32.

MEDITATION ON THE IDEA OF PONCELET'S THEOREM.

[*Philosophical Magazine*, XX. (1860), pp. 307—316.]

HITHERTO Poncelet's theorem has been regarded as a method *sui generis* and complete in itself; but in truth it is but the first germ or rudiment of a vast and prolific algebraical theory; and not only so, but the principle which it contains admits of applications of the utmost value in various dynamical and analytical questions, which it is surprising should have been allowed to lay so long dormant. For the present, however, I mean to confine myself to a very brief indication of one direction in which the theorem admits of being generalized. And first I will make a remark upon so simple a matter as the extraction of the square root, which seems to have escaped observation, and at all events is so far from being generally known, that two of the highest authorities for mathematical erudition in this country whom I have consulted on the subject provisionally accept it as new.

Let r be an approximate value of $\sqrt{N}$; then by that mode of application of Newton's method of approximation to the equation $x^2 = N$ which is equivalent to the use of continued fractions, we may easily establish the following theorem, namely, that

$$\frac{r^2+N}{2r},\ \frac{r^3+3rN}{3r^2+N},\ \frac{r^4+6r^2N+N^2}{4r^3+4rN},\ \frac{r^5+10r^3N+5rN^2}{5r^4+10r^2N+N^2},\ \ldots *$$

will be successive approximations to $\sqrt{N}$, whose limits of error can be

* In other words, if r be the first approximation to $\sqrt{N}$, the ith approximation will be

$$\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\sqrt{N},$$

so that the relative error becomes

$$\frac{2(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i},$$

in which form the theorem is self-subsistent, and needs no proof. But the fact remains interesting, that the application of Newton's method of approximation to the equation $x^2 = N$ will be found to lead to the form above written at the ith step of the process conducted after the continued-fraction fashion.

assigned when a limit to the error of the first approximation r is given. The coefficients of the qth approximation, it will be observed, are for the numerator the alternate binomial coefficients

$$1,\ q\frac{q-1}{2},\ q\frac{q-1}{2}\frac{q-2}{3}\frac{q-3}{4},\ \text{\&c.};$$

and for the denominator the intermediate ones,

$$q,\ q\frac{q-1}{2}\frac{q-2}{3},\ \text{\&c.}$$

Mr Cayley has reminded me that the third approximation, $\frac{r^3+3rN}{3r^2+N}$, is a special case of a formula for *any* root of N given in the books; and to Mr De Morgan I am indebted for a hint which has led me to notice that all these forms may be deduced from the Newtonian method of approximation*.

If we call the ith approximation $\phi(i, r)$, we shall find that the functional equation $\phi\{j, \phi(i, r)\} = \phi(ij, r)$ will be satisfied; which is not so mere a truism as might at first sight be supposed, as any one may satisfy himself by studying the analogous theory for cubic or higher roots, a part of the subject to which I may hereafter return.

Now as to the limits of accuracy afforded by the successive approximations. Let e be a known limit to the relative error of the first approximation r, by which I mean that $\left(\frac{\sqrt{N}-r}{\sqrt{N}}\right)^2 < e^2$. For greater simplicity, I take separately the cases where r is too great and r is too small.

1. Let $\sqrt{N} < r < (1+\epsilon)\sqrt{N}$; then the errors will be throughout in excess; and we may assign as a limit of error to the ith approximation a quantity, say ϵ_i, which is a known function of ϵ, namely, $\frac{2}{(2\epsilon^{-1}+1)^i-1}$, which it may be noticed is less than $\frac{\epsilon^i}{2^{i-1}}$.

2. Let $\sqrt{N} > r > (1-\eta)\sqrt{N}$; then the errors will be alternately in defect and excess, and to the ith approximation we may assign a limit of error η_i, where $\eta_i = \frac{2}{(2\eta^{-1}-1)^i-(-1)^i}$ †.

* The expansion (after Newton) of $\sqrt{N}$ introduces the binomial coefficients—a curious fact! What are the analogous integers which the continued-fraction process applied to $\sqrt[3]{N}$ will produce?

