48.

CONCLUDING PAPER ON TACTIC.

[*Philosophical Magazine*, XXII. (1861), pp. 45—54.]

In my tactical paper in the May Number of this *Magazine* [p. 264 above], I considered the number of groupings and of types of groupings of *synthemes* formed out of triads of three *nomes* of three elements each. The first example of considering the *ensemble* of the groupings of a defined species of synthemes (each of such groupings being subjected to satisfy a certain exhaustive condition) was, as already stated, furnished by me in this *Magazine*, April, 1844. In that case the synthemes consisted of duads belonging to a single nome of 6 elements, and the total number of the groupings was observed to be 6, all contained in one type or family. The total number of synthemes in that instance being 15, and there being 6 groupings of 5 synthemes each, it followed that in the whole family every syntheme is met with twice over; once in one grouping, and once in another. In the case treated of in my last communication to this *Magazine*, the total number of the synthemes of the kind under consideration is 36 (for it may easily be shown that the number of synthemes of n-nomial n-ads of n nomes of q elements each is $(1.2.3...q)^{n-1}$); and as each grouping contains 9 synthemes, these 36 are distributed *without repetition* between the 4 groupings of the smaller of the two natural species,—a phenomenon of a kind here met with for the first time in the study of *syntax*. If now we go on (as a natural and irrepressible curiosity urges) to ascertain the groupings of the synthemes of *binomial* triads of the same 3 nomes of three elements each, we advance just one step further in the direction of type-complexity; that is to say, we meet with the existence of 3, and not more than 3, types or species in which all such groupings are comprised. The investigation by which this is made out appears to me well worthy to be given to the world, as affording an example of a new and beautiful kind of analysis proper to the study of *tactic*, and thus lighting the way to the further opening up of this fundamental doctrine of mathematic, the science of necessary relations, of which, combined with logic (if indeed the two be not identical), tactic appears to me to constitute the main stem from which all others, including even arithmetic itself, are derived and secondary branches. The key to success in dealing with the problems of this incipient science (as I suppose of most others) must be sought for in the construction of an apt and expressive notation, and in the discovery of language by force of which the mind may be enabled to lay hold of complex operations and mould them into simple and easily transmissible forms of thought. I must then entreat the indulgence of the reader if, in this early grappling with the difficulties of a new language and a new notation, I may occasionally appear wanting in absolute clearness and fulness of expression.

Let us, as before, represent the nine elements by the numbers from 1 to 9, and suppose the *nomes* to be 1, 2, 3 : 4, 5, 6 : 7, 8, 9.

If we take any syntheme formed out of the *binomial* triads belonging to the above nomes, and if out of such syntheme we omit the elements 1, 2, 3 (belonging to the 1st nome) wherever they occur, the slightest consideration will serve to show that the synthemes thus denuded will assume the *form* $l.m.r, p.q, n$, where l, m, r may be regarded as belonging to one of the remaining nomes, and p, q, n to the other. The total number of synthemes in a grouping which contains all the binomial triads is 18, because the total number of these triads is 54; and consequently it will be seen that every grouping will in fact consist of the same *framework*, so to say, of combinations of elements belonging to the second and third nomes variously compounded with the elements of the first nome.

This framework may be advantageously divided into two parts, each containing nine terms, and which I shall call respectively U and $\dot{U}$. Thus by U I shall understand the nine arrangements following:—

4.5.7, 8.9, 6; 4.5.8, 7.9, 6; 4.5.9, 7.8, 6
5.6.7, 8.9, 4; 5.6.8, 7.9, 4; 5.6.9, 7.8, 4
6.4.7, 8.9, 5; 6.4.8, 7.9, 5; 6.4.9, 7.8, 5

each *imperfect* or defective syntheme being separated from the next by a semicolon, or else by a change of line. So by $\dot{U}$ I shall understand the *complementary* part of the framework, namely:—

8.9.6, 4.5, 7; 7.9.6, 4.5, 8; 7.8.6, 4.5, 9
8.9.4, 5.6, 7; 7.9.4, 5.6, 8; 7.8.4, 5.6, 9
8.9.5, 6.4, 7; 7.9.5, 6.4, 8; 7.8.5, 6.4, 9.

