50.

ON A GENERALIZATION OF A THEOREM OF CAUCHY ON ARRANGEMENTS*.

[*Philosophical Magazine*, XXII. (1861), pp. 378—382.]

IN a paper "On the Theory of Determinants" in the *Philosophical Magazine* for March in this year, Mr Cayley has referred and added to a theorem of Cauchy deduced from the latter's method of *arrangements*, namely, that if we resolve an integer n in every possible way into parts, to wit α parts of a, β parts of b, ... λ of l, (a, b, c ... l being all distinct integers), then

$$\Sigma \frac{1}{\Pi\alpha \,.\, a^\alpha \Pi\beta \,.\, b^\beta \ldots \Pi\lambda \,.\, l^\lambda} = 1.$$

Now both Cauchy's theorem and Mr Cayley's addition to it (which essentially consists in the observation, that if before the numerator 1 in the above quantity under the sign of summation we write $(-)^{\alpha+\beta+\ldots+\lambda}$, the sum becomes zero) are no more than particular cases of the following theorem: namely, that if instead of 1 we write $\rho^{\alpha+\beta+\ldots+\lambda}$ in the numerator of the quantity under the sign of summation (ρ being any quantity whatever), the sum becomes expressible as a known function of ρ. Nothing can be easier than the proof.

Let the $\alpha, \beta, \gamma, \ldots \lambda$ in the preceding statement be supposed subject to the further condition that their sum is r; then for any assigned value of r (a positive integer) it is easy to see that the sum of the terms within the sign of summation in Cauchy's theorem is

$$S\left(\frac{1}{x_1 x_2 \ldots x_r} \,.\, \frac{\rho^r}{\Pi(r)}\right),$$

where $x_1, x_2, \ldots x_r$ mean *every* system of values of $x_1, x_2, \ldots x_r$ (permutations *admitted*) which satisfy the equation

$$x_1 + x_2 + \ldots + x_r = n.$$

(It should here be observed that $\alpha, \beta, \gamma, \ldots \lambda$; $a, b, c, \ldots l$ are the systems

[* See above p. 245.]

which satisfy $\alpha a + \beta b + \gamma c + \ldots + \lambda l = n$, permutations being *excluded*; that is to say, if, for example, α, β, γ should happen to be equal for any partition of n, the values α, a; α, b; α, c would figure only *once*, and not *six* times, among the systems included under the sign of Σ.) Hence then we see that

$$\Sigma \frac{\rho^{\alpha+\beta+\gamma+\ldots+\lambda}}{\Pi\alpha \,.\, a^\alpha\, \Pi\beta \,.\, b^\beta \ldots \Pi\lambda \,.\, l^\lambda} = \sum_{r=\infty}^{r=1} S_r \frac{\rho^r}{\Pi(r)} \, *,$$

where S_r is the coefficient of t^n in $\left(\frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{3} + \text{\&c. } \textit{ad infin.}\right)^r$, that is, in $\left\{\log\left(\frac{1}{1-t}\right)\right\}^r$; and the total sum designated by Σ will be consequently the coefficient of t^n in

$$\log\left(\frac{1}{1-t}\right)\rho + \left(\log\frac{1}{1-t}\right)^2 \frac{\rho^2}{1\,.\,2} + \text{\&c.},$$

that is, in $e^{\rho \log\left(\frac{1}{1-t}\right)}$, that is, in $\left(\frac{1}{1-t}\right)^\rho$.

