17.

ON A DISCOVERY IN THE PARTITION OF NUMBERS*.

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LET $a_1, a_2, \ldots a_r$ be any given system of integer elements; I call the number of ways of composing the number n with these elements the *quotity*† of n in respect to the given elements. Let the least common multiple of $a_1, a_2, a_3, \ldots a_r$ be called p and let the roots of $\frac{x^p - 1}{x - 1} = 0$ be called ρ, then we may express the quotity in question under the form

$$A + U,$$

* From the last foot-note at p. 87, it follows that the non-periodical part of the analytical expression for the number of ways in which n can be composed of the r elements $a, b, c \ldots l$, is the coefficient of $\frac{1}{t}$ in the expansion, in a series of ascending powers of t, of the fraction $\frac{e^{nt}}{(1-e^{-at})(1-e^{-bt})\ldots(1-e^{-lt})}$. Moreover, if we suppose $\frac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)} = \frac{P}{(1-x)^r} +$ fractions not containing $(1-x)$ in the denominator, it further follows that, for values of n not less than r, the coefficient of x^n in P will be the coefficient of $\frac{1}{t}$ in the expression

$$\frac{e^{(n-r)t}\cdot(e^t-1)^r}{(1-e^{-at})(1-e^{-bt})\ldots(1-e^{-lt})},$$

which is evidently zero, as it ought to be.

† Thus the quotity of n in respect to a and 1 is the integer next greater than $\frac{n}{a}$; the complete expression for this quantity, it may be mentioned, is

$$\frac{1}{a}(n+\tfrac{1}{2}) - \frac{1}{a^2}\Sigma\{(a-1)+(a-2)\rho+\ldots+\rho^{a-2}\}\rho^{n+1},$$

where ρ is a prime root of $\frac{\rho^a-1}{\rho-1}=0$.

The quotity of n in respect to the consecutive elements 1, 2, 3 … r is equal to the number of ways of partitioning $n+r$ into r parts.

where A is an algebraical function of n and the elements, which is clear of all exponential expressions, and U is of the form

$$\Sigma\,(A_0 + A_1\rho + A_2\rho^2 + \text{\&c.} + A_{p-1}\rho^{p-1})\,\rho^n *.$$

I call U the quot-undulant†; A the quot-additant‡. I shall say nothing at present about the former, although I can express its value completely for any system of elements which are prime each to each, or of which the relations of identity existing between the prime factors are given: my theorem, for present purposes, is confined to the quot-additant, which may be written under the form following, namely,

$$\frac{1}{a_1 a_2 \ldots a_r}\left\{B_1 + B_2 n + B_3\frac{n^2}{1\,.\,2} + \text{\&c.} \ldots + B_{r-1}\frac{n^{r-2}}{1\,.\,2 \ldots (r-2)} + B_r\frac{n^{r-1}}{1\,.\,2 \ldots (r-1)}\right\},$$

where $$B_\omega = \Sigma C_{\omega_{\theta_1}} C_{\omega_{\theta_2}} \ldots C_{\omega_{\theta_r}} S\,(a_1^{\omega_{\theta_1}} a_2^{\omega_{\theta_2}} \ldots a_r^{\omega_{\theta_r}}),$$

S denoting as usual that a symmetrical function is to be formed, of which the quantity which follows it is the type of the general term, and the symbol Σ referring to a summation to be effected in respect to all the distinct systems of integer values (zeros included in the number) of $\omega_{\theta_1}, \omega_{\theta_2}, \ldots \omega_{\theta_r}$, whose sum is $r - \omega$, and where, in general, C_m denotes the coefficient of t^m in

$$\frac{te^t}{e^t - 1}\,§.$$

* The coefficients A_0, A_1, &c., are, in general, algebraical functions of n and of the elements whose degree in n is one unit inferior to the greatest number of the elements having the same common measure.

† The quot-undulant, although for present purposes presented as a single whole, is in fact a collective quantity made up, and most simply and naturally expressed by means of the sum of a series of analogous periodic or periodico-progressive functions, whose number is the same as that of the distinct elements, and whose periods are respectively measured by the number of units in each such element; it may be compared with a great wave, composed of a number of wavelets, whose lengths are either the same as or submultiples of its own. This is the view first taken by Mr Cayley, who, in his researches, has followed in the footsteps of Euler, but to which, also, I have been independently and unavoidably conducted by the method of investigation peculiar to myself. Sir John Herschel and Mr Kirkman have not taken this view, and accordingly there is an unnecessary complexity in their statements of results.

‡ If we suppose the fraction $\dfrac{1}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_r})}$ thrown under the form

$$\frac{P}{(1-x)^r} + \frac{Q}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_r}) \div (1-x)^r},$$

the quot-additant of n is the coefficient of x^n in $\dfrac{P}{(1-x)^r}$ which gives the means of expressing P, and consequently also Q. Compare Note iv. in M. Serret's excellent *Cours d'Algèbre supérieure*, 2nd edition.

§ The theorem may also be stated as follows. Let A_0, A_1, A_2, &c., denote the successive coefficients in the expansion in a series of ascending powers of x of the reciprocal of the product of $1-e^{-ax}$, $1-e^{-bx}$, ... $1-e^{-lx}$, then will the quot-additant of n in respect of the r elements $a, b, c \ldots l$, be expressed by $A_{r-1} + A_{r-2}\,.\,n + A_{r-3}\,.\,\dfrac{n^2}{1\,.\,2} + \ldots + A_0\,.\,\dfrac{n^{r-1}}{1\,.\,2\,.\,3 \ldots (r-1)}$. [See note * of p. 86.]

