34.

ON THE PRESSURE OF EARTH ON REVETMENT WALLS.

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PART I.

Critique of the Hypothesis of Parallel "Planes of Rupture."

THE ensuing investigation deals with the pressure of *Mathematical* earth. By mathematical earth, I mean earth treated according to the idea of Coulomb, namely, as a *continuous** mass separable by planes in all directions, but whose separating surfaces exert upon one another forces consisting of two parts, one of the nature of ordinary friction, the other of so-called cohesion. Of the latter, for greater simplicity, I shall commence with taking no account, so that the matter with which we have to deal becomes, so to say, "a frictional fluid." If we isolate in idea any element of this fluid—suppose, to fix the ideas, a molecule bounded by plane faces, this molecule will be kept at rest by its own weight, the pressures on the several faces, and the forces of friction acting along these faces: these last-named forces are limited not to exceed the product of the corresponding pressures by a certain coefficient, termed the coefficient of friction.

In order to render the inquiry before us quite definite, let us begin with supposing two vertical side walls and a back of solid immoveable masonry, between which the earth is piled up in a determinate form, fronted by a pier of given specific gravity, whose *minimum* thickness is to be determined by the condition that it may just suffice to prevent the pier from being either forced forward or turned round over its further edge. The earth is thus of course supposed to have only one free face, being entirely supported at the sides and the back by the masonry just spoken of. The problem then that we have to solve is evidently the following:—"Of all the possible states of equilibrium of the earth consistent with the assigned conditions, to determine that one which shall make the greater of two quantities to be

* The only *essential* quality of our mathematical earth which differentiates it from actual vulgar earth is this of *continuity*.

named the least possible,"—one of these quantities being the thickness of the wall determined by the condition that its friction with the ground shall be just equal to the sum of the horizontal pressures on the wall, the other by the condition that its moment about the edge most remote from the earth shall be just equal to the sum of the moments of the entire thrust upon the wall at each several element thereof in respect to the same edge.

Whenever Coulomb's method leads to a right solution of the problem of revetments, the thrusts on the several elements of the wall will be all parallel; and it may easily be seen that, in solving the problem for this case, we are solving the problem of making the statical sum of the thrusts a minimum; and the result will be the same, whether the pier can only be pushed bodily on its base, or can only turn over an edge, or can do both one and the other. But it must obviously be erroneous to assume as a universal principle, that in the state bordering upon motion, or what is going still further, in a state antecedent to this, the statical sum of the pressures will be a minimum; and if Mr Moseley's "principle of least resistance," quoted by Professor Rankine, means this, I have no scruple in proclaiming my entire dissent from such an assumption. I do not here enter at all into the question of determining pressure, except in the state of equilibrium bordering upon motion; and in that state common sense points out that it is not the pressure or sum of pressures, but the effect of such pressure or pressures in inducing motion in a certain possible manner, or in any one out of a choice of possible manners, that governs the determination of the minimum. This principle of least resistance is one of the shoals upon which Mr Rankine's investigation appears to me to have split.

Be it observed that the only physical assumption which I propose is this, that if *equilibrium can be preserved consistently with the imposed conditions, equilibrium will be preserved.* Without such a supposition the question would be incapable of treatment without further laws regulating the interior forces than we suppose given. The legitimacy of such an assumption cannot, I think, be seriously called into question, and once made, the problem of determining the wall's thickness becomes a purely mathematical question; one undoubtedly of great difficulty, but perfectly determinate, and falling under the dominion of the Calculus of Variations, as will easily be recognized from the circumstance that the integration of the general equations of equilibrium, if it could be performed, would necessarily contain arbitrary functions, whose form would have to be assigned so as to make a certain quantity or the greatest of a set of quantities a minimum; but the peculiar manner in which the internal forces are defined as subject to satisfy not an equation or system of equations, but a law of inequality, must render it a task exceeding the present powers, at all events, of the writer of this paper, to arrive at a result by the direct application of the Calculus referred to. In order to pave the way to the discussion of the more general inquiry, I shall commence with examining whether under any and what circumstances the forced solution of Coulomb and his followers, founded upon the notion of what have been (it seems to me incautiously) termed planes of fracture or rupture (but which really mean no more than planes for which friction at each point thereof is acting with its utmost energy, that is, if we please so to say, planes of greatest frictional energy*), is the true solution; that is to say, I shall investigate under what conditions the surfaces of "rupture" or "of greatest energy of friction" are or can be planes; and I shall easily be able to ascertain these conditions, and to prove that when they are satisfied (but not otherwise) the results of the received theory are exact.

