33.

NOTES TO THE MEDITATION ON PONCELET'S THEOREM, INCLUDING A VALUATION OF THE TWO NEW DEFINITE INTEGRALS

$$\int_0^{\frac{\pi}{2}} \frac{\log \cos \phi \, d\phi}{\sqrt{\{1 - b^2 (\cos \phi)^2\}}}, \quad \int_0^{\frac{\pi}{2}} \frac{\log [1 + \sqrt{\{1 - b^2 (\cos \phi)^2\}}] \, d\phi}{\sqrt{\{1 - b^2 (\cos \phi)^2\}}}.$$

[*Philosophical Magazine*, XX. (1860), pp. 525—533.]

Note A.

The method given in the October Number of the *Magazine* for approximately representing a quadratic surd by a rational fraction is equally applicable to a surd of any degree. To fix the ideas, suppose we wish to approximate in this manner to $\sqrt[3]{R}$.

If we assume P as the first approximation, and make

$$L = P + \sqrt[3]{R}, \quad M = P + \rho \sqrt[3]{R}, \quad N = P + \rho^2 \sqrt[3]{R},$$

where $\rho^3 = 1$, and write

$$F_1 = L^i + \ \ M^i + \ \ N^i,$$

$$F_2 = L^i + \rho^2 M^i + \rho N^i,$$

$$F_3 = L^i + \rho \, M^i + \rho^2 N^i,$$

$$U_1 = \frac{F_1}{F_2} R^{\frac{1}{3}}, \quad U_2 = \frac{F_2}{F_3} R^{\frac{1}{3}}, \quad U_3 = \frac{F_3}{F_1} R^{\frac{1}{3}},$$

$$V_1 = \frac{F_2}{F_1} R^{\frac{2}{3}}, \quad V_2 = \frac{F_3}{F_2} R^{\frac{2}{3}}, \quad V_3 = \frac{F_1}{F_3} R^{\frac{2}{3}},$$

we may easily establish the following propositions, which indeed are almost self-evident :—

(1) Each U and V is a rational fraction.

(2) When $i = \infty$, each $U = R^{\frac{1}{3}}$, each $V = R^{\frac{2}{3}}$.

(3) For all finite values of i, $R^{\frac{1}{3}}$ is intermediate between the least and greatest U, and $R^{\frac{2}{3}}$ between the least and greatest V.

So in general if k is any prime number, we may form $(k-1)$ cycles, each cycle containing k fractions possessing precisely analogous properties as regards representing approximately and limiting the successive powers of $R^{\frac{1}{k}}$. By means of these formulæ [the theory of which might be extended to algebraic quantities of every order (in Abel's sense of the word)], we obtain a complete command over the integration of surd quantities in general as they may appear in any physical problem, being thereby enabled to represent the integrals, not merely arithmetically, but analytically (which is of much higher importance) by logarithmic and circular functions to any degree of accuracy that may be required, and with known assignable numerical limits of error.

NOTE B.

This note relates to the concluding paragraph of the long note at page 313 in the October Number of the *Magazine* [203, above]. I find that the ith inferior limit to $F(c) - \log \frac{2}{b}$, when c differs indefinitely little from unity given by the method therein explained, is

$$\log \frac{2}{b} + \frac{2}{i} \sum_{k=E\frac{i}{2}}^{k=1} \frac{\cos \frac{2k-1}{2k}\pi}{\sqrt{\left\{1+\left(\sin \frac{2k-1}{2i}\pi\right)^2\right\}}} \cos^{-1}\left(\sin \frac{2k-1}{2i}\pi\right)^2,$$

and that the superior limit is

$$\log \frac{2}{b} + \frac{\pi}{2i} + \frac{2}{i} \sum_{k=E\frac{i}{2}}^{k=1} \frac{\cos \frac{k\pi}{i}}{\sqrt{\left\{1+\left(\sin \frac{k\pi}{i}\right)^2\right\}}} \cos^{-1}\left(\sin \frac{k\pi}{i}\right)^2.$$

When $i = \infty$ these limits of course come together, and the finite sums resolve themselves into the definite integral

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\tau \frac{\cos \tau}{\sqrt{\{1+(\sin \tau)^2\}}} \cos^{-1}(\sin \tau)^2,$$

of which, therefore, the value must be log 2. Hence, writing $(\sin \tau)^2 = \cos 2\theta$, we obtain

$$\int_0^{\frac{\pi}{4}} dt \frac{\theta \sin \theta}{\sqrt{(\cos 2\theta)}} = \frac{\log 2}{\sqrt{2}} \frac{\pi}{4}.$$

NOTE C.