It is of cardinal importance to notice that the order in which the *imperfect synthemes* are arranged in U and $\dot{U}$ is one of absolute reciprocity. It is in this reciprocity, and in the fact of U or $\dot{U}$ being each in *strict regimen* (so to say) with the other, that the cause of the success of the method about to be applied essentially resides.

The slightest reflection will serve to show that every *complete* syntheme of the kind required will be of the form

$$\begin{vmatrix} U \times P \\ \dot{U} \times \dot{P} \end{vmatrix},$$

where the symbolical multipliers P and $\dot{P}$ are each of them some one of the forms (by no means necessarily the *same*) represented generally by the framework of defective synthemes hereunder written (defective in the sense that all the elements of the second and third nomes are supposed to be omitted),

$$\begin{array}{lll} ,\ a,\ bc\,; & ,\ b,\ ca\,; & ,\ c,\ ab \\ ,\ b,\ ca\,; & ,\ c,\ ab\,; & ,\ a,\ bc \\ ,\ c,\ ab\,; & ,\ a,\ bc\,; & ,\ b,\ ca, \end{array}$$

or else by the cognate framework

$$\begin{array}{lll} ,\ a,\ bc\,; & ,\ c,\ ab\,; & ,\ b,\ ca \\ ,\ b,\ ca\,; & ,\ a,\ bc\,; & ,\ c,\ ab \\ ,\ c,\ ab\,; & ,\ b,\ ca\,; & ,\ a,\ bc, \end{array}$$

where a, b, c are identical in some order or another with the elements of the first nome, namely, 1, 2, 3; so that there are six different systems of a, b, c in each of these two frameworks.

No other combination of the elements in U or $\dot{U}$ (all of which belong to the second and third nomes) with the elements in the first nome is possible; for any such combination would involve the fact of a *repetition* of the same *triad* or triads in the same grouping, contrary to the nature of a grouping. Hence, then, the number of forms of P and of $\dot{P}$ being twice six, or 12, we at once perceive that the total number of groupings is 12×12, or 144.

But now comes the more difficult question of ascertaining between how many distinct species or types these groupings are distributed. If we study the form of P or $\dot{P}$, it is obvious that it will be completely and distinctively *denoted in brief* by the twelve forms arising from the development of

$$\begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \quad \text{and} \quad \begin{array}{ccc} a & c & b \\ b & a & c \\ c & b & a \end{array}; \quad \text{videlicet}$$

(1)	(2)	(3)	(4)	(5)	(6)
1 2 3	2 3 1	3 1 2	2 1 3	1 3 2	3 2 1
2 3 1	3 1 2	1 2 3	1 3 2	3 2 1	2 1 3
3 1 2	1 2 3	2 3 1	3 2 1	2 1 3	1 3 2
(7)	**(8)**	**(9)**	**(10)**	**(11)**	**(12)**
1 3 2	2 1 3	3 2 1	2 3 1	1 2 3	3 1 2
2 1 3	3 2 1	1 3 2	1 2 3	3 1 2	2 3 1
3 2 1	1 3 2	2 1 3	3 1 2	2 3 1	1 2 3

which we may for facility of future reference denote by

$$\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5 \ldots \pi_{12}.$$

Now as regards the types: since the order of the elements in one nome is entirely independent of the order of the elements in any other, it is obvious that it is not the particular form of P or of $\dot{P}$ which can have any influence on the form of the type, but solely the relation of P and $\dot{P}$ to one another. In order then to fix the ideas, I shall for the moment consider P equal to

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2. \end{matrix}$$

This at once enables us to fix a *limit* to the number of distinct types. In the first place, the essentially distinct FORMS of the first column in $\dot{P}$, with respect to that of P, may be sufficiently represented by taking the two columns identical, or differing by a single interchange, or else having no two elements in the same place. Hence $\dot{P}$, so far as the ascertainment of types is concerned, may be limited to the six forms following:—

$$\begin{matrix} (\alpha) & & (\gamma) & & (\epsilon) \\ 1\ 2\ 3 & & 2\ 1\ 3 & & 2\ 3\ 1 \\ 2\ 3\ 1 & & 1\ 3\ 2 & & 3\ 1\ 2 \\ 3\ 1\ 2 & & 3\ 2\ 1 & & 1\ 2\ 3 \\ (\beta) & & (\delta) & & (\eta) \\ 1\ 3\ 2 & & 2\ 3\ 1 & & 2\ 1\ 3 \\ 2\ 1\ 3 & & 1\ 2\ 3 & & 3\ 2\ 1 \\ 3\ 2\ 1 & & 3\ 1\ 2 & & 1\ 3\ 2. \end{matrix}$$