Professor Rankine, in the light in which he appears in a paper published in the *Transactions* of the Royal Society, is not to be ranked among those whom I have called the followers of Coulomb. He is entitled to the merit of

* It is obvious that the notion of the planes in question being the planes in which the earth would begin with crumbling, if the equilibrium were disturbed by the wall giving way (for such is the idea intended to be conveyed by their being called planes of rupture), is quite irrelevant to the determination of their position, and to the solution of the question of the thrust in the wall. But such a notion in itself is objectionable, as assuming a physical fact for which there is no just ground. The idea, or rather I may say the metaphysical process, which unconsciously has swayed Coulomb and his followers to give them this name, appears to me to be the following. "Since it is only along these planes that friction is acting at its full energy, and since, when motion ensues, friction must be acting at its full energy, therefore a change must have taken place in the friction of any other plane before motion can take place along it, which change does not take place along the planes in question. Now every change must operate in time, therefore the motion must have begun along the planes of greatest friction before it can have taken place along any other." But it is a most dangerous proceeding, and fraught with errors familiar to mathematicians, to attempt to reason from the conditions of equilibrium to those of incipient motion; and that dynamical considerations, and not statical, must decide the incipient directions of the motion in the case before us, will be obvious when we reflect that the friction might be supposed to become *nil*, and then we should be treating of a perfect fluid, in which case the planes of rupture disappear, but none the less would motion take place in determinate directions on any wall of the reservoir containing the fluid giving way. A notable example of the important distinction between rest and equilibrium is afforded by the question (which, I am informed, originated in Caius College, Cambridge) of finding the tension of a rope by which a bucket full of water, with a cork tied to its bottom, is fastened to a fixed point, at the moment when the fastening is cut or gives way. At that moment the vertical pressure in the bottom of the bucket, supposing the specific gravity of the cork to be one-fourth that of water, if it could be estimated on statical principles, that is with reference to the elevation of the surface of the fluid [and some non-mathematical physicists might easily suppose it could be so estimated, since motion has not *yet taken place*, but is only *imminent*], would be the weight of the bucket together with that of the water, together with four times that of the cork, and so it would appear as if the tension would be increased by the cutting of the string, whereas, in fact, precisely the contrary effect will take place; for since downward momentum must result from the impending motion of the cork upwards and the water downwards, part of the weight of the water and cork is spent as downward moving force, and consequently only a portion remains to act as vertical pressure upon the bucket, just as an air-cushion will press with less force than its weight on the seat which bears it, when, in consequence of the air being let out, part of the weight is being expended in lowering the top of the cushion.

having perceived that the received hypothesis rested on no solid foundation, and of having been the first (publicly at least) to assert that the equations of internal equilibrium must be resorted to for the satisfactory discussion of the question; but, notwithstanding the sincere esteem in which I hold the great abilities of this gentleman, I have been compelled to come to the conclusion, and trust to be able to satisfy himself, that the use he has made of these equations is illusory, and that his results bear upon their very face a demonstrable character of error.