It may be shown that any of the expressions for $N^{\frac{1}{k}}$ derived from making $i = \infty$ in the general formulæ given in Note A, are in fact tantamount to its representation as a definite integral of a very simple kind. I shall not go into the proof of this here; it may be sufficient to indicate that it depends upon the fact that the equation of infinite degree $(\phi x)^i + (\psi x)^i + (\vartheta x)^i + \&c.$, may be resolved into sets of factors of a known form. In the question before us, the function to be so resolved is the denominator of any one of the quantities analogous to U or V in Note A; and ϕx, ψx, $\vartheta x \ldots$ become linear functions of x with imaginary coefficients. Its resolution into factors is rendered possible by the circumstance that only two of the quantities $\phi, \psi, \vartheta \ldots$ can bear a finite ratio to each other for any given value of x, and consequently all the roots of the equation

$$(\phi x)^i + (\psi x)^i + (\vartheta x)^i \ldots = 0\,*$$

are contained *among* the roots of several binary equations

$$(\phi x)^i = (\psi x)^i, \quad (\phi x)^i = (\vartheta x)^i, \ \&c.:$$

which are the roots of any one of these equations (as for example of the first) that belong to the *given* equation will be determined by the condition that they must make the norms of all the other functions (for example of ϑx) indefinitely small as compared with the norms of those two which appear in it (for example ϕx, ψx). In this manner, if the total number of the functions is k, supposing $\phi, \psi, \vartheta \ldots$ to be all linear functions of x, each binomial equation out of its entire stock of i roots will contribute $\dfrac{i}{k\dfrac{k-1}{2}}$ roots available towards the solution of the given equation. Mr Cayley has remarked to me the analogy between this determination and Newton's method of finding the form of the several parabolic equations $y = cx^\lambda$ which represent the branches of a given algebraical curve at its origin. In the equation to the given curve cx^λ is to be substituted for y; the terms will then all become powers of x (an infinitesimal) whose indices will be linear functions of λ; every pair of them in turn is equated to zero, and of all the values of λ thus obtained only those will be preserved which cause the two equated linear functions of λ belonging to any given pair of terms to be less than all the others, and consequently the terms themselves (whose indices the linear functions are) infinitely greater than all the other terms.

Linear functions of a variable figure in both investigations, namely, in Newton's as indices of the same infinitesimal quantity, in mine as quantities whose infinite index is the same†; but the logic and mode of procedure (utterly unlike as are the questions in their origin and subject matter) is the same in either case.

* My friend, M. Jordan, of the École des Mines (author of a remarkable thesis on *groups*), has developed some interesting geometrical consequences arising out of the study of this equation, which I hope he may be induced to publish.

† In a word, Newton's equation is an exponential one made up of nothings, mine an algebraical one made up of infinities.

NOTE D.

The remark contained in the preceding note, as to the effect of representing $N^{\frac{1}{k}}$ by an infinite rational fraction being identical with that of expressing it as a definite integral, combined with a consideration of the cause of the success of the particular method referred to in Note B, has led me to the investigation following, of the value of the complete elliptic function of the first species. As usual denoting it by $F(c)$, we have

$$F(c)=\int_0^{\frac{\pi}{2}} d\theta \frac{1}{\sqrt{\{1-c^2(\sin\theta)^2\}}}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} dx \frac{\cos\theta}{1-c^2(\sin\theta)^2+(\cos\theta)^2x^2}$$

$$=\frac{2}{\pi}\int_0^{\infty} dx I,$$

where

$$I=\int_0^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{(1+x^2)-(c^2+x^2)(\sin\theta)^2}$$

$$=\frac{1}{\sqrt{\{(1+x^2)(c^2+x^2)\}}}\left[\log\{\sqrt{(1+x^2)}+\sqrt{(c^2+x^2)}\}-\log\sqrt{(1-c^2)}\right].$$

Let $x=\tan\phi$, $b=\sqrt{(1-c^2)}$; then

$$F(c)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{1-b^2(\cos\phi)^2\}}}\{\log[\sec\phi+\sqrt{\{(\sec\phi)^2-b^2\}}]-\log b\}$$

$$=\frac{2}{\pi}\log\frac{1}{b}F(b)+\frac{2}{\pi}R,$$

where

$$R=\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{1-b^2(\cos\phi)^2\}}}\log[\sec\phi+\sqrt{\{(\sec\phi)^2-b^2\}}]$$

$$=\int_{\frac{\pi}{2}}^0 d\phi\left\{\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}\right\}+\int_0^{\frac{\pi}{2}} d\phi \frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}}.$$

It will presently appear that these two definite integrals are equal to one another!