Thus if $\rho = 1$, we have Cauchy's theorem, namely $\Sigma = 1$;

Thus if $\rho = -1$, we have Cayley's theorem, namely $\Sigma = 0$†;

and in general for any value of ρ,

$$\Sigma = \frac{\rho(\rho+1)\ldots(\rho+n-1)}{1\,.\,2\ldots n} \ddagger.$$

* For if we take a system of values satisfying the above equation, consisting of α equal values a, β equal values b, ... λ equal values l, such a system will give rise in $\Sigma \frac{1}{x_1 x_2 \ldots x_r}$ to $\frac{\pi(r)}{(\pi\alpha)(\pi\beta)\ldots(\pi\lambda)}$ repetitions of the term $\frac{1}{a^\alpha \,.\, b^\beta \ldots l^\lambda}$, and consequently in $\Sigma \frac{1}{x_1 x_2 \ldots x_r} \,.\, \frac{1}{\pi r}$ to a total value $\frac{1}{(\pi\alpha)\, a^\alpha\, (\pi\beta)\, b^\beta \ldots (\pi\lambda)\, l^\lambda}$, condensed into a single term in Cauchy's theorem.

† Provided, however, that n exceeds 1, a limitation accidentally omitted in Mr Cayley's paper; and so in general

$$\Sigma \frac{(-\rho)^{\alpha+\beta+\ldots+\lambda}}{\Pi\alpha \,.\, a^\alpha \ldots \Pi\lambda \,.\, l^\lambda} = 0,$$

ρ being *any* positive integer provided n is greater than ρ.

‡ If $\rho = \frac{1}{2}$, we obtain

$$\Sigma = \frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n};$$

from which it is easy to infer that the number of substitutions of $2n$ things representable by the product of cyclical substitutions, all of an even order, is $\{1\,.\,3\,.\,5\ldots(2n-1)\}^2$. If $\rho = -\frac{1}{2}$, we obtain

$$\Sigma = \frac{1\,.\,1\,.\,3\ldots(2n-1)}{2\,.\,4\,.\,6\ldots(2n)},$$

combining which with the preceding result, it is easy to infer that the number of substitutions of $2n$ things representable by the product of an odd number of cyclical substitutions, all of an even order, is to the number of such representable by the product of an even number of cyclical substitutions, all of an even order, in the ratio of n to $(n-1)$. The former of these two theorems

In this theorem is in fact included another, namely, that if

$$\alpha a + \beta b + \ldots + \lambda l = n \text{ and } \alpha + \beta + \ldots + \lambda = r$$

(permutations *not* admissible), then

$$\Sigma \frac{\Pi n}{\Pi\alpha . a^{\alpha} . \Pi\beta . b^{\beta} \ldots \Pi\lambda . l^{\lambda}}$$

is equal to the coefficient of ρ^{r-1} in

$$(\rho+1)(\rho+2)\ldots(\rho+n-1).$$

This coefficient is accordingly (to return to Cauchy's theory of arrangements) the number of substitutions of n elements capable of being expressed by the product of r cyclical substitutions. As, for instance, the number of substitutions of four elements a, b, c, d capable of expression by the product of two cyclical substitutions ought to be the coefficient of λ in $(\lambda+1)(\lambda+2)(\lambda+3)$, that is, 11, which is right; for the number of substitutions of the form $(a, b)(c, d)$ will be 3, and of the form $(a, b, c)(d)$ will be 8. In conclusion, I may notice that by an obvious deduction from this last theorem, we are led to the well-known one in the theory of numbers, that every coefficient in the development of

$$\Sigma(\rho+1)(\rho+2)\ldots(\rho+n-1),$$

except the first and last, and the sum of these two, is divisible by n when n is a prime number; and indeed we can actually express by aid of it the quotient of every intermediate coefficient divided by n as the sum of separate integer terms free from the sign of addition.

Postscript. By an extension of the method of generating functions contained in the text above, it may easily be seen that the number* of substitutions of n letters represented by the products of r cyclical substitutions, where the number of letters of each cycle leaves a given residue e in respect

is intimately allied with Mr Cayley's celebrated theorem on "skew," or what, for good reasons hereafter to be alleged, I should prefer to call *polar* determinants, namely, that every such of the $2n$th order is the square of a *Pfaffian*. A Pfaffian is in fact a sum of quantities typifiable completely, both as to sign and magnitude, by a duadic *syntheme* of $2n$ elements, the number of which is readily seen to be $1.3.5\ldots(2n-1)$. I believe I shall soon be in a condition to announce a remarkable extension of this theory to embrace the case of Polar *Commutants* and *Hyperpfaffians*.