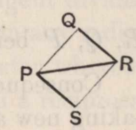
Under the supposed *data*, it is, if not obvious, at all events assumed by all writers on the subject, that the equilibrium of every vertical section of the earth, parallel to the side walls, may be determined *per se*, and that we may treat the question as one regarding space of only two dimensions. I shall therefore, with a view to clearness, treat of the equilibrium of any one such section; the molecules, whose equilibrium is to be considered, will be spoken of as bounded by lines instead of planes, and so we shall speak of lines instead of planes of "rupture," and we may thus conform our language to the relations of the figure actually represented upon the paper.

For the benefit of those to whom the conditions of molecular equilibrium are new, it may be well to indicate briefly how they may be obtained, still keeping within our prescribed framework of two-dimensioned space (although the reader will not experience the slightest difficulty in extending them to space of three dimensions)*. Through any point in the interior of the plane-mass at rest, imagine a small rectilinear element to be drawn. The entire molecular force exerted on this actual element might be termed the thrust; but by the thrust I shall understand *the unit of thrust*, corresponding to the well-known conception of *unit of pressure* for the particular case of a fluid mass. This thrust (or unit of thrust) may be imagined separated into two parts, one perpendicular to the element, which may be

* I have purposely begun with the beginning, because I wish to give perfect precision to the terms Thrust, Pressure, and Stress, as I shall use them. Some recent authors on mechanics have wished to distinguish force measured statically from force measured by acceleration, by giving to the former the name of pressure. But surely unnecessary confusion is introduced into mechanical language when we are thereby reduced to speak of the pressure of friction, and ought to enunciate the cardinal law of friction by stating that the *pressure of friction* bears to the *pressure of pressure* a certain limiting relation. I acknowledge an objection scarcely less valid (except that it has antiquity to plead in excuse) to the use of the term accelerating force; as we may be thereby reduced to speak of the accelerating force of a retarding influence, as friction, or of an influence which does not necessarily either accelerate or retard, as in the case of a centripetal pull upon a body moving uniformly in a circle. I think this difficulty in language may be met to some extent by giving to force, usually called accelerative, the designation of alternative, and to force measured by weight or momentum that of quantitative force. There is no magic in names, however well selected, but there may be a great deal of mischief arising out of a confused and uncertain nomenclature.

termed the *pressure*, the other parallel to it, which may be termed *face-force* [for the case we shall have more especially to consider, the face-force receives the name of *friction*, and is limited to be less than the *pressure* multiplied by the so-called coefficient of friction]. As the element acted upon turns round, the thrust changes in magnitude and direction, and to the totality of the thrusts going forth in all directions from a given point we may give the name of *stress**. We shall now be able to obtain two sorts of conditions—one giving the necessary law connecting the various *thrusts* of the same *stress*, the other expressing the law of the variation of the pressure and facial force (together constituting the thrust) upon an element given in direction in passing from one stress to another; we may call these respectively the equations of distribution and the equations of variation.

Let $PQRS$ be any infinitely small molecule bounded by lines at right angles to one another. Since this is kept at rest by its own weight, by the lines of pressures perpendicular to QR and PS, and the other pair perpendicular to PQ and RS, and by the facial forces acting along PQ, QR, RS, SP respectively, if we call f the face-force [that is, the unit of face-force] on PS, it is obvious that the corresponding quantity for QR will differ from it by an infinitely small quantity; in like manner f' may be taken as the face-force on QR, PS respectively. Hence the couples whose moments $(f.QR)\times QP$ and $(f'.PQ)\times QR$ respectively must be equal and opposite, or in other words, f being understood to act to or from Q, according as f' acts to or from Q, we must have $f=f'$. To fix the ideas, conceive the face-forces to tend towards Q. Let us now consider the equilibrium of the triangular molecule PQR. Call the pressure on PQ (R), the pressure on RQ (P), the face-force on PQ or QR (Q). In comparison with the thrusts on the faces of our triangular molecule, gravity or other impressed forces may be neglected as giving rise to quantities of an inferior order of smallness.