† If we write

$$\epsilon_i = \theta(\epsilon, i) \text{ and } \eta_i = \vartheta(\eta, i),$$

then if i be any *odd* number,

$$\theta\{\theta(\epsilon, i), j\} = \theta(\epsilon, ij),$$
$$\vartheta\{\vartheta(\eta, i), j\} = \vartheta(\eta, ij);$$

and if i be any *even* number,

$$\vartheta\{\theta(\epsilon, i), j\} = \theta(\epsilon, ij),$$
$$\theta\{\vartheta(\eta, i), j\} = \vartheta(\eta, ij).$$

We may now apply these results to Poncelet's linear approximate representation of $\sqrt{(a+bx+cx^2)}$. Suppose $f+gx$ is the first approximation, as found by Poncelet's method, with a maximum relative error e, then

$$\frac{(f+gx)^2+(a+bx+cx^2)}{2(f+gx)}$$

will be a much closer approximation, with a relative error never exceeding $\frac{e^2}{2+e}$ in excess, nor $\frac{e^2}{2-e}$ in defect. So a still nearer approximation will be $\frac{(f+gx)^3+3(f+gx)(a+bx+cx^2)}{3(f+gx)^2+a+bx+cx^2}$, with a relative error never exceeding $\frac{e^3}{4+6e+3e^2}$ in excess, nor $\frac{e^3}{4-6e+3e^2}$ in defect, and so on. The marvellous facility which these formulæ afford for the calculation of elliptic and ultra-elliptic functions, and not merely for their computation as by a method of quadratures, but (which is of far greater importance) their quasi-representation under circular and logarithmic forms, with assignable limits of proportional error, will be illustrated in a future communication. As regards the idea of substituting rational for irrational functions, I have only to-day learned from Mr Cayley that I am anticipated in this by Mr Merrifield*,

Or more simply, if the error in excess be treated as positive, and in defect as negative, and δ be the first and δ_i the ith limit of error, we shall have

$$\delta_i=\frac{2\delta^i}{(2+\delta)^i-\delta^i};$$

and calling $\delta_i=\theta(i, \delta)$,

$$\theta\{j, \theta(i, \delta)\}=\theta(ij, \delta).$$

Thus, then, if we call $\frac{N+x^2}{2x}=\psi x$, $\psi^q x$ will correspond to the (2^q)th order of approximation, and the absolute value of the error will be less than

$$\frac{2\delta^{2^q}}{(2+\delta)^{2^q}-\delta^{2^q}}.$$

By way of example, suppose we take 6 as our first approximation to $\sqrt{31}$, then

$$\delta<\frac{\frac{1}{2}}{5\frac{1}{2}}<\frac{1}{11};$$

and if we make $\psi x=\frac{31+x^2}{2x}$, we shall have

$$\psi^4 6 : \sqrt{31} :: 1+\omega : 1,$$

where

$$\omega<\frac{2}{23^{16}-1},$$

which serves to exemplify the prodigious rapidity of the approximation in this method of extracting the square roots of numbers.

* I quite concur with Mr Merrifield, and in fact before being made acquainted with the existence of his paper, had emitted the same opinion (among others to Dr Borchardt of Berlin), that the substitutive method, consisting in the employment of rational functions in place of the radical, affords by far the most expeditious means for the calculation of elliptic functions of all orders, especially the third, and supersedes the necessity for the construction of special

in a paper very recently read before the Royal Society, but not yet printed in the *Transactions**.

auxiliary tables. I believe, however, that my substitutions, founded on Poncelet's views, are in general the best that can be employed for the purpose. In addition to other advantages they possess this, which deserves notice—that as we know *à priori* a superior limit to the proportional error, the arithmetical values of the integrals to which they are applied may be brought out correct to any required place of decimals, without its being necessary to calculate and compare a superior and inferior limit to the integral, either one of these being sufficient in my method to indicate its own reliable degree of precision.

