21.

DEVELOPMENT OF AN IDEA OF EISENSTEIN.

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EISENSTEIN has remarked, in a note among his collected works, that the expansion of any negative power of a series of ascending powers of x may be made to depend upon the expansions of the positive powers of the same series. The following method which reposes upon the most elementary principles of algebra serves to establish this practically important proposition.

Let
$$u = 1 + A_1 x + A_2 x^2 + \&c.$$
Then
$$\frac{1}{u^i} = \frac{1}{\{1-(1-u)\}^i} = 1 + i(1-u) + \frac{i(i+1)}{2}(1-u)^2 + \frac{i(i+1)(i+2)}{2.3}(1-u)^3 + \&c.$$

If now we wish to express the nth power of x or, in fact, any power of x lower than the nth by means of this series, it is obvious that we may stop at the term containing the nth power of $1-u$. In general, then, denoting by $C_{i,\nu}$ the coefficient of x^ν in u^i, provided ν is greater than unity and not greater than n, we have

$$C_{-i,\nu} = -i\left(1 + (i+1) + \frac{(i+1)(i+2)}{2} + \&c. + \frac{(i+1)(i+2)\ldots(i+n-1)}{1.2\ldots(n-1)}\right)C_{1,\nu}$$
$$+ i.\frac{i+1}{2}\left(1 + (i+2) + \&c. + \frac{(i+2)(i+3)\ldots(i+n-1)}{1.2\ldots(n-2)}\right)C_{2,\nu}$$
$$\&c., \&c., \&c.$$
$$\pm \frac{i(i+1)\ldots(i+n-1)}{1.2\ldots n}C_{n,\nu}$$
$$= -i.\frac{(i+2)(i+3)\ldots(i+n)}{1.2\ldots(n-1)}C_{1,\nu} + i\frac{i+1}{2}.\frac{(i+3)\ldots(i+n)}{1.2\ldots(n-2)}C_{2,\nu}$$
$$\mp \&c. \pm \frac{i(i+1)\ldots(i+n-1)}{1.2\ldots n}C_{n,\nu}.$$

In practice it will, of course, be always most expedient (on the score of brevity of expression) to assume $n=\nu$.

If
$$u = 1 + A_\omega x^\omega + A_{\omega+1}x^{\omega+1} + \text{\&c.}$$
the condition for the truth of the above equation will be that ν shall not exceed ωn; and then in practice it will be expedient to take n equal to $\frac{\nu}{\omega}$, if that be an integer, or, if not, to take n, the integer next above $\frac{\nu}{\omega}$.

We may with propriety denote by $C_{0,n}$ the coefficient of x^n in $\log u$*; and since
$$\log u = -\{(1-u) + \tfrac{1}{2}(1-u)^2 + \tfrac{1}{3}(1-u)^3 + \text{\&c.}\}$$
we shall have, subject to the same conditions as before,
$$\begin{aligned} C_{0,\nu} &= (1+1+\text{\&c. to } n \text{ terms})\, C_{1,\nu} \\ &\quad - \tfrac{1}{2}(1+2+\text{\&c.}+(n-1))\, C_{2,\nu} \\ &\quad + \tfrac{1}{3}\left(1+3+6+\text{\&c.}+\frac{(n-2)(n-1)}{2}\right) C_{3,\nu} \\ &\quad \text{\&c., \&c.} \\ &= nC_{1,\nu} - \frac{(n-1)n}{2^2}C_{2,\nu} + \frac{(n-2)(n-1)n}{2.3^2}C_{3,\nu}\ \text{\&c.} + (-)^{n-1}\frac{C_{n,\nu}}{n}. \end{aligned}$$

In the general theorem suppose $i=1$, $\nu=n$. Then
$$C_{-1,n} = -\frac{(n+1)n}{1.2}C_{1,n} + \frac{(n+1)n(n-1)}{1.2.3}C_{2,n}\ \text{\&c.} \pm C_{n,n}.$$

Thus, to take the example alluded to by Eisenstein, suppose
$$u = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} + B_3\frac{x^6}{6!}\ \text{\&c., \&c.},$$
so that by the formula
$$\frac{(-)^{n+1}B_n}{(2n)!} = -\frac{(2n+1)2n}{1.2}C_{1,2n} + \frac{(2n+1)(2n)(2n-1)}{1.2.3}C_{2,2n}\ \text{\&c.} + C_{2n,2n}.$$

Here
$$\begin{aligned} C_{u,2n} &= \text{coefficient of } x^{2n} \text{ in } \left(\frac{1-e^{-x}}{x}\right)^u \\ &= \text{coefficient of } x^{2n+u} \text{ in } 1 - ue^{-x} + u\frac{u-1}{2}e^{-2x}\ \text{\&c.} + (-)^u e^{-ux} \\ &= \frac{(-)^u}{(2n+u)!}\left\{-u + u\frac{u-1}{2}2^{2n+u} \mp \text{\&c.} + (-)^u u^{2n+u}\right\} \\ &= \frac{\Delta^u 0^{2n+u}}{(2n+u)!}. \end{aligned}$$

* The rule for any power, positive or negative, of $\log u$ deserves investigation; the case of $C_{0,n}$ (using that symbol in a more extended sense than in the text above) containing, as it were, a microcosmical reiteration of the whole theory under discussion.