Examples. The quot-additants of n, in respect to the systems a; a, b; a, b, c; a, b, c, d, &c., respectively, are as follows:

$$\frac{1}{a},\ \frac{1}{ab}\left(\frac{a+b}{2}+n\right),$$

$$\frac{1}{abc}\left(\frac{a^2+b^2+c^2+3ab+3ac+3bc}{12}+\frac{a+b+c}{2}n+\frac{n^2}{1.2}\right),$$

$$\frac{1}{abcd}\left(\frac{\Sigma a^2b+3\Sigma abc}{24}+\frac{(\Sigma a)^2+3\Sigma ab}{12}n+\frac{\Sigma a}{4}n^2+\frac{n^3}{1.2.3}\right)^*.$$

So, again, the constant term in the quot-additant to the system of elements a, b, c, d, e will be

$$\frac{1}{abcde}\Sigma\left\{-\frac{a^4}{720}+\frac{a^2b^2}{144}+\frac{a^2bc}{48}+\frac{abcd}{16}\right\},$$

and to the system of six elements it will be

$$\frac{1}{abcdef}\Sigma\left\{-\frac{a^4b}{1440}+\frac{a^2b^2c}{288}+\frac{a^2bcd}{96}+\frac{abcde}{32}\right\};$$

it will be seen that the latter quantity under the sign of summation is obtained term for term from the one above by introducing a new element with the index unity in the numerator and doubling each denominator; this law is general, and is an immediate consequence of the fact that for a coefficient of i dimensions in the elements the only partitionments of i which appear in the groups of indices are those which are made up of the elements 1, 2, 4, 6, &c., all the odd elements except 1 from the nature of Bernouilli's numbers giving rise to the coefficient zero, so that, consequently, the partitionments of $2\iota+1$ which enter into the expression in question, are all derived from those of 2ι by the addition of a single distinct unit.

The series of fractions $\frac{1}{2}, \frac{1}{12}, 0, \frac{1}{120}, 0$, &c., arise in my method as the results of substituting $\dfrac{1}{\omega+1}$ in place of $\dfrac{\delta^\omega\phi}{\phi}$ in the expansion of the successive variations of $\log\phi$.

Thus, $\quad \delta\log\phi=\dfrac{\phi'}{\phi}=\frac{1}{2},\ \delta^2\log\phi=\dfrac{\phi''}{\phi}-\left(\dfrac{\phi'}{\phi}\right)^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12},$

$$\delta^3\log\phi=\frac{\phi'''}{\phi}-3\frac{\phi''\phi'}{\phi^2}+2\frac{\phi'^3}{\phi^3}=\tfrac{1}{4}-\tfrac{3}{6}+\tfrac{2}{8}=0, \text{ \&c. \&c.}$$

* When the elements $a, b, c \ldots l$ are prime each to each, the quot-undulant will not contain n, that is, will be strictly periodic. For this case, therefore, the difference between the quot-additant of n and that of $n-(a.b.c\ldots l)$ will represent the difference between the entire quotity of n and that of $n-(a.b.c\ldots l)$ in respect to the system supposed. We have consequently an easy method of verifying, by actual decompositions of numbers, the general expression for the additant part without knowing the value of the undulant part in the complete expression for the quotity. For the case in question the quot-additant may also be defined and calculated as the algebraical expression whose mean value is the same as the mean value of the quotity when the partible number passes through a period of $a.b.c\ldots l$ consecutive integer values.

Hence it may easily be collected, that if we write

$$\frac{te^t}{e^t-1}=1+K_1t-2K_2t^2+3K_3t^3 \mp \&c.,$$

we ought to have

$$K_i=\frac{1}{i}E_0-\frac{1}{i-1}E_1+\frac{1}{i-2}E_2 \;\&c. \pm E_{i-1},$$

where E_ω denotes, in general, the coefficient of h^ω in

$$\left(\frac{1}{2}+\frac{h}{2\,.\,3}+\frac{h^2}{2\,.\,3\,.\,4}+\&c.\right)^{i-\omega},$$

or, which is the same thing, K_i ought to be equal to the coefficient of t^i in $\log\frac{e^t-1}{t}$ as is easily demonstrable to be the case by Maclaurin's Theorem.

In general, if the quot-additant of n, in respect to the roots of

$$x^r+p_1x^{r-1}+p_2x^{r-2}+\&c.+p_r,$$

be expressed as a function of $p_1, p_2, \dots p_r$ and n, and be called $\frac{1}{p_r}Q_r$, we have the following equations existing, namely,

$$Q_r=\int dn\, Q_{r-1} \text{ and } \left(\frac{d}{dp_1}+p_1\frac{d}{dp_2}+\dots+p_{r-1}\frac{d}{dp_r}\right)Q_r=-\frac{1}{2}Q_{r-1}.$$

Observation. My method which has led me to the preceding theorem reposes upon the axiom, which I believe is quite new, that the mean value of the $a\,.\,b\,.\,c \dots l$ sums of homogeneous powers and products (all affected with the coefficient unity) of n dimensions in $\alpha, \beta, \gamma, \dots, \lambda$, where $\alpha^a=1$, $\beta^b=1, \dots \lambda^l=1$, is equal to the quotity of n in respect to $a, b, c, \dots l$.