* In general it is obvious, if ϕx between the limits a and b retain always the same sign, and ψx within these limits be sometimes greater and sometimes less than ϕx, but the difference between them be always less than $\epsilon\phi x$, then $\int_b^a dx\,\psi x$ will differ from $\int_b^a dx\,\phi x$ by *considerably* less than $\epsilon\int_b^a dx\,\phi x$. Paradoxical, however, as it may at first sight appear, there are extreme cases where this difference tends to a ratio of equality with $\epsilon\int_b^a dx\,\phi x$. The complete elliptic function of the first order may be made to furnish an example of this. Let

$$\phi x = \frac{1}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}} = \frac{\sqrt{\{(1-x^2)+b^2x^2\}}}{(1-c^2x^2)\sqrt{(1-x^2)}}$$

(so that $b^2 = 1 - c^2$), and let

$$\psi x = \frac{f\sqrt{(1-x^2)}+gbx}{(1-c^2x^2)\sqrt{(1-x^2)}};$$

if we make

$$f = \frac{2}{1+\sqrt{2}},\quad g = \frac{2}{1+\sqrt{2}},\quad \epsilon = \frac{\sqrt{2}-1}{\sqrt{2}+1},$$

it follows from Poncelet's theorem, that for all values of x intermediate between 0 and 1, ψx will differ from ϕx by less than $\epsilon\phi x$.

Now it will easily be found by ordinary integration that

$$\int_0^1 dx\,\psi x = \frac{f}{2c}\log\frac{1+c}{1-c} + \frac{g}{c}\tan^{-1}\frac{b}{c}.$$

Hence $\int_0^1 dx\,\phi x$ must be always less than

$$\frac{f}{2(1-\epsilon)c}\log\frac{1+c}{1-c} + \frac{g}{(1-\epsilon)c}\tan^{-1}\frac{b}{c},$$

that is,

$$< \frac{1}{2c}\log\frac{1+c}{1-c} + \frac{1}{c}\tan^{-1}\frac{b}{c},$$

when c becomes indefinitely near to unity; that is, when b becomes indefinitely small, this approaches indefinitely near to $\log\frac{2}{b}+\frac{\pi}{2}$. But we know, by a theorem of Legendre, that the approximate value for the integral in such case is $\log\frac{4}{b}$; so that the superior limiting value of $\int_0^1 dx\,\phi x$, found by the application of Poncelet's method, approaches in this instance indefinitely near to the value itself. The explanation of this is easy. As c approximates to unity, the only important values of x in the integral

$$\int_0^1 \frac{dx}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}},$$

are those which lie in the *immediate* vicinity of 1; and for all such values the relative error is at a *negative maximum*.

The method, however, of Mr Merrifield in working out this conception is, I believe, entirely different from that here indicated: how the many mathematicians of a practical stamp, English and foreign, who have worked with

It is not a little remarkable that so rude an application of Poncelet's method should serve to indicate almost with the force of rigorous demonstration the approximate formula

$$F(c) = \log\frac{1}{b} + \text{constant},$$

when c approaches indefinitely near to unity, the constant left undetermined being known to be less than $\log 2 + \frac{\pi}{2}$.

Nay, the demonstration may be made absolutely rigid if we set about to find an inferior limit. To this end make

$$\psi x = \frac{1}{\{f\sqrt{(1-x^2)} + gbx\}\sqrt{(1-x^2)}},$$

we shall find without difficulty

$$\int_0^1 \psi dx = \frac{1}{f\gamma}\log\frac{1+\gamma-b}{\gamma-1+b}\,\frac{\gamma+b}{\gamma-b}, \text{ where } \gamma = \sqrt{(1+b^2)},$$

and consequently we shall obtain as an *inferior* limit to $F(c)$ the expression

$$\frac{1}{\gamma}\log\frac{1+\gamma-b}{\gamma-1+b}\,\frac{\gamma+b}{\gamma-b},$$

