83.

ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

[*Monthly Notices of the Royal Astronomical Society*, XXVI. (1865), pp. 27—29.]

THE original demonstration by Lambert of the celebrated theorem which bears his name was a geometrical one, see *Monthly Notices*, vol. XXII. p. 238, where this demonstration is reproduced by Mr Cayley. Lagrange has given no less than three distinct demonstrations of the same: one a sort of verification by aid of trigonometrical formulae, another founded on a property of integrals, and a third, perhaps the most remarkable of all, derived from the general expressions for the time in an orbit described about two centres of force varying according to the law of nature by supposing one of them to be situated in the orbit itself, and to become zero. Notwithstanding this plethora of demonstration, the following direct algebraical method of proving from the ordinary formulae for the time of a planet passing from one point to another, that, when the period is given, the time is a function only of the sum of the distances of these points from the centre of force, and of their distance from one another, may be deemed not wholly undeserving of notice.

Let ρ, ρ' be the distances of the two positions from the Sun, c their distance from one another, v, v' the true, u, u' the excentric, m, m' the mean anomalies thereunto corresponding, e the excentricity,

$$\omega = m - m',\ s = \rho + \rho',\ \Delta = \tfrac{1}{2}(s^2 - c^2):$$

then

$$\rho = 1 - e\cos u,\ \rho' = 1 - e\cos u',\ m = u - e\sin u,\ m' = u' - e\sin u',$$

$$\rho\cos v = \cos u - e,\quad \rho\sin v = \sqrt{(1-e^2)}\sin u,$$

$$\rho'\cos v' = \cos u' - e,\ \rho'\sin v' = \sqrt{(1-e^2)}\sin u',$$

$$c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(v' - v).$$

Writing for brevity c, c', s, s', for $\cos u$, $\cos u'$, $\sin u$, $\sin u'$, and to avoid confusion putting also for the moment $\bar{s}, \bar{c}$ in place of the original s and c, we have

$$\bar{s} = 2 - ec - ec', \ \omega = u - u' - es + es',$$

$$\Delta = \rho\rho' + \rho\rho' \cos(v' - v) = 1 + cc' + ss' - 2e(c + c') + e^2(1 + cc' - ss').$$

Let $J = \dfrac{d(\Delta, \bar{s}, \omega)}{d(e, u, u')}$; then J is the determinant

$$\begin{vmatrix} -2(c+c')+2e(1+cc'-ss'); & cs'-c's+2es-e^2(cs'+c's); & c's-cs'+2es'-e^2(cs'+c's) \\ -c-c' \quad ; & es \quad ; & es' \\ -s+s' \quad ; & 1-ec \quad ; & -1+ec' \end{vmatrix}.$$

Denoting this determinant by

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

we find

$$(A, B, C) - 2H(D, E, F) + 2E(G, H, K) = (0, B, -B),$$
$$(A, B, C) - 2K(D, E, F) + 2F(G, H, K) = (0, -C, C),$$

so that
$$J = \begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix} = 0.$$

Hence restoring s, c, instead of $\bar{s}, \bar{c}$, it appears that $d\omega$ is a linear function of ds and $d\Delta$; that is, ω is a function of s and Δ, or what is the same thing of s and c, independent of e. If then, when $e = 1$, the corresponding values of $\rho, \rho', v, v', u, u'$ are $r, r', \theta, \theta', \phi, \phi'$, we have $\cos\theta = -1$, $\cos\theta' = -1$, $\sin\theta = 0$, $\sin\theta' = 0$, $r - r' = c$, $r + r' = s$, whence writing

$$1 - \cos\phi = \frac{s + c}{2}, \ 1 - \cos\phi' = \frac{s - c}{2},$$

we have finally $\omega = \phi - \phi' - \sin\phi + \sin\phi'$ as was to be proved.

Essentially this demonstration is of the same value as the first of Lagrange's three methods of proof above referred to, but with the difference that it leads up to and accounts beforehand for the success of the transformations therein employed.