of equations and any number of connecting variables, the two systems may be supposed to be related.