Let QP, QR be regarded as two fixed rectangular axes, and let $QPR=\theta$. Let the pressure and face-force on PR (always understanding thereby the units of such forces) be called N and F respectively (F, to fix the ideas, being taken to act from P to R). Then resolving the forces perpendicular to PR, we obtain

$$N.PR=R.PQ.\cos QPR+P.QR\cos QRP$$
$$+Q.PQ\sin QPR+Q.QR\sin QRP,$$

or

$$N=R(\cos\theta)^2+2Q\cos\theta\sin\theta+P(\sin\theta)^2;$$

* Thus, stress stands in somewhat the same relation to its component thrusts, as a radiant point to the luminous rays which it emits.

and resolving parallel to PR, we have

$$F \times PR = R \,.\, PQ \sin QPR - P \,.\, QR \sin QRP + Q \,.\, PQ \cos QPR - Q \,.\, QR \cos QRP,$$

or

$$F = (R - P) \sin\theta \cos\theta.$$

Imagine now QPR to be represented by a single point O. R, P are respectively the pressures, and N the face-force ("units of pressures and of face-force") on elements drawn in the orthogonal directions OX, OY; N the pressure, and F the face-force on an element drawn in the direction OP, making an angle θ with OX. Obviously, therefore, if we draw in all directions from O lines whose lengths are as the inverse square roots of the pressure-part of the thrust acting on those lines, calling the length of line corresponding to θ, r, we have

$$\frac{1}{r^2} = R (\cos\theta)^2 + 2Q \sin\theta \cos\theta + P (\sin\theta)^2,$$

R, Q, P being constant quantities.

Consequently the locus of the extremities of these lines is a conic; and taking new axes of coordinates in the directions of the principal axes of this conic, and understanding by R and P the pressures perpendicular to those axes respectively, the equations obtained assume the form

$$N = R (\cos\theta)^2 + P (\sin\theta)^2, \qquad (1)$$

$$F = (R - P) \sin\theta \,.\, \cos\theta; \qquad (2)$$

showing that in elements in the directions of the principal axes the face-forces vanish, and the thrusts become purely pressures, that is, forces perpendicular to the surfaces upon which they act. R and P are of course essentially positive, as otherwise the molecules would be subject to a force of separation instead of compression, and consequently the conic in question is an ellipse. The total value of the thrust $= \sqrt{(N^2 + F^2)}$

$$= \sqrt{\{R^2 (\cos\theta)^2 + P^2 (\sin\theta)^2\}}. \qquad (\alpha)$$

R and P will evidently be in the directions in which, for a given point, the entire thrust, as well as the pressure-part of it, is the least and greatest. These directions may be said to be those of "principal thrust." If we start from any point and proceed from that point always in the direction of a line of principal thrust so as to form a continuous curve, two such curves cutting each other at right angles will intersect every point of the mass at rest, of which, in the case of mathematical earth, I may state, by way of anticipation, that only one can cut the free surface when that surface is supposed to form part of a horizontal plane.

These lines may also be termed the principal lines of pressure, or simply the lines of pressure; and this name may be considered indifferently to have reference either to the fact that the thrust *in the direction of the tangent* at any point in any such curve is the thrust acting upon the normal, or to the fact that the thrust *upon the tangent* at any point is in the direction of the normal; as either one of such conditions implies the other.

The cosine of the angle between the pressure and the thrust will be

$$\frac{R(\cos\theta)^2 + P(\sin\theta)^2}{\sqrt{\{R^2(\cos\theta)^2 + P^2(\sin\theta)^2\}}};$$

which, calling the principal semiaxes of the ellipse referred to a and b respectively, and the rectangular coordinates of any point therein x and y, becomes

$$\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{\sqrt{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)}} = \frac{1}{\sqrt{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)}}$$

which is equal to the perpendicular from the centre on the tangent divided by the radius vector, showing that the direction of the thrust on any radius of the ellipse in question is in the direction of the conjugate diameter, whereby it is seen that the line of thrust, and the line thrust upon, stand in a reciprocal relation to each other.