* For this number, divided by $\Pi(n)$, is the coefficient of x^n in

$$\frac{1}{\Pi r}\left(\int_0^x \frac{dx\, x^{e-1}}{1-x}\right)^r, \text{ say } \frac{1}{\Pi r}(\phi x)^r,$$

and therefore of $x^n \rho^r$ in $e^{\rho\phi x}$, say $\psi(x, \rho)$, and therefore $\left(\text{since } \frac{d\psi}{dx} = \frac{x^{e-1}}{1-x^m} \text{ and } \psi \text{ may be put}\right.$ under the form $\left.\Sigma \frac{u_n}{n} x^n\right)$ of ρ^r in $\frac{u_n}{n}$, where u_n is defined as in the text.

to a given *modulus* μ, may be made to depend on the solution of the equation in differences

$$u_n - u_{n+\mu} = \frac{\rho}{n-e} u_{n-e}.$$

The case where $e=1$ is deserving of particular notice.

It may be shown by means of the above equation in differences, that the number of substitutions of n letters formed by r cycles each of the form $\mu K+1$ (μ being constant), say $\phi(n, r, \mu, 1)$, where $\frac{n-r}{\mu}$ is necessarily an integer, may be found by taking in every possible way $\frac{n-r}{\mu}$ *distinct* groups of μ consecutive terms of the series $1, 2, 3, \ldots (n-1)$; the sum of the products of every such combination of groups is the value required. For example, if

$$n=8, \quad r=3, \quad \mu=2,$$

$$\begin{aligned}\phi(8, 3, 2, 1) = {} & 1.2.3.4.5.6 + 1.2.3.5.6.7 + 1.2.3.5.6.8 \\ & + 1.2.3.6.7.8 + 2.3.4.5.6.7 + 2.3.4.6.7.8 \\ & + 3.4.5.6.7.8.\end{aligned}$$

And as a corollary, since it may easily be seen that $\phi(n, r, \mu, e)$ is always divisible by n when n is a prime and $\mu r + e < n$, it follows that the sum of all the possible products of (any given number) i distinct groups of a given number r of consecutive terms of the series $1, 2, 3, \ldots (n-1)$ will be divisible by n when n is a prime and $ir < n-1$*. When $r=1$, this theorem becomes identical with Wilson's, already referred to.

Finally, it may be noticed that the number of substitutions of n letters formed by *any* number of cycles, all of an *odd* order, will be the coefficient of x^n in $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$, that is, $\{1.3.5\ldots(n-1)\}^2$ (the same as the number that can be formed with cycles all of an *even* order) when n is even, and

$$\{1.3.5\ldots(n-2)\}^2 n$$

when n is odd†.

* For instance, making $n=7$, $r=2$, $i=2$,

$$1.2.3.4+1.2.4.5+1.2.5.6+2.3.4.5+2.3.5.6+3.4.5.6=784$$

and is *divisible by* 7.

† By taking $\mu=2$ in the general theorem, it is an easy inference that if we write

$$(\tan^{-1}x)^r = x^r - \frac{A_2 x^{r+2}}{(r+1)(r+2)} + \frac{A_4 x^{r+4}}{(r+1)(r+2)(r+3)(r+4)} \mp \&c.,$$

A_{2i} will be the sum of all the products of $2i$ integers comprised between 1 and $r+2i-1$ that can be formed with combinations of i distinct pairs of consecutive integers; thus, for example, the coefficient of x^{2m} in $(\tan^{-1}x)^2$ ought to be

$$\frac{1}{m}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2m-1}\right),$$

which may be easily verified.