18.

ON THE PARTITION OF NUMBERS.

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I MUST reluctantly content myself for the present (unexpected events, which have robbed me of the leisure and calm of mind necessary for composition, and the due evolution and embodiment of ideas on any extensive scale, forbid me to do more) with a brief statement of the general solution of this important question, which (as known to my thrice-distinguished friend, Mr Cayley) I succeeded in completing almost immediately after the appearance of the last number of the Journal.

It must be clearly understood that the methods of Euler, De Morgan, Herschel, Kirkman, and Cayley (the last a great advance upon all that went before) have only afforded the means (with more or less generality) of determining the *quotity* of a number in respect of given elements in any particular case; the existence of a universal algebraical representation of this quotity seems not even to have been suspected. Moreover, it will be found that the general formula, which I am about to give, possesses an immense practical advantage in point of facility of computation over the methods previously employed. Thus, for example, I have been able to compute by it, in a moderate space of time, the number of ways of partitioning n into nine parts; the enormous complexity of the calculations required by the methods of Herschel and Cayley had induced those distinguished authors to rest satisfied with stopping short at the formula for only five parts.

My result has been erected upon a completely independent basis, and deduced by an equally original method, namely, the axiom contained in the observation at the end of my former paper combined with a simple theorem for expressing, by means of partial fractions, the sum of the homogeneous powers and products of any number of quantities, not merely for the *special* case of these quantities being all unlike, but for the *general* case of their being made up of any sets of equals. MM. Cayley and Terquem have both suggested, what is no doubt true, the possibility of obtaining my result otherwise, and perhaps a little more simply, by aid of M. Cauchy's Theory of Residues.

I now proceed to enunciate the theorem.

$a_1, a_2, \ldots a_r$ (all positive integers) are supposed to be the elements, n the partible number, and the object in view is the expression of the quotity of n quâ the elements $a_1, a_2, \ldots a_r$, that is, of the number of solutions of the equation in integers $a_1x_1 + a_2x_2 + \&c. + a_rx_r = n$, in which equation it may be observed no further condition is imposed upon the coefficients $a_1, a_2, \ldots a_r$ than that of their being positive integers. There is no restriction upon their being equal in any manner *inter se*. Call Q the quotity in question: then we may consider Q as made up of an infinite number of waves, of which, however (as it will immediately be seen), only a finite number have an *actual* existence, the rest will be *abortive*.

Let $\frac{p}{q}$ be any rational numerical fraction whatever, not exceeding unity, in its lowest terms, and use $w_{\frac{p}{q}}$ to denote the coefficient of $\frac{1}{t}$ in the development of the expression

$$e^{nw}(1-e^{a_1u})^{-1}(1-e^{a_2u})^{-1}\ldots(1-e^{a_ru})^{-1},$$

where $$w = \frac{2\pi ip}{q} + t, \quad u = \frac{2\pi ip}{q} - t, \quad i = (-1)^{\frac{1}{2}};$$

then $$Q = \Sigma w_{\frac{p_i}{q}}.$$

If $p_1, p_2, \ldots p_i$ be all the numbers (unity included) less than q, and prime to it, and if we write

$$w_{\frac{p_1}{q}} + w_{\frac{p_2}{q}} + \ldots + w_{\frac{p_i}{q}} = W_q,$$

we shall have more simply

$$Q = \Sigma W_q.$$

W_q again may be expressed* under a more easily intelligible form as the coefficient of $\frac{1}{t}$ in the development in ascending powers of t of

$$\Sigma \frac{\rho^n e^{nt}}{(1-\rho^{a_1}e^{-a_1t})(1-\rho^{a_2}e^{-a_2t})\ldots(1-\rho^{a_r}e^{-a_rt})},$$

where ρ is in succession each of the roots of the *prime factor* of $\rho^q - 1$; and

[* Cf. p. 157 below.]

consequently since this development will contain no term where $\frac{1}{t}$ enters, unless some one at least of the quantities $\rho^{a_1}, \rho^{a_2}, \ldots \rho^{a_r}$ is unity, it follows that $W_q = 0$, except for those values of q which are contained in some one or more of the elements $a_1, a_2, \ldots a_r$. The number of *actual* waves expressing a given quotity is consequently the number of distinct integers, unity included, which enter into the composition of the elements to which the quotity has reference.