Hence

$$(-)^{n+1} B_n = (2n\,!)\left\{-\frac{(2n+1)\,2n}{2\,!}\frac{\Delta 0^{2n+1}}{(2n+1)\,!}\right.$$
$$\left.+\frac{(2n+1)(2n)(2n-1)}{3\,!}\frac{\Delta^2 0^{2n+2}}{(2n+2)\,!}+\&c.\right\}$$

$$= 2n\left\{\frac{-\Delta 0^{2n+1}}{2\,!}+\frac{2n-1}{2n+2}\frac{\Delta^2 0^{2n+2}}{3\,!}-\frac{(2n-1)(2n-2)}{(2n+2)(2n+3)}\frac{\Delta^3 0^{2n+3}}{4\,!}+\&c.\right.$$
$$\left.+\frac{(2n-1)(2n-2)\ldots 1}{(2n+2)(2n+3)\ldots 4n}\frac{\Delta^{2n} 0^{4n}}{(2n+1)\,!}\right\}.$$

Thus, if $n=1$,

$$B_1 = 2\left\{-\frac{\Delta 0^3}{2}+\frac{1}{4}\frac{\Delta^2 0^4}{6}\right\}$$
$$= 2\left\{-\frac{1}{2}+\frac{1}{24}(2^4-2\,.\,1^4)\right\}$$
$$= 2\left\{-\frac{1}{2}+\frac{14}{24}\right\}=\frac{1}{6}.$$

If $n=2$,

$$-B_2 = 4\left\{\frac{-\Delta 0^5}{2}+\frac{1}{2}\frac{\Delta^2 0^6}{6}-\frac{1}{7}\frac{\Delta^3 0^7}{24}+\frac{1}{56}\frac{\Delta^4 0}{120}\right\}$$
$$= 4\left\{-\frac{1}{2}+\frac{1}{12}(2^6-2)-\frac{1}{168}(3^7-3\,.\,2^7+3)\right.$$
$$\left.+\frac{1}{6720}(4^8-4\,.\,3^8+6\,.\,2^8-4)\right\}$$
$$= 4\left\{-\frac{1}{2}+\frac{31}{6}-\frac{43}{4}+\frac{243}{40}\right\}$$
$$= 4\left\{-\frac{1}{2}+\frac{1}{6}+\frac{1}{4}+\frac{3}{40}\right\}$$
$$= \frac{1}{30}\{-60+20+30+9\}$$
$$= -\frac{1}{30}, \text{ or } B_2 = \frac{1}{30}, \text{ and so on.}$$

The annexed independent demonstration of the formula for $\frac{1}{fu}$ appertains to Mr Cayley.

Write $$x = u + hfx,$$

then, by Lagrange's theorem,

$$Fx = Fu + \frac{h}{1}F'u\,fu + \frac{h^2}{1\,.\,2}\frac{d}{du}F'u\,(fu)^2 + \frac{h^3}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^2 F'u\,(fu)^3 + \ldots,$$

or, differentiating with respect to u,

$$\frac{F'x}{1-hf'x}=F'u+\frac{h}{1}\frac{d}{du}F'ufu+\frac{h^2}{1\,.\,2}\left(\frac{d}{du}\right)^2 F'u\,(fu)^2+\ldots;$$

whence, putting $h=\dfrac{x-u}{fx}$,

$$\frac{F'x}{1-(x-u)\dfrac{f'x}{fx}}=F'u+\frac{x-u}{fx}\frac{d}{du}F'ufu+\frac{1}{1\,.\,2}\left(\frac{x-u}{fx}\right)^2\frac{d^2}{du^2}F'u\,(fu)^2+\ldots,$$

which is true identically.

Suppose now $F'u=\dfrac{u}{fu}$, we have

$$\frac{x}{fx-(x-u)f'x}=\frac{u}{fu}+\frac{x-u}{fx}\frac{d}{du}u+\frac{1}{1\,.\,2}\left(\frac{x-u}{fx}\right)^2\frac{d^2}{du^2}ufu+\&\text{c.},$$

or, if $\qquad fu=1+bu+cu^2+du^3+\&\text{c.},\quad fx=1+bx+cx^2+\&\text{c.},$

and $x=0$, that is, $fx=1$, the formula becomes

$$0=\frac{u}{fu}-u+\frac{u^2}{1\,.\,2}\left(\frac{d}{du}\right)^2 ufu-\frac{u^3}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^3 u\,(fu)^2+\ldots.$$

Whence

$$\frac{1}{fu}=1-\frac{u}{1\,.\,2}\left(\frac{d}{du}\right)^2 ufu+\frac{u^2}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^3 u\,(fu)^2$$
$$-\frac{u^3}{1\,.\,2\,.\,3\,.\,4}\left(\frac{d}{du}\right)^4 u\,(fu)^3+\&\text{c.},$$

which gives the expansion of $\dfrac{1}{fu}$ when the expansions of the positive powers $(fu)^2$, $(fu)^3$, &c. are known.