But again, since (β) and (η) are each derivable from (α) (the assumed form of P) by an interchange of two columns *inter se*, it is clear that, as regards distinction of type, $\eta = \beta$, and consequently there are only *at utmost* five types remaining, which may be respectively described by the symbols

$$\left|\begin{matrix} U\alpha \\ \dot{U}\alpha \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\beta \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\gamma \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\delta \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\epsilon \end{matrix}\right|.$$

It must be noticed that α comprehends or typifies the squares numbered 1; β those numbered 7, 8, 9; γ those numbered 4, 5, 6; δ those numbered 10, 11, 12; ϵ those numbered 2, 3.

I say designedly that the number of types is *at utmost* limited to these five. But it by no means follows that the number is so great as five; for it will not fail to be borne in mind that these differences have reference to the peculiar mode in which we have chosen to decompose in idea each syntheme, by viewing it as a symbolical product of an arrangement containing only the elements of the second and third nomes by an arrangement containing only those of the first nome. But the nomes are interchangeable, and therefore it may very well be the case that two types which appear to be distinct are in reality identical, their elements in the groupings appertaining to such types having absolutely analogous relations to different orderings of the nomes, so that the groupings will be convertible into each other by permutations among the given elements. We must therefore ascertain how the above types, or any specific forms of them, come to be represented when we interchange the first nome with either of the other two, or, to fix the ideas, let us say with the second.

To effect this, let $U\alpha$, $\dot{U}\alpha$, $\dot{U}\beta$, $\dot{U}\gamma$, $\dot{U}\delta$, $\dot{U}\epsilon$ be actually expanded; by the performance of the symbolical multiplications we obtain—

$U\alpha=$	4.5.7 8.9.1 6.2.3;	4.5.8 7.9.2 6.1.3;	4.5.9 7.8.3 6.2.1
	5.6.7 8.9.2 4.1.3;	5.6.8 7.9.3 4.1.2;	5.6.9 7.8.1 4.2.3
	6.4.7 8.9.3 5.1.2;	6.4.8 7.9.1 5.2.3;	6.4.9 7.8.2 5.1.3
$\dot{U}\alpha=$	8.9.6 4.5.1 7.2.3	7.9.6 4.5.2 8.1.3	7.8.6 4.5.3 9.2.1
	8.9.4 5.6.2 7.1.3	7.9.4 5.6.3 8.1.2	7.8.4 5.6.1 9.2.3
	8.9.5 6.4.3 7.2.1	7.9.5 6.4.1 8.2.3	7.8.5 6.4.2 9.1.3
$\dot{U}\beta=$	8.9.6 4.5.1 7.2.3	7.9.6 4.5.3 8.1.2	7.8.6 4.5.2 9.1.3
	8.9.4 5.6.2 7.1.3	7.9.4 5.6.1 8.2.3	7.8.4 5.6.3 9.2.1
	8.9.5 6.4.3 7.2.1	7.9.5 6.4.2 8.1.3	7.8.5 6.4.1 9.2.3
$\dot{U}\gamma=$	8.9.6 4.5.2 7.1.3	7.9.6 4.5.1 8.2.3	7.8.6 4.5.3 9.1.2
	8.9.4 5.6.1 7.2.3	7.9.4 5.6.3 8.1.2	7.8.4 5.6.2 9.1.3
	8.9.5 6.4.3 7.1.2	7.9.5 6.4.2 8.1.3	7.8.5 6.4.1 9.2.3
$\dot{U}\delta=$	8.9.6 4.5.2 7.1.3	7.9.6 4.5.3 8.1.2	7.8.6 4.5.1 9.2.3
	8.9.4 5.6.1 7.2.3	7.9.4 5.6.2 8.1.3	7.8.4 5.6.3 9.1.2
	8.9.5 6.4.3 7.1.2	7.9.5 6.4.1 8.2.3	7.8.5 6.4.2 9.1.3
$\dot{U}\epsilon=$	8.9.6 4.5.2 7.1.3	7.9.6 4.5.3 8.1.2	7.8.6 4.5.1 9.2.3
	8.9.4 5.6.3 7.1.2	7.9.4 5.6.1 8.2.3	7.8.4 5.6.2 9.1.3
	8.9.5 6.4.1 7.2.3	7.9.5 6.4.2 8.1.3	7.8.5 6.4.3 9.1.2