It will readily be seen that on making $q = 1$ we shall obtain the expression for the so-called quot-additant (a name only adopted for provisional purposes, and which I now discard) given [p. 87 above] in the preceding number of the Journal. The generating function for this part of the quotity becomes, from the general formula,

$$\frac{e^{nt}}{(1-e^{-a_1t})(1-e^{-a_2t})\ldots(1-e^{-a_rt})}.$$

The coefficient of $\frac{1}{t}$ in this expression will give the formulæ contained in the body of the paper referred to; but far more expeditious formulæ of computation may be substituted in lieu of these. For we may write

$$W_1 = \text{coefficient of } \frac{1}{t} \text{ in}$$

$$e^{nt - \{\log(1-e^{-a_1t}) + \log(1-e^{-a_2t}) + \ldots + \log(1-e^{-a_rt})\}}.$$

But in general

$$\log(1-e^{-t}) = \log t - \frac{t}{2} + \frac{t^2}{24} + \&c.$$

$$= \log t - \frac{t}{2} + \frac{B_1}{1.2^2}t^2 - \frac{B_2}{1.2.3.4^2}t^4 + \&c.,$$

B_1, B_2, &c., denoting Bernouilli's numbers, namely,

$$\frac{1}{6}, \ \frac{1}{30}, \ \frac{1}{42}, \ \frac{1}{30}, \ \&c.$$

Hence

$$W_1 = \frac{1}{a_1a_2\ldots a_r} \times \text{coefficient of } t^{r-1} \text{ in } e^{(n+\frac{1}{2}s_1)t - \frac{B_1s_2}{1.2^2}t^2 + \frac{B_2s_4}{1.2.3.4^2}t^4\ \&c.},$$

where s_ω in general denotes the sum of the ωth powers of the elements $a_1, a_2, \ldots a_r$.

Hence, writing $n + \frac{1}{2}s_1 = \nu$, we have

$$W_1 = \text{coefficient of } t^{r-1} \text{ in}$$

$$(a_1 a_2 \ldots a_r)^{-1} \left(1 + \nu t + \nu^2 \frac{t^2}{1.2} + \nu^3 \frac{t^3}{1.2.3} + \&c.\right)$$
$$\times \left(1 - \frac{1}{24} s_2 t^2 + \frac{1}{1152} s_2{}^2 t^4 + \&c.\right)$$
$$\times \left(1 + \frac{1}{2880} s_4 t^4 + \frac{1}{165888} s_4{}^2 t^8 + \&c.\right)$$
$$\times \left(1 - \frac{1}{181440} s_6 t^6 + \&c.\right)$$
$$\times \&c.^*$$

The wave W_2 is also deserving of particular notice, on account of it also being free from the sign of summation, and involving only the Bernouillian numbers. To find this wave we have to take ρ, the root of the prime factor of $\rho^2 - 1$, that is, we have simply $\rho = -1$.

And if we distinguish the elements $a_1, a_2 \ldots a_r$ into two groups, say $\alpha_1, \alpha_2 \ldots \alpha_l$, all even, and $\beta_1, \beta_2 \ldots \beta_m$ all odd, we have

$$W_2 = \text{coefficient of } \frac{1}{t} \text{ in the generator}$$

$$e^{nt}(-)^n \frac{1}{(1 - e^{-\alpha_1 t})(1 - e^{-\alpha_2 t}) \ldots (1 - e^{-\alpha_l t})} \frac{1}{(1 + e^{-\beta_1 t})(1 + e^{-\beta_2 t}) \ldots (1 + e^{-\beta_m t})},$$

which $$= (-)^n e^{nt - R},$$

where $$R = \Sigma \log(1 - e^{-\alpha_1 t}) + \Sigma \log(1 + e^{-\beta_1 t}).$$

But $\log(1 - e^{-t})$ has been already expressed, and

$$\log(1 + e^{-t}) = \log 2 - \frac{t}{2} + \frac{1}{8} t^2 + \&c.$$

$$= \log 2 - \frac{t}{2} + \frac{3}{4} B_1 t^2 - \frac{15}{16} \frac{B_2}{1.2.3} t^4 + \&c.$$

