64.

OBSERVATIONS ON THE METHOD FOR FINDING THE CENTRE OF GRAVITY OF A QUADRILATERAL GIVEN IN THE PRESENT NUMBER OF THE JOURNAL.

[*Quarterly Journal of Mathematics*, VI. (1863), pp. 130—133.]

THE method given in *Mechanical Solutions of Geometrical Problems*, p. 127, for finding the centre of gravity of a quadrilateral, leaves nothing to be wished for in point of elegance and conciseness; it is new* to the Editors and stands in advantageous contrast with all other methods of effecting the same end. It involves only four lines of construction and two bisections; in some elementary works on Mechanics, in use at our Universities, a method is given involving no less than 9 or 11 auxiliary lines. It must henceforth take rank as the best method of effecting the end in view; the second best is that which has been treated analytically, by Mr Stephen Fenwick, of the Royal Military Academy, in the *Mathematician*, 1847, Vol. II., p. 292, but admits of a simple and pleasing geometrical proof.

Let us call the intersection of the two diagonals the *cross-centre*, and the intersection of the two bisectors of opposite pairs of sides the *mid-centre* of a quadrilateral.

If we take the centres of gravity of the four triangles into which a given quadrilateral is divided by its two diagonals, it is clear that the cross-centre of the new quadrilateral, of which these four points are the summits, will be the centre of gravity of the original quadrilateral. But it may easily be seen that this new quadrilateral is only a miniature image of the original one, and that each of the two quadrilaterals has the same mid-centre; in a word, the new quadrilateral may be obtained by reducing the linear dimensions of the original one in the ratio of 1 to 3, and then swinging it through half a revolution round the mid-centre. Hence the new cross-centre will be in opposition with the original one, in respect to the mid-centre, and at a distance from it equal to one-third of the distance of the former one from the same.

* I should say new in *form*; in substance it is identical with that given, Vol. II., p. 292, of the *Mathematician*.

This method involves only four auxiliary lines, but requires four bisections and one trisection, instead of merely two bisections, according to the method of the text above.

The substitution of heavy points for areas or volumes admits of an extension which the author of this note believes to be new, and which occurred to him incidentally in treating of the extension of Gauss' method of approximation from simple to multiple quadratures.

It will be convenient to call the sum of the masses of any system of bodies into the nth powers of their distances from a fixed plane their nth *moments* in respect to the plane. (Thus the second moments will mean the sum of the masses into their squared distances.) It may then be affirmed as a universal proposition that such nth moments of a line, triangle, and tetrahedron (and so on for the higher dimensions of space) may always be replaced by suitable weights at fixed points symmetrically situated about the centre of gravity of such figures. For example, the *second* moments of lines, triangles, and tetrahedra (say each of mass unity) in respect to any plane may be replaced by masses of $\frac{1}{6}$ at the two angular points for the line, of $\frac{1}{12}$ at the three angular points for the triangle, and of $\frac{1}{20}$ at the four angular points for the tetrahedron, the balance $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ being of course placed at the centre of gravity in these cases respectively. Hence it follows obviously that the same law will be true for the *moments of inertia* of a line, triangle, or tetrahedron about any *axis*, and consequently the centres and times of oscillation of these figures about any axis will be the same as for equal weights placed at the angles and weights respectively 4, 9, and 16 times as great placed at their centres of gravity. The ingenious author of the matter which has called forth these observations may probably be able to draw interesting inferences from this equivalence, and also from combining his own unrivalled method for finding the centre of gravity of a quadrilateral with the miniature-image method hereinbefore explained.

One word more before I conclude; the rule given in the text may be expressed in general terms by aid of a simple verbal definition. Let two points situated in a limited line be said to be *opposite* when their respective distances from opposite ends of the line are equal.

The centre of gravity of a quadrilateral may then be stated to be identical with the centre of gravity of a triangle whose apices are the point of intersection of the two diagonals and the opposite points thereto on these two diagonals*.

* It is difficult to resist the impression that some similar construction must apply to the determination of the centre of gravity of the frustum of a pyramid. Two points in a line with the centre of gravity of a triangle and at equal distances on opposite sides may be defined as opposite points in respect to the triangle. As a mere conjecture to be subjected to ulterior verification I suggest the possibility of the following construction being applicable; if not true it may at least serve to set the mind athinking in the right direction for the discovery of the truth.

The Barycentric principle employed in the text leads me to make an observation which will be found somewhat prolific in consequences and may be made instrumental (as I have satisfied myself by actual trial) in the edification of a complete theory of the parabola by processes greatly exceeding in simplicity those depending on Cartesian coordinates.

The secret of the utility of the Barycentric principle consists essentially in the plasticity of the axes to which the moments may be referred. Equally advantageous will be found the introduction of the laws of motion into the theory of the parabola, aided by the *plastic* condition that the motion of a projectile acted on by a constant force, *reckoned in any direction*, depends only on the actual velocity and force respectively estimated in such direction.