which approaches indefinitely near to $\log\frac{2}{b}$ as c approaches indefinitely near to unity. It is thus seen that Legendre's $F(c)$, when c is indefinitely near to 1, lies between $\log\frac{2}{b}$ and $\log\frac{2}{b} + \frac{\pi}{2}$; the arithmetical mean between these limits is $\log\frac{2}{b} + \frac{\pi}{4}$, that is, $\log\frac{1}{b} + 1{\cdot}4785$, differing by only $\cdot 0923$ from the true value $\log\frac{1}{b} + \log 4$. Of course, when the form of $F(c)$ in the case supposed is known, namely, $\log\frac{1}{b} + C$, there is no difficulty in determining C (as may be seen in Verhulst's *Traité des Fonctions Elliptiques*); but the process above given of throwing the general value of $F(c)$ between limits, is, I believe, by far the easiest and most natural method of obtaining this form. The limits themselves, it should be noticed, have virtually been found by the method, simple to *naïveté*, of writing $\sqrt{(1-c^2x^2)} = \sqrt{(p^2+q^2)}$, where $p = \sqrt{(1-x^2)}$ and $q = bx$, and then substituting for $\frac{1}{\sqrt{(p^2+q^2)}}$, $\frac{1}{p+q}$ as an inferior, and $\frac{p+q}{p^2+q^2}$ as a superior limit in the quantity to be integrated. *Closer* and *calculable* limits *ad libitum* to the integral may be arrived at by substituting for $\frac{1}{\sqrt{(p^2+q^2)}}$ one or the other of the two following rational functions of p, q, according as we wish to obtain an inferior or superior limit to the integral, namely,

$$\frac{\{p+q+\sqrt{(p^2+q^2)}\}^i - \{p+q-\sqrt{(p^2+q^2)}\}^i}{\{p+q+\sqrt{(p^2+q^2)}\}^i + \{p+q-\sqrt{(p^2+q^2)}\}^i}\,\frac{1}{\sqrt{(p^2+q^2)}},$$

or

$$\frac{\{p+q+\sqrt{(p^2+q^2)}\}^i + \{p+q-\sqrt{(p^2+q^2)}\}^i}{\{p+q+\sqrt{(p^2+q^2)}\}^i - \{p+q-\sqrt{(p^2+q^2)}\}^i}\,\frac{1}{\sqrt{(p^2+q^2)}},$$

in which formulæ the greater i is taken the closer will be the approximation. I am not aware that any of these limits to $F(c)$ (even the simplest of which, namely, those given above, may have some value for computational purposes, and have fallen thus very incidentally in my way) have ever before been noticed.

It is not unworthy of notice that the second superior limit to $\frac{1}{\sqrt{(p^2+q^2)}}$, namely, $\frac{p^2+pq+q^2}{(p+q)(p^2+q^2)}$, is an arithmetic mean between the first superior and first inferior limits, and

Poncelet's method during the last quarter of a century, should have managed to overlook so obvious and important an extension of the principle and its applications, I find hard to realize; and my wonder is even greater that I should not have been anticipated twenty years ago, than that I should have been anticipated so recently. But the algebraical theory to which this extension points the way is replete with interest of a far higher order than its applications to practice; for plainly the derived approximate fractions, however sufficient for the purposes of computation, are not, nor ever can be the best and closest of their respective kinds*. To fix the ideas, let us

consequently our second superior limit to the integral when b is indefinitely small becomes $\log\frac{2}{b}+\frac{\pi}{4}$, which brings the constant much nearer to its true value than did the use of the first limit; and as this approximation will evidently not stop at the second step of the process, we may safely infer that the integral derived from either formula when $i=\infty$ (for all values of b, whether finite or indefinitely small), not merely bears to $F(c)$ a ratio differing infinitely little from that of equality, but is absolutely equal to, and may for all analytical purposes be employed to represent $F(c)$.