* To save circumlocution, I have not expressed in the text the general value of the coefficient of t_i, but of course there is not the slightest difficulty in so doing; let i be thrown in every possible way under the form $K_1 + 2K_2 + 4K_4 + 6K_6 + \&c.$ (that is to say, all the partitions of i quâ the elements 1, 2, 4, 6, &c., are to be written down), then the coefficient in question is

$$\Sigma\phi(K_1, K_2, K_4, \&c.)\, \nu^{K_1} . s_2{}^{K_2} . s_4{}^{K_4} . s_6{}^{K_6} \&c.,$$

where $$\phi(K_1, K_2, K_4, \&c.) = \frac{\pm 1}{\Pi K_1 \, \Pi K_2 \, \Pi K_4 \, \&c.} \left(\frac{B_1}{1.2^2}\right)^{K_2} \left(\frac{B_2}{1.2.3.4^2}\right)^{K_4} \left(\frac{B_3}{1.2.3.4.5.6^2}\right)^{K_6} \&c.$$

It is deeply interesting to observe how, in the very formula for expressing partitions, a class of partitions reappears,—in fact, partitions constitute the sphere in which analysis lives, moves, and has its being; and no power of language can exaggerate or paint too forcibly the importance of this till recently almost neglected, but vast, subtle, and universally permeating element of algebraical thought and expression.

Happy ought I to feel in the reflection of having been the appointed instrument to make so great an advance in a doctrine which contains a large part of the future of pure analysis, and to have impressed upon it a form which must inevitably give rise to an illimitable host of the most important applications and consequences.

Hence, using $s_1, s_2 \ldots$ to denote the sums of the 1st, 2nd ... powers of $\alpha_1, \alpha_2 \ldots$ and $\sigma_1, \sigma_2 \ldots$ to denote the sums of the 1st, 2nd ... powers of $\beta_1, \beta_2 \ldots$ and writing $n + \frac{1}{2}(s_1 + \sigma_1) = \nu$, we have

$$nt - R = -\log(2^m \alpha_1\alpha_2 \ldots \alpha_l) + l \log t + \nu t - \frac{B_1}{1 \, . \, 2^2}(s_2 + 3\sigma_2) t^2$$

$$+ \frac{B_2}{1 \, . \, 2 \, . \, 3 \, . \, 4^2}(s_4 + 15\sigma_4) t^4 + \&c.$$

and consequently, $W_2 = \dfrac{1}{2^r \left(\dfrac{\alpha_1}{2} \cdot \dfrac{\alpha_2}{2} \ldots \dfrac{\alpha_l}{2}\right)} \times$ coefficient of t^{l-1} in

$$\left(1 + \nu t + \nu^2 \frac{t^2}{1 \, . \, 2} + \nu^3 \frac{t^3}{1 \, . \, 2 \, . \, 3} + \&c.\right)$$

$$\times \left(1 - \frac{1}{24}(s_2 + 3\sigma_2) t^2 + \frac{1}{1152}(s_2 + 3\sigma_2)^2 t^4 + \&c.\right)$$

$$\times \left(1 + \frac{1}{2880} s_4 t^4 + \&c.\right)$$

$$\times \&c.$$

So in general if we wish to find the wave W_q, we must distinguish the elements into two groups,

$\alpha_1, \alpha_2 \ldots \alpha_l$, all exactly divisible by q,

and $\beta_1, \beta_2 \ldots \beta_m$ not so divisible;

W_q will then be the coefficient of $\dfrac{1}{t}$ in

$$\Sigma \frac{\rho^n e^{nt}}{(1 - \rho^{\alpha_1} e^{-\alpha_1 t})(1 - \rho^{\alpha_2} e^{-\alpha_2 t}) \ldots (1 - \rho^{\alpha_l} e^{-\alpha_l t}) \times (1 - \rho^{\beta_1} e^{-\beta_1 t})(1 - \rho^{\beta_2} e^{-\beta_2 t}) \ldots (1 - \rho^{\beta_m} e^{-\beta_m t})}$$

where the sign of summation indicates that all the values are to be taken in succession of the prime roots of $\rho^q - 1 = 0$; this, again, may be expressed as the coefficient of t^{l-1} in a quantity of the form ρ^{nt-R} where R may be expressed by means of the prime qth roots of unity and the known numerical coefficients which enter into the expansion in ascending powers of t of the quantity $\dfrac{1}{1 - c\rho^{-t}}$; but I do not propose here to enter into the details of the method. It will be enough for present purposes to illustrate it by an example. Suppose, then, that we take the elements 1, 2, 3, 4, 5, 6; in other words, that we propose to express algebraically the number of ways in which n can be divided into six or a smaller number of parts.