Let $$V_{2r}=\int_{\frac{\pi}{2}}^0 (\cos\phi)^{2r}\log(\cos\phi)\,d\phi.$$

Then we may easily establish the formula of reduction,

$$V_{2r}=\frac{2r-1}{2r}V_{2r-2}-\frac{1.3.5..(2r-3)}{2.4.6..(2r-2)2r}\frac{\pi}{2};$$

and since (as is well known) $V_0 = \frac{\pi}{2}\log 2$, we have

$$V_2 = \frac{1}{2}\frac{\pi}{2}\left(\log 2 - \frac{1}{1\,.\,2}\right),$$

$$V_4 = \frac{1\,.\,3}{2\,.\,4}\frac{\pi}{2}\left(\log 2 - \frac{1}{1\,.\,2} - \frac{1}{3\,.\,4}\right),$$

$$V_6 = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\pi}{2}\left(\log 2 - \frac{1}{1\,.\,2} - \frac{1}{3\,.\,4} - \frac{1}{5\,.\,6}\right),$$

$$\&c. = \&c.$$

Hence, by expanding the denominator in a series proceeding according to powers of $(\cos\phi)^2$, it is readily seen that the first integral becomes

$$\frac{\pi}{2}\left\{\log 2 + \left(\frac{1}{2}\right)^2\left(\log 2 - \frac{1}{1\,.\,2}\right)b^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\left(\log 2 - \frac{1}{1\,.\,2} - \frac{1}{3\,.\,4}\right)b^4 + \&c.\right\}.$$

To find the second integral, we must obtain the general term in the expansion in a series of powers of t of

$$\frac{\log\{1 + \sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}}$$

(where t stands for $b\cos\phi$), that is, of

$$\frac{1}{\sqrt{(1-t^2)}}\int dt\left(\frac{1}{t} - \frac{1}{t\sqrt{(1-t^2)}}\right),$$

say of $\phi t = \frac{1}{\sqrt{(1-t^2)}}\psi t$. Now

$$\left(\frac{d}{dt}\right)^2\{(1-t^2)\,\phi t + \int dt\,(t\phi t)\} = \left(\frac{d}{dt}\right)^2\left(\sqrt{(1-t^2)}\,\psi t + \int dt\,\frac{t\psi t}{\sqrt{(1-t^2)}}\right)$$

$$= \psi t\left\{\left(\frac{d}{dt}\right)^2\sqrt{(1-t^2)} + \frac{d}{dt}\frac{t}{\sqrt{(1-t^2)}}\right\} + \psi' t\left(\frac{-2t}{\sqrt{(1-t^2)}} + \frac{t}{\sqrt{(1-t^2)}}\right) + \sqrt{(1-t^2)}\,\psi'' t$$

$$= -\frac{t}{\sqrt{(1-t^2)}}\psi' t + \sqrt{(1-t^2)}\,\psi'' t$$

$$= -\frac{t}{\sqrt{(1-t^2)}}\left\{\frac{1}{t} - \frac{1}{t\sqrt{(1-t^2)}}\right\} - \sqrt{(1-t^2)}\left\{\frac{1}{t^2} + \frac{2t^2-1}{t^2(1-t^2)^{\frac{3}{2}}}\right\}$$

$$= -\frac{1}{\sqrt{(1-t^2)}} - \frac{\sqrt{(1-t^2)}}{t^2} + \frac{1}{1-t^2} - \frac{2t^2-1}{t^2(1-t^2)}$$

$$= \frac{-1}{t^2\sqrt{(1-t^2)}} + \frac{2}{t^2}$$

$$= \frac{1}{t^2} - \frac{1}{2} - \frac{1\,.\,3}{2\,.\,4}t^2 - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}t^4,\ \&c.$$

Hence, writing

$$\frac{\log\{1 + \sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}} = \log 2 + K_2 t^2 + \ldots + K_{2i-2}t^{2i-2} + K_{2i}t^{2i} + \&c.,$$

and equating the coefficients of t^{2i}, we obtain

$$2i(2i-1)(K_{2i}-K_{2i-2})+(2i-1)K_{2i-2}=-\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i},$$

that is, $$K_{2i}=\frac{2i-1}{2i}K_{2i-2}-\frac{1.3.5\ldots 2i-3}{2.4.6\ldots(2i-2)(2i)^2}.$$