Let Aa, Bb, Cc be the edges of the frustum.

Let the three diagonal triangles Abc, Bca, Cab intersect in the point P and let the opposites to P in these three triangles be respectively P', P'', P'''. Similarly, by means of the other system of diagonal planes aBC, bCA, cAB, let a second system of four points p, p', p'', p''' be obtained. I conjecture that the centre of gravity of equal weights at these eight points, or, which is the same thing, the centre of gravity of the pyramid, whose apices bisect respectively the lines Pp, $P'p'$, $P''p''$, $P'''p'''$, *may be* the centre of gravity of the frustum. This intermediate pyramid appears to be the natural measure of the distortion of the frustum from the prismatic form, as the triangle formed by the cross-centre and its two opposites is that of the distortion of the quadrilateral, which may be regarded as the frustum of a triangle.

I have verified the conjectural construction for the case where the frustum becomes a prism and also for the case where it becomes a tetrahedron by the vanishing of one of its triangular faces.

À propos of the relation between the trapezium and the pyramidal frustum, I am not aware whether it has been observed that as a trapezium may be divided in two ways into a pair of triangles, so may the frustum of a pyramid be divided in six ways into a triplet of tetrahedrons. Using the same letters as before, one such division will be represented by the table following:

$$\begin{array}{cccc} a & b & c & A \\ \hline b & c & A & B \\ \hline c & A & B & C \end{array}$$

and permuting simultaneously and conformably the two systems of letters a, b, c; A, B, C, we obtain all the six systems in question. This stereotomic division leads to a direct and almost instantaneous geometrical proof of the known expression

$$\delta = \frac{h}{4}\,\frac{a^2+2ab+3b^2}{a^2+ab+b^2},$$

for finding the position, of the centre of gravity of a frustum bounded by parallel faces, in the line joining the centres of the parallel faces.

Obviously for space of any number of dimensions, say n, an analogous dissection may be effected in $n!$ different ways; the scheme above given serving fully to disclose the tactical law of the symbols.

For example, for space of four dimensions one such dissection out of the twenty-four will be denoted by the scheme

$$\begin{array}{ccccc} a & b & c & d & A \\ \hline b & c & d & A & B \\ \hline c & d & A & B & C \\ \hline d & A & B & C & D \end{array}$$

By aid of this principle I have reconstructed all the essential properties of the curve in respect to its directrix, focus, and tangents, and obtained, as it were instantaneously, various theorems, some of which, if not new, could only be obtained by long processes through the ordinary methods, whether of Geometry or of Cartesian coordinates.

Since penning the above observations, the author has found without difficulty the two geometrical constructions for the centre of gravity of a pyramidal frustum, precisely analogous to those alluded to for the centre of gravity of a quadrilateral: which will probably appear* in the August Number (or, if not, in the September Number) of the *Philosophical Magazine*. The true *mid-centre* is the centre of gravity of six equal weights placed at the six angles of the frustum; the true *cross-centre* is the point of intersection of either of two ternary systems of planes, which have the property of intersecting in the same point; one of these planes, for example, will be the plane passing through the middle point of ab, the middle point of aC, and the middle point of BC.

This brings to mind an analogous generalization long ago made known by the writer of this note, namely, that as a quadratic surface is cut by any tangent plane in two straight lines, so is a cubic hyper-surface by a tangent hyper-plane in six, a quartic transhyper-surface by a tangent transhyper-plane in twenty-four right lines, and so on indefinitely. Passing by an abrupt flight from a transcendental analogy to what many may regard as a mere platitude, let me notice that it is not a truism but a proposition and no insignificant one to affirm that a convex figure of five [plane] faces capable of being formed by joining conformably the angles of one triangle with those of another can only be the *frustum of a pyramid*; it is in fact equivalent to the assertion that three right lines of which every two intersect must either lie in one plane or pass through one point.

I ought not to conclude without alluding to a second conjectural method for finding the centre of gravity of such frustum which will in all probability stand or fall with that already given, bearing to it the same relation as the second best bears to the best determination of the analogous problem for the quadrilateral. Taking as Q (the *cross-centre*) the point mid-way between P, p the respective intersections of the two systems of diagonal planes, and as O (the *mid-centre*) the point where the axis joining the centres of gravity of the two triangular faces meets the plane containing the centres of gravity of the three quadrilateral faces, QO being joined and produced through O to G, so that $GO=\frac{QO}{4}$, G will be the conjectural position of the centre of gravity of the frustum. This is easily verified for the case where the frustum becomes an entire pyramid.

P.S.—Since the above was in press I have ascertained that each of the above two conjectural methods is erroneous. Apparently the *euristic* problem to be solved is to discover in the pyramidal frustum, the analogue to the cross-centre in the quadrilateral; this, there is every reason to believe, is closely connected with the points P and p above described; it is, however, certainly not the point mid-way between them.

[* Below, p. 342.]