The expression will here consist of six parts, which I shall reckon in inverse order, beginning with W_6.

W_6 will be the coefficient of $\frac{1}{t}$ in

$$\Sigma \frac{e^{nt}\rho^n}{(1-e^{-6t})\times(1-\rho e^{-t})(1-\rho^2 e^{-2t})(1-\rho^3 e^{-3t})(1-\rho^4 e^{-4t})(1-\rho^5 e^{-5t})},$$

where ρ is either root of $\rho^2-\rho+1=0$;

this is evidently equal to

$$\frac{1}{6}\times\Sigma\rho^n\frac{1}{(1-\rho)(1-\rho^2)(1-\rho^3)(1-\rho^4)(1-\rho^5)}$$

$$=\Sigma\frac{1}{6}\rho^n\frac{1}{(1-\rho)(1-\rho^2)\,2\,(1+\rho)(1+\rho^2)}$$

$$=\Sigma\frac{\rho^n}{12}\frac{1}{(1-\rho^2)(1-\rho^4)}$$

$$=\Sigma\frac{\rho^n}{12}\frac{1}{(2-\rho)(1+\rho)}=\Sigma\frac{\rho^n}{36}.$$

In like manner,

$$W_5=\frac{1}{5}\,.\,\Sigma\frac{\rho^n}{(1-\rho)(1-\rho^2)(1-\rho^3)(1-\rho^4)(1-\rho)},$$

where ρ is any root of

$$\rho^4+\rho^3+\rho^2+\rho+1=0.$$

Hence $$W_5=\frac{1}{25}\Sigma\frac{\rho^n}{1-\rho}$$

$$=\frac{1}{125}\Sigma\rho^n(4+3\rho+2\rho^2+\rho^3)$$

$$=\frac{1}{125}\Sigma\rho^n(2+\rho-\rho^3-2\rho^4).$$

Again,

$$W_4=\frac{1}{4}\Sigma\frac{\rho^n}{(1-\rho)(1-\rho^2)(1-\rho^3)(1-\rho)(1-\rho^2)},$$

where ρ is either root of $\rho^2+1=0$.

Hence $$W_4=\frac{1}{16}\Sigma\frac{\rho^n}{(1-\rho)^2(1+\rho)}$$

$$=\frac{1}{64}\Sigma\frac{\rho^n(1-\rho)}{-\rho}$$

$$=\frac{1}{64}\Sigma(\rho^n-\rho^{n-1}).$$

Again, $W_3=$ coefficient of $\frac{1}{t}$ in

$$\Sigma\frac{\rho^n e^{nt}}{(1-e^{-3t})(1-e^{-6t})\times\&\text{c.}},$$

where $$\rho^2+\rho+1=0,$$

$$=\frac{1}{18}\Sigma\frac{\rho^n\nu}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)}=\frac{1}{18}\cdot\frac{\Sigma\rho^n\nu}{\{(1-\rho)(1-\rho^2)\}^2}$$

$$=\frac{\Sigma\rho^n\nu}{162},$$

where $$\nu=n+\frac{3}{2}+\frac{6}{2}+\frac{\rho}{\rho-1}+\frac{2\rho^2}{\rho^2-1}+\frac{4\rho^4}{\rho^4-1}+\frac{5\rho^5}{\rho^5-1}$$

$$=n+\frac{9}{2}+\frac{\rho}{\rho-1}+\frac{2\rho^2}{\rho^2-1}+\frac{4\rho}{\rho-1}+\frac{5\rho^2}{\rho^2-1}$$