Thus $$K_0=\log 2,\quad K_2=\frac{1}{2}\left(\log 2-\frac{1}{1.2}\right),\quad K_4=\frac{1.3}{2.4}\left(\log 2-\frac{1}{1.2}-\frac{1}{3.4}\right),$$

$$\&c.=\&c.,$$

and consequently $$\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$=\frac{\pi}{2}\left\{\log 2+\left(\frac{1}{2}\right)^2\left(\log 2-\frac{1}{1.2}\right)b^2+\left(\frac{1.3}{2.4}\right)^2\left(\log 2-\frac{1}{1.2}-\frac{1}{3.4}\right)b^4+\&c.\right\}.$$

Thus, then, we obtain the following remarkable equalities*:

$$\frac{\pi}{2}F(c)=\log\frac{1}{b}F(b)+2\int_{\frac{\pi}{2}}^{0} d\phi\,\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$=\log\frac{1}{b}F(b)+2\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}},$$

or $$\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}}=\int_{\frac{\pi}{2}}^{0} d\phi\,\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$\left[=\int_0^{\frac{\pi}{2}} d\phi F(b,\phi)\cot\phi\right]=\frac{\pi}{4}F(c)+\frac{1}{2}\log bF(b).$$

When b is indefinitely small, it is obvious from either of these equations that

$$F(c)=-\frac{2}{\pi}\log b\,\frac{\pi}{2}+2\log 2=\log\frac{4}{b},$$

Legendre's well-known formula previously referred to.

The equality of the first two definite integrals in the *sorites* above given, is, as we have seen, a consequence of the equality

$$\frac{\log\{1+\sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}}=\frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} d\theta\,\{\log\cos\theta+\log\cos\theta\,(\cos\theta)^2t^2$$
$$+\log\cos\theta\,(\cos\theta)^4t^4+\&c.\}.$$

[* The reader may be glad to have the references: *Schlömilch Zeitschrift*, II. (1857), pp. 49, 414; *Tortolini Annali*, III. (1860), p. 254. See also below, p. 298.]

Hence we have

$$\int_{\frac{\pi}{2}}^{0} \frac{\log \cos\theta}{1-b^2(\cos\theta)^2}\,d\theta = \frac{\pi}{2}\frac{\log\{1+\sqrt{(1-b^2)}\}}{\sqrt{(1-b^2)}} *.$$

The extreme facility and brevity with which the method in the text gives the value of $F(c)$ for b indefinitely small is worthy of notice, as in the usual text-books it is obtained by a very indirect and circuitous process. We may obtain in like manner the value of

$$\int_0^{\frac{\pi}{2}} d\theta\, \frac{1}{(1-e\sin\theta)} \cdot \frac{1}{\sqrt{\{1-c^2(\sin\theta)^2\}}}$$

on the same supposition as to c, whether $1-e$ vanishes with $1-c$ or remains finite when $c=1$. On the latter supposition, the definite integral in question has for its value

$$\frac{1}{1-e}\log\frac{2}{b} + \frac{1}{1-e^2}\log\frac{2}{(1+e)^e}.$$

When $e=1$, this becomes infinite; when $e=-1$, the second term becomes $\frac{1}{4}+\frac{1}{2}\log 2$, and the entire integral is $\frac{1}{2}\log\frac{4}{b}+\frac{1}{4}$; when $e=0$, it is $\log\frac{4}{b}$. Subtracting the half of the latter integral from the former, we shall obtain

$$\int_0^{\frac{\pi}{2}} d\theta\, \frac{(1-\sin\theta)^2}{(\cos\theta)^3} = \frac{1}{2},$$

which is easily verified.

By taking successively $e=\sqrt{(-n)}$, $e=-\sqrt{(-n)}$, and adding together the halves of the two integrals corresponding to these suppositions, we obtain the *ultimate* value of the complete elliptic integral of the third kind, namely,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+n(\sin\theta)^2} \cdot \frac{1}{\sqrt{\{1-c^2(\sin\theta)^2\}}},$$

from the general formula above given, always of course subject to the condition that c is supposed indefinitely near to 1†.

* From this it will readily be seen that when n is any integer we may obtain

$$\int_0^{\frac{\pi}{2}} \frac{\log\cos\theta}{\{1-b^2(\cos\theta)^2\}^n}$$

by processes of differentiation in a form involving only algebraical and logarithmic quantities, and so, from what precedes, when n is any half-integer, in terms of such quantities and of complete elliptic functions.

† It seems to be expected of every pilgrim up the slopes of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock. The author of these notes has been somewhat late in acquitting himself of this debt of honour, but ventures to hope that the principal results contained in the text above may be thought not unworthy of a place in some future edition of that noble and sumptuous monument of Dutch learning, industry, and fine taste, the invaluable collection of definite integrals by M. Bierens de Haan.