I have been at the trouble of calculating the inferior limit afforded by the second approximation, and find that for b indefinitely small it is $\log\frac{2}{b}+\frac{\pi}{3\sqrt{3}}$, that is, $\log\frac{1}{b}+1\cdot2977$; the superior limit has been shown to be $\log\frac{1}{b}+1\cdot4785$, the mean is therefore $\log\frac{1}{b}+1\cdot3881$, differing by only $\cdot0018$ from the true value! As the constant continues for all values of i to be a multiple of π, the ith approximations *à suprà* and *ab infrà*, which are always effectible, will give (on making $i=\infty$) two new expansions for π, one infinitesimally in excess, the other infinitesimally in defect of its true value expressed as a multiple of log 2, which it might well repay the trouble of some young analyst to develope.

* That the fractional forms derived from the linear substitutive form are not the best of their respective kinds, appears immediately, so far as the derivatives of the odd order (subsequent to the first) are concerned, from the consideration that the limits of error in excess and in defect will be actually attained for values of x lying within the prescribed limits; but these errors, ϵ_i and η_i (when $\epsilon=\eta$, which is true by hypothesis), are never equal, the former (the extreme error in defect) being always the greater of the two; but if any such derivative were the best of its kind, the absolute values of the extreme errors of excess and defect ought to be *equal* to each other. But more generally, if possible, let the ith derivative to $L(x)$ (where $L(x)$ represents the radical linear approximant $fx+b$ to $\sqrt{(a+bx+cx^2)}$, say $Q(x)$), namely,

$$\frac{(Lx+Qx)^i+(Lx-Qx)^i}{(Lx+Qx)^i-(Lx-Qx)^i}Q(x),$$

be supposed the best of its kind: then the relative error is $\frac{2(Lx-Qx)^i}{(Lx+Qx)^i-(Lx-Qx)^i}$, and the maximum value of this must be equal (to the sign *près*) to the value which it has when we give to x either of its extreme connecting values. Now obviously the above is a maximum only when $\frac{Lx+Qx}{Lx-Qx}$ is a minimum, and therefore when $\frac{Lx}{Qx}$ is a maximum; but by hypothesis, the value of x, say m, which makes this a maximum, gives to $\frac{Lx}{Qx}-1$ the same value with the opposite sign to that which it would have in writing for x either of its limiting values, say k or k'.

Thus we have two equations for determining $\frac{Lk}{Qk}$, $\frac{Lm}{Qm}$, namely,

$$\frac{Lm}{Qm}-1=1-\frac{Lk}{Qk},$$

confine ourselves to the *second* Ponceletic approximation to $\sqrt{(a+bx+cx^2)}$, namely, that which has the form $\frac{\lambda+\mu x+\nu x^2}{1+qx}$, where λ, μ, ν are to be determined. The problem to be solved is the following.

Let
$$\lambda+\mu x+\nu x^2 = V,$$
$$(1+qx)\sqrt{(a+bx+cx^2)} = U;$$

it is required to assign the four constants λ, μ, ν, q, so that the maximum value of $\left(\frac{V}{U}-1\right)^2$ for values of x intermediate between a and b shall be the least possible. Some little way, but only a little way, into the solution of this problem we can look in advance. In the first place, if we seek for the maximum values of $\left(\frac{V}{U}-1\right)^2$, we obtain the rational equation

$$\sqrt{(a+bx+cx^2)}\left(U\frac{dV}{dx}-V\frac{dU}{dx}\right)=0,$$

which will easily be seen to be a *cubic* (not a *biquadratic*) equation in x. Call $\left(\frac{V}{U}-1\right)=\phi(x)$; then the three roots of this equation being named x_1, x_2, x_3, the law of equality explained in my preceding paper would seem to show* that we must be able to satisfy the following equations,

$$(\phi x_1)^2=(\phi x_2)^2=(\phi x_3)^2=(\phi a)^2=(\phi b)^2,$$

which amount to four independent equations, the precise number of constants λ, μ, ν, q to be determined. So in like manner the ith rational *approximant* will contain $2i$ disposable constants; the differentiation of the quantity analogous to $\frac{V}{U}$ will give rise to an equation of the $(2i-1)$th degree; and

and $$\left(\frac{Lm}{Qm}+1\right)^i-\left(\frac{Lm}{Qm}-1\right)^i=(-)^{i-1}\left\{\left(\frac{Lk}{Qk}+1\right)^i-\left(\frac{Lk}{Qk}-1\right)^i\right\}.$$