$$=n+\frac{9}{2}-\frac{1}{3}\{5(2+\rho)+7(2+\rho^2)\}$$

$$=n+\frac{9}{2}-\frac{1}{3}(24+5\rho+7\rho^2)$$

$$=n-\frac{1}{6}(21+5\rho+7\rho^2)=n-\frac{1}{3}(7-\rho).$$

$$W_2=\frac{(-)^n}{2^6(1.2.3)}\times\text{coefficient of } t^2 \text{ in }\left(1+\nu t+\frac{\nu^2t^2}{2}\right)\left(1-\frac{1}{24}(s_2+3\sigma_2)t^2\right),$$

where $$\nu=n+\frac{1}{2}(1+2+3+4+5+6)$$

$$=n+\frac{21}{2},$$

$$s_2=2^2+4^2+6^2=56;\ \ \sigma_2=1^2+3^2+5^2=35;\ \ 3\sigma_2=105.$$

Hence $$W_2=(-)^n\frac{1}{384}\times\left(\frac{\nu^2}{2}-\frac{161}{24}\right)$$

$$=(-)^n\left(\frac{\nu^2}{768}-\frac{161}{9216}\right).$$

Finally $$W_1=\frac{1}{1.2.3.4.5.6}\times\text{coefficient of } t^5 \text{ in}$$

$$\left(1+\nu t+\nu^2\frac{t^2}{2}+\nu^3\frac{t^3}{6}+\nu^4\frac{t^4}{24}+\nu^5\frac{t^5}{120}\right)$$

$$\times\left(1-\frac{1}{24}s_2t^2+\frac{1}{1152}s_2{}^2t^4\right)$$

$$\times\left(1+\frac{1}{2880}s_4t^4\right)$$

$$=\frac{1}{720}\left\{\frac{\nu^5}{120}-\frac{\nu^3}{144}s_2+\nu\left(\frac{s_2{}^2}{1152}+\frac{s_4}{2880}\right)\right\},$$

where $$\nu = n + \frac{21}{2}$$

$$s_2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$$

$$s_4 = 1 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 = 2275$$

$$s_2{}^2 = 8281.$$

Hence $$\frac{s_2{}^2}{1152} + \frac{s_4}{2880} = \frac{1}{1152}(8281 + 910)$$

$$= \frac{1}{1152}(9191),$$

and $$W_1 = \frac{\nu}{720}\left(\frac{\nu^4}{120} - \frac{91\nu^2}{144} + \frac{9191}{1152}\right).$$

As no useful object would be attained by substituting for ν its value $n + \frac{21}{2}$, I leave the expressions for W_1, W_2 in their present form as explicit functions of ν.

It is well worthy of observation that the exponent of the generating function of W_1, namely, $nt - R$, when the elements are taken the consecutive numbers 1, 2, 3 ... r, consists exclusively of Bernouillian numbers and sums of powers of 1, 2, 3 ... r; but as these latter are themselves expressible by Euler's theorem, in terms of powers of r and the Bernouillians, R is for this case a quadratic function of the numbers of Bernouilli.

If we express the quantities of the form $\Sigma\rho^w$, which occur in the different modes in terms of Herschel's circulating functions, then our expression assumes the very same form in which Mr Cayley had observed it was the most advantageous to express the quotity, and to which he has given the name of "prime radical circulators*."

* Thus what with Mr Cayley was an invention, with me becomes a theorem. Mr Cayley was led to the use of prime circulators from a perception of their affording the best analytical means of giving determinateness to the representation of the results; in my method they offer themselves spontaneously, and cannot be rejected.

Supposing that Mr Cayley could claim a right to the exclusive use of these forms, we should have an instructive instance of one of the mischiefs ascribed to the general system of patent law, namely, of blocking up the necessary march of invention. For the benefit of foreign readers of the *Journal*, I should add that r_n is used by Herschel to denote a quantity which is unity when n contains r as a factor, and is otherwise zero; and that any function of $r_n, r_{n-1}, r_{n-2} \dots r_{n-r+1}$ is called a circulating function, and may, of course, be expressed as a linear function of the above quantities.

Suppose ρ to be any factor whatever of r, i to be less than ρ, and $r = \rho\sigma$. Then if for all admissible values of ρ and i

$$A_{n-i} + A_{n-i-\rho} + A_{n-i-2\rho} + \&\text{c.} + A_{n-i-(\sigma-1)\rho} = 0,$$

$A_n . r_n + A_{n-1} . r_{n-1} + \&\text{c.} + A_{n-r+1}$ (where A_n, A_{n-1} ... are ordinary constants) is, according to Cayley, a prime radical circulator (prime circulator would be quite as specific and more convenient).