Let us form a *framework* with the nomes 1 . 2 . 3, 7 . 8 . 9 exactly similar to that which we formed before with 4 . 5 . 6, 7 . 8 . 9, and let V, $\dot{V}$ be its two parts respectively analogous to U, $\dot{U}$, we thus obtain for $\dot{V}$,

1.2.7, 8.9, 3; 1.2.8, 7.9, 3; 1.2.9, 7.8, 3
2.3.7, 8.9, 1; 2.3.8, 7.9, 1; 2.3.9, 7.8, 1
3.1.7, 8.9, 2; 3.1.8, 7.9, 2; 3.1.9, 7.8, 2,

and for V,

8.9.3, 1.2, 7; 7.9.3, 1.2, 8; 7.8.3, 1.2, 9
8.9.1, 2.3, 7; 7.9.1, 2.3, 8; 7.8.1, 2.3, 9
8.9.2, 3.1, 7; 7.9.2, 3.1, 8; 7.8.2, 3.1, 9.

We must now perform the unwonted process of symbolical division, and obtain the quotients of $U\alpha$ by V, and of $\dot{U}\alpha$, $\dot{U}\beta$, $\dot{U}\gamma$, $\dot{U}\delta$, $\dot{U}\epsilon$ by $\dot{V}$ (it will of course be perceived that it is known *à priori* that the dividend forms of arrangement are actual multipliers of the divisors V and $\dot{V}$). In writing down the results of these divisions, which will consist exclusively of elements belonging to the nome 4 . 5 . 6, and of which each term will be of the form $d, e.f$, we may, analogously to what we have done before for greater brevity, write down only the single element (d), and omit the residue (ef), which is determined when (*d*) is determined. We shall thus obtain the quotients following :—

$$\frac{U\alpha}{V} = \begin{matrix} 5 & 4 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \end{matrix}$$

$$\frac{\dot{U}\alpha}{\dot{V}} = \begin{matrix} 5 & 4 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \end{matrix} \qquad \frac{\dot{U}\beta}{\dot{V}} = \begin{matrix} 5 & 6 & 4 \\ 6 & 4 & 5 \\ 4 & 5 & 6 \end{matrix} \qquad \frac{\dot{U}\gamma}{\dot{V}} = \begin{matrix} 5 & 4 & 6 \\ 4 & 6 & 5 \\ 6 & 5 & 4 \end{matrix}$$

$$\frac{\dot{U}\delta}{\dot{V}} = \begin{matrix} 5 & 6 & 4 \\ 4 & 5 & 6 \\ 6 & 4 & 5 \end{matrix} \qquad \frac{\dot{U}\epsilon}{\dot{V}} = \begin{matrix} 4 & 6 & 5 \\ 5 & 4 & 6 \\ 6 & 5 & 4. \end{matrix}$$

It may be observed that these divisions may be effected with great rapidity ; because when three out of the nine figures (in any quotient) not in the same line or column are known, all the rest are known. Thus, for example, to find $\frac{\dot{U}\epsilon}{\dot{V}}$ it is only necessary to seek in $\dot{U}\epsilon$ the syntheme which contains 1 . 2 . 7, and then to take out the figure in that syntheme associated with 8 . 9 in that line, namely, 4 ; then again to seek the syntheme which contains 1 . 2 . 8, and to take out the figure in that syntheme associated with 7 . 9, which is 6 ; and finally to seek the syntheme which contains 2 . 3 . 7, and then to take out the figure associated with 8 . 9, namely 5 ; we thus obtain the three corner figures of the square which represents $\frac{\dot{U}\epsilon}{\dot{V}}$ as thus :

$$\begin{matrix} 4 & 6 & . \\ 5 & . & . \\ . & . & . \end{matrix}$$

of which the six remaining figures are given by the condition that in no line and in no column must the same two figures be found. In order to compare these quotients, or rather the relations of the first of them to the remaining five with those of α to α, β, γ, δ, ϵ, it will be convenient to subtract the constant number 3 from each figure, and to transpose the first and second columns ; we thus obtain