Thus, suppose $i=2$, we should obtain from the second equation $\frac{Lm}{Qm}=-\frac{Lk}{Qk}$, which is inconsistent with the first; so if $i=3$, we should obtain $\left(\frac{Lm}{Qm}\right)^2=\left(\frac{Lk}{Qk}\right)^2$, and therefore, on account of the first equation, $\frac{L(m)}{Q(m)}=1$; and so in like manner for any value of i, we should derive one or more *numerical* values for $\frac{Lm}{Qm}$, which is absurd, since this quantity is a function of k, k', the two connecting values of x.

* Is it not, however, somewhat uncertain whether the equalities

$$(\phi x_1)^2=(\phi x_2)^2=(\phi x_3)^2$$

must all, in all cases (that is to say, for all given values of the limits) subsist? since the law of equality will not apply to such values of x as lie without the prescribed limits, and *non constat à priori* that the roots of the cubic do all lie within these limits. The subject at the very threshold is beset with doubts and difficulties of a peculiar kind, which we can hardly hope to overcome without calling in geometrical imagination to our aid.

there will be $2i-1+2$, that is, $2i+1$ functions of these $2i$ quantities to be equated, which furnish precisely the required number of equations to make the problem definite. It is, however, apparent that in solving these equations we shall find a *multiplicity* of systems, by which I mean a *definite* number of systems of values of the disposable constants which will equally well satisfy the equations. For instance, in the theory of the second approximation, the equalities

$$(\phi x_1)^2 = (\phi x_2)^2 = (\phi x_3)^2$$

will be satisfied by supposing $x_1 = x_2 = x_3$*. But it is by no means evident *à priori* that this system of equalities will correspond to the absolute minimum of which we are in quest: nay, though even we had $\phi x_1 = \phi x_2 = \phi x_3$, those equations do not necessarily imply $x_1 = x_2 = x_3$. Of the multiplicity of solutions referred to, one only gives the true minimum; but to assign *à priori* the distinguishing marks of this truest and best, *hic labor, hoc opus est.* It will be delightful to find, if it turn out to be true, that for the best form, $\frac{P}{Q}$ representing $\sqrt{X}$ (P being a rational function of the ith degree, and Q of the $(i-1)$th in x), the rational quantity

$$XQ\frac{dP}{dx} - \sqrt{X}\,P\frac{d}{dx}(Q\sqrt{X})$$

must be a perfect $(2i-1)$th power of a linear function of x; but in the present state of my ignorance I dare not do more than affirm that there is a bare probability in favour of this being true: whoever shall first succeed in discovering the true form of the expression will have established a remarkable theorem. Here for the moment I break off, contented with having pointed to a theory as yet, if the expression may be allowed, sleeping in its cradle, but destined, I am persuaded, at no distant day to set in motion as large a mass of algebraical thought as has been set in motion by the never-to-be-forgotten Hessian discussion of the flexures of the cubic curve,—the turning-point between the old algebra and the new.

Henceforward Poncelet's theorem figures no longer as a detached method, a mere stroke of art in aid of the computer, but becomes integrally attached to the grand and progressive body of doctrine of the modern algebra.

* If this is so, we shall have for determining the four constants the following equations:

$$x_1 = x_2 = x_3, \quad \phi a = \phi b = -\phi x_1.$$

But more probable than this seems the conjecture, that, supposing x_1, x_2, x_3 to be arranged in the order of their relative magnitudes, the determining equations might be

$$x_1 = x_3, \quad \phi a = \phi b = \phi x_2 = -\phi x_1.$$

Or is it possible that the *character* of the solution may be discontinuous, and may depend upon the magnitudes, relative or absolute, of the given limits a and b? Probably Dr Tchebitcheff would be able better than any other living analyst to answer these queries. But what an endless vista of future research does the prosecution of the Ponceletic method open out to us!