I shall conclude this very brief notice of my theory by converting the W waves in the example above treated into the form of these prime circulators.

For W_6 ρ is any root of $\rho^2-\rho+1=0$.

Hence

$$\Sigma\rho^0=2;\ \Sigma\rho=1;\ \Sigma\rho^2=-1;\ \Sigma\rho^3=-2;\ \Sigma\rho^4=-1;\ \Sigma\rho^5=1.$$

Hence in the notation of Herschel

$$W_6=\frac{6_n}{18}+\frac{6_{n-1}}{36}-\frac{6_{n-2}}{36}-\frac{6_{n-3}}{18}-\frac{6_{n-4}}{36}+\frac{6_{n-5}}{36}.$$

For W_5 ρ is any root of $\rho^4+\rho^3+\rho^2+\rho+1=0$.

Hence

$$\Sigma\rho^0=4;\ \Sigma\rho=-1;\ \Sigma\rho^2=-1;\ \Sigma\rho^3=-1;\ \Sigma\rho^4=-1.$$

Hence

$$\begin{aligned}W_5=\frac{1}{125}\{&(8-1+1+2)\,5_n+(-2-1+1-8)\,5_{n-1}\\&+(-2-1-4+2)\,5_{n-2}+(-2-1+1+2)\,5_{n-3}\\&+(-2+4+1+2)\,5_{n-4}\}\\=&\frac{2}{25}\,5_n-\frac{2}{25}\,5_{n-1}-\frac{1}{25}\,5_{n-2}+\frac{1}{25}\,5_{n-3}.\end{aligned}$$

In W_4 ρ is either root of $\rho^2+1=0$.

Hence $\Sigma\rho^0=2;\ \Sigma\rho=0;\ \Sigma\rho^2=-2;\ \Sigma\rho^3=0,$

and hence

$$\begin{aligned}W_4&=\frac{1}{16}(2\,.\,4_n+2\,.\,4_{n-1}-2\,.\,4_{n-2}-2\,.\,4_{n-3})\\&=\frac{4_n}{8}+\frac{4_{n-1}}{8}-\frac{4_{n-2}}{8}-\frac{4_{n-3}}{8}.\end{aligned}$$

In W_3 $\rho^2+\rho+1=0$.

Hence $\Sigma\rho^0=2;\ \Sigma\rho=-1;\ \Sigma\rho^2=-1.$

Hence

$$\begin{aligned}W_3&=\left(\frac{1}{81}\,3_n-\frac{1}{162}\,3_{n-1}-\frac{1}{162}\,3_{n-2}\right)\left(n-\frac{7}{3}\right)\\&\quad-\frac{1}{486}\,3_n-\frac{1}{486}\,3_{n-1}+\frac{1}{243}\,3_{n-2}\\&=\left(\frac{1}{81}\,3_n-\frac{1}{162}\,3_{n-1}-\frac{1}{162}\,3_{n-2}\right)n\\&\quad-\left(\frac{5}{162}\,3_n-\frac{1}{81}\,3_{n-1}-\frac{3}{162}\,3_{n-2}\right).\end{aligned}$$

Finally

$$W_2=\left(\frac{\nu^2}{768}-\frac{161}{9216}\right)(2_n-2_{n-1}).$$

W_1 has already been expressed in its simplest terms; and the solution of the question of the partition of n into six parts is now complete.

The same causes which have interposed to prevent my setting forth at length the method above sketched out, have also interposed to preclude me from extending the exposition which I had intended of the application of the principles contained in my paper on Differential Transformations to the general question of the solution of equations, or systems of equations, containing any number of variables, thereby entirely superseding the necessity of all special considerations whatever in obtaining Lagrange's and Laplace's Theorems, either in their developed or ordinary form; and theorems to many degrees of infinity more general than these; for their method being, by the aid of this most desiderated discovery, now capable of being substituted for artifice, which, in general, may be said to stand in the same relation to method as what instinct is to reason, or the craft of the savage to the wisdom of the civilised man.