I may add the cursory remark as regards the value of the total thrusts in the case more immediately before us, that [as is apparent from the equation (α)] they will be represented in relative magnitude by the radius vector drawn in the direction of the line thrust upon, to meet, not the ellipse of pressures just described, but another ellipse whose major and minor axes are to one another in the duplicate ratio of the other two.

If we wish, however, to present the above results in a form more immediately translateable into the actual case of nature, I mean that of space with three dimensions, it becomes expedient to use a different ellipse, or rather the same ellipse in another position, to represent the stress at any point.

In the equations above found, connecting N and F with P, Q, R, θ is the angle made with a fixed axis, not by the line of pressure R, but by the element on which this pressure is exerted. Let ϕ be the angle made by the pressure itself, so that $\phi = \theta + \frac{\pi}{2}$, then we have

$$N = P(\cos\phi)^2 - 2Q\sin\phi\cos\phi + R(\sin\phi)^2$$
$$F = (P - R)\sin\phi\cos\phi.$$

And the same process as has been already employed will serve to show that we may construct an ellipse such that the inverse square of the radius vector in every direction may represent the magnitude of the *pressure* in that direction (that is the magnitude of the normal part of the thrust upon the element perpendicular to that direction), and in this ellipse the radius vector and perpendicular to the tangent at each point will represent the corresponding directions of pressure and thrust, which obviously will coincide for the directions of greatest and least pressure.

If, now, we go out into space of three dimensions, it will readily be anticipated, and may easily be proved, that an ellipsoid whose radii vectores represent the relative magnitudes of the inverse square roots of the pressures takes the place of the ellipse, the thrusts and pressures correspond respectively (in direction) to the normal and radius vector at each point, and in three directions, at right angles to each other, these latter come together.

It is desirable that the reader should bear in mind that the ellipse of which I have spoken is in fact only a principal section of this ellipsoid. The assumption which (following in the track of my predecessors) I shall make, that the greatest energy of friction exerted at any point will be exerted in some direction in a vertical plane parallel to the revetment wall, will be seen from what follows a little further on, to imply that every such plane contains the radius vector which makes the greatest angle with the normal, and consequently the section of the ellipsoid of stress with which we are dealing will be the plane of greatest and least thrust, or greatest and least pressure. By way of aid to the imagination in seizing this subtle conception of stress (a real conquest in physical ideology due to the last quarter of the present century, although its first germ may be recognized in the much earlier molecular view of the circumambient pressures round about each internal point of a perfect fluid), I have gone thus briefly into the generation of the ellipse and ellipsoid above described; but I shall have very little occasion, except for occasional facility of reference, to have resort to them, as the equations (1) and (2) will suffice for my purpose in the present inquiry.

These are the equations which govern the distribution of stress; and it may be convenient to confer upon the ellipse whose radii vectores are in length inversely as the square roots of the pressures acting upon them, the name of the ellipse of *pressures*, in order to obviate any possibility of the position of this ellipse being confounded with that of the one which would, I believe, more ordinarily go by the name of the ellipse of *stress*. Every point in the mass is the centre of such an ellipse; and those ellipses, if properly drawn, will represent completely, and on the same scale, the magnitude and distribution of the pressures round about any point. It is almost needless to add that for a perfect fluid these ellipses would become circles.

Let us now proceed to establish the law of the variation of the stresses, or, to speak more accurately, of the thrusts acting on planes drawn in any given directions, on passing from one point of the mass to another. Returning to our little rectangular element $PQRS$, and considering the lines PQ, PS to be given in direction, so that we may consider $PQ = dx$ and $PS = dy$, and calling the units of pressure on PQ and RQ L and N, the unit of face-force M, the impressed forces of acceleration X in the direction of x, and Y in the direction of y, and the unit of mass ρ, by simple estimation of the forces in the directions of x and y respectively we obviously obtain, due attention being paid to the mode of fixing the positive directions of X and Y,

$$\frac{dL}{dy} + \frac{dM}{dx} = \rho Y,$$

$$\frac{dN}{dx} + \frac{dM}{dy} = \rho X.$$

If, as in the case with which we shall have to deal, the sole impressed force is that of gravity, and if we treat the weight of a unit of the mass as unity, and make the axis of x horizontal and that of y vertical, the equations become

$$\frac{dL}{dy} + \frac{dM}{dx} = 1,$$

$$\frac{dN}{dx} + \frac{dM}{dy} = 0.$$

These, being the equations which control the law of the variation of the thrusts estimated in given directions in passing from one stress to another, I call the equations of variation of stress.