$$\frac{U\alpha}{V} \equiv \left| \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \right| \equiv \pi_1 \equiv \alpha,$$

$$\frac{\dot{U}\alpha}{\dot{V}} \equiv \left| \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \right| \equiv \pi_1 \equiv \alpha, \qquad \frac{\dot{U}\beta}{\dot{V}} \equiv \left| \begin{matrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{matrix} \right| \equiv \pi_9 \equiv \beta,$$

$$\frac{\dot{U}\gamma}{\dot{V}} \equiv \left| \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \right| \equiv \pi_{11} \equiv \delta, \qquad \frac{\dot{U}\delta}{\dot{V}} \equiv \left| \begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{matrix} \right| \equiv \pi_6 \equiv \gamma,$$

$$\frac{\dot{U}\epsilon}{\dot{V}} \equiv \left| \begin{matrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} \right| \equiv \pi_3 \equiv \epsilon.$$

Thus, for greater brevity, considering the five types to be represented by

$$\begin{matrix} \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \beta & \gamma & \delta & \epsilon, \end{matrix}$$

or still more briefly by

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon;$$

and calling the nomes N_1, N_2, N_3, we find that the effect of interchanging N_1 and N_2 with each other is to change

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

into

$$\alpha \quad \beta \quad \delta \quad \gamma \quad \epsilon.$$

In like manner it may be ascertained (and the student is advised to satisfy himself by actual trial of the fact) that the effect of interchanging N_1 and N_3 with each other is to convert

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

into

$$\alpha \quad \delta \quad \gamma \quad \beta \quad \epsilon.$$

From these two calculations it follows that the effect of any permutation between N_1, N_2, N_3 is to produce a permutation in β, γ, δ *inter se*, but will leave α and ϵ unaltered*. Hence then we have arrived at the goal of our inquiry, having demonstrated that

$$\left| \begin{matrix} V\alpha \\ \dot{V}\alpha \end{matrix} \right|$$

indicates one type,

$$\left| \begin{matrix} V\alpha \\ \dot{V}\beta \end{matrix} \right|, \quad \left| \begin{matrix} V\alpha \\ \dot{V}\gamma \end{matrix} \right|, \quad \left| \begin{matrix} V\alpha \\ \dot{V}\delta \end{matrix} \right|$$

each of them another *the same* type, and

$$\left| \begin{matrix} V\alpha \\ \dot{V}\epsilon \end{matrix} \right|$$

a third type,—and bearing in mind that

(α)	belongs to	π_1	exclusively,
(ϵ)	"	π_2, π_3	"
(β)	"	π_7, π_8, π_9	"
(γ)	"	π_4, π_5, π_6	"
(δ)	"	$\pi_{10}, \pi_{11}, \pi_{12}$	"

* This result, by the aid of a fine observation, may be more rapidly established *uno ictu* (I mean by *one* calculation instead of two) as follows. Let $N_1 N_2 N_3$ be made to undergo a *cyclical* interchange, then it will be found that β, γ, δ also undergo a cyclical interchange, whilst α and ϵ remain unchanged. This proves that β, γ, δ are only different phases of the same type, which is *sufficient*; for as regards α and ϵ, the fact of the number of individuals which they represent being unequal *inter se*, and also unequal to the number contained in β, γ, δ, renders it *à priori* impossible to allow that they can either pass into each other or into the forms β, γ, δ, by virtue of any interchange among the elements.

and that each form of π comprehends 12 groupings due to the 12 forms of $V\alpha$, we are enabled to affirm that the total number of groupings of the binomial triads of 3 nomes of 3 elements each is 144, and that the number of types or species between which these 144 are distributed is 3, comprising 12, 24, and 108 respectively,—a conclusion which it would almost have exceeded the practical limits of human labour and perspicuity to have established by the direct comparison of the 144 groupings of 18 synthemes each with each other, with a view to ascertain which admit of being permutable into each other, and which not.

The largest species of 108 groupings, it may be observed, is subdivisible into 3 *varieties*, not really allotypical, of 36 each,—the characteristic of those groupings which belong to the same variety being that they permute *exclusively* into each other when the permutations of the elements are confined to perturbations of the order of the elements in the same nome or nomes, and the different nomes are subject to no interchange of elements between themselves.

Just so the species of 36 groupings of trinomial triads, treated of in my preceding paper, subdivides into 3 varieties of sub-families characterized by a similar property.

The total number of modes of subdivision of 9 elements between 3 nomes being 280, it follows, from considerations of the same kind as stated in the May Number of this *Magazine* [p. 264 above], that there exist transitive substitution-groups belonging to 9 elements of

$$\frac{\pi(9)}{280 \times 12}, \quad \frac{\pi(9)}{280 \times 24}, \quad \frac{\pi(9)}{280 \times 108},$$

that is, 108, 24 and 12 substitutions respectively.

Again, let us consider the question of forming the synthemes of the triads of a *single nome* of 9 elements into groupings where *every* triad shall be found without repetition. We may obtain such groupings by choosing arbitrarily any one of the 280 sets of 3 nomes into which the 9 elements may be segregated*, and then forming one syntheme with the three monomial triads (corresponding to such set so chosen), 18 synthemes (in any one of the 144 possible ways) of exclusively binomial triads, and 9 synthemes (in any one of the 40 possible ways) of exclusively trinomial triads; we shall thus obtain in all 280 × 144 × 40, or 1,612,800 solutions of the question proposed; I mean

* 280 is also evidently the number of synthemes of triads belonging to one nome of 9 elements. In general the number of r-ads belonging to one nome of mn elements is

$$\frac{\pi(mn-1)\,\pi\{(m-1)n-1\}\,\pi\{(m-2)n-1\}\ldots\pi(n-1)}{\{\pi(n-1)\}^m\,\pi\{(m-1)n\}\,\pi\{(m-2)n\}\ldots\pi(n)}.$$

1,612,800 groupings, all satisfying the imposed condition, and reducible to 6 *genera**, comprising respectively

$$4 \times 12 \times 280, \quad 4 \times 24 \times 280, \quad 4 \times 108 \times 280, \quad 36 \times 12 \times 280,$$
$$36 \times 24 \times 280, \quad 36 \times 108 \times 280,$$

that is, 13,440, 26,880, 120,960, 120,960, 241,920, 1,088,640 individual groupings. I conclude with putting a grand question, more easy to propose than to answer, namely, are these one million six hundred thousand (and upwards) groupings (classifiable under six distinct genera) all the possible modes and types of grouping which will satisfy the conditions of the question? and if not, what other mode or type of grouping can be found? Were I compelled to give an answer to this question, I would say that the balance of my mind leans to the opinion that the six types in question are the sole possible types of solution; but I do not pretend to rest this judgment upon any solid grounds of demonstration, nor to entertain it with any strong degree of assurance. It is a question which the effort to resolve cannot but react powerfully on our knowledge of the principles of tactic in general, and of the theory of substitution-groups in particular; and as such I submit it to the consideration of the rising chivalry of analysis, seeking myself meanwhile fresh fields and pastures new of meditation.

* The above *genera* must not be confounded with types or species. (In my preceding communications I may inadvertently have used the word *family* as coincident with type: *species* is the proper term.) The type of a total grouping in the problem referred to in the text will depend not only on the particular combination of the types of the binomial and trinomial partial groupings which give rise to these 6 ($=2 \times 3$) genera, but also on the relative *phases* of the types so combined. The number of groupings in one type or species is always a submultiple of the number of permutations of the elements; whereas it will be seen that the number of groupings in one of the above genera greatly exceeds that number, which in the present case is only

$$1.2.3.4.5.6.7.8.9, \text{ or } 362,880.$$

Whatever may be the case in natural history, the nature of a type or species, as distinguished from a genus, family, or any other higher kind of aggregation of individuals, in *pure syntax* is perfectly clear and unambiguous; those groupings form a species which are commutable into one another by an interchange of elements: thus the different *phases* of the same type or species are in analogy with the different values of the same function arising out of a change in a constant parameter. If it should turn out that the above sixteen hundred thousand and odd groupings are not the sole solutions of which the question admits, then it will follow that even in this early instance we shall have an example not only of species and genera, but of distinct families of genera, for it is certain that the above six genera constitute within themselves a complete natural family. It will form an interesting subject of inquiry to ascertain how many types are included within each of the six genera belonging to this family; and be it never forgotten that to each species corresponds, and from it is, so to say, capable of being extracted or sublimated, a Cauchian substitution-group.