I now proceed to the application of the principles above set forth to the treatment of the particular question in hand.

Let μ be the coefficient of friction of the earth upon itself, and $\mu = \tan \lambda$, so that λ is the angle of repose; by this is to be understood that the thrust on any element can never make, with the perpendicular to that element, an angle greater than λ. Now the general law of the distribution of stress proves that the actual angle between the perpendicular to the element and its thrust will in two directions be zero. Hence at any given point it will pass through all gradations, from zero up to a certain limit. Here presents itself the question, Is that limit λ, or can it be λ for every point in the mass? As we have no right to assume *à priori* that this limiting angle in that state of equilibrium which we wish to determine must be equal to λ throughout the mass, and obviously it will not be so for actual cases of equilibrium which arise, we want a name to distinguish the maximum ratio which friction bears to pressure in any specified stress from the absolute maximum which this ratio is capable of attaining. We may name the former the coefficient of frictional energy; and for every point where this is equal to the absolute coefficient of friction, we may say the friction of the stress is at its maximum energy. Let (μ) be the coefficient of frictional energy for any given stress, and $(\lambda) = \tan^{-1}(\mu)$ the corresponding angle of repose. [We may also, if we please, term (μ) and (λ) the relative coefficient and relative angle of repose respectively, that is, relative to any assigned stress.] Let the ratio between the maximum and minimum thrust of any stress be called γ^2: a simple relation connects γ and (λ)*.

For calling, as before, L the pressure, and M the face-force (now the friction), we have by equation (1),

$$L = P(\cos\theta)^2 + R(\sin\theta)^2,$$
$$M = (P - R)\sin\theta\cos\theta,$$
$$R = P\gamma^2,$$
$$\tan(\lambda) = (\mu) = \text{maximum value of } \frac{M}{L}.$$

To find this maximum, we have

$$\delta\{\cot\theta + \gamma^2\tan\theta\} = 0.$$

Hence $$\gamma\tan\theta = 1,$$

and therefore $$\cot(\lambda) = \frac{2\gamma}{1-\gamma^2},$$

therefore $$(1-\gamma^2) - 2\gamma\tan\lambda = 0,$$

or $$\gamma = \sec(\lambda) - \tan(\lambda) = \frac{1-\sin(\lambda)}{\cos(\lambda)} = \tan\left(45^\circ - \frac{(\lambda)}{2}\right).$$

This equation expresses the universal relation between the form of the ellipse of pressures for any stress and the relative angle of repose for such stress.

The problem we have just solved may be presented advantageously, in order to make the impression of it more vivid (as it is of cardinal importance), under a geometrical point of view. Taking any radius vector of the ellipse of pressures, the angle between it and its conjugate radius is 90° at any vertex; at some point therefore it will be at a minimum, and this minimum will be the complement of the relative angle of repose.

From the preceding investigation, it will easily be seen that, to find the ray-directions which give this minimum, we have only to construct a rectangle circumscribing the ellipse, and either of its two diagonals will be in the direction required, and the angle between either such ray and the principal axes *plus* or *minus* half the angle between it and the normal (which angle is the relative angle of repose) will be half a right angle†.

* This relation and its importance are well known to Professor Rankine.

† In fact the diameters which coincide with the directions of these diagonals are conjugate diameters, equally inclined to the principal axes; and these, as I suppose must be well known, are the conjugate diameters whose inclination to each other is a minimum.