84.

ON AN ELEMENTARY PROOF AND GENERALIZATION OF SIR ISAAC NEWTON'S HITHERTO UNDEMONSTRATED RULE FOR THE DISCOVERY OF IMAGINARY ROOTS*.

[Syllabus of Lecture delivered at King's College, London†, June 28, 1865. *Proceedings of the London Mathematical Society*, I. (1865—1866), pp. 1—16.]

LET $fx = 0$ be an algebraical equation of degree n.

Suppose $fx = a_0x^n + na_1x^{n-1} + \frac{1}{2}n(n-1)a_2x^{n-2} + \ldots\ldots\ldots + na_{n-1}x + a_n$;

$a_0, a_1, a_2 \ldots\ldots\ldots a_n$ may be termed the simple elements of fx.

Suppose

$A_0 = a_0^2,\quad A_1 = a_1^2 - a_0a_2,\quad A_2 = a_2^2 - a_1a_3, \ldots A_{n-1} = a_{n-1}^2 - a_{n-2}a_n,\quad A_n = a_n^2$;

$A_0, A_1, A_2, \ldots\ldots\ldots A_n$ may be termed the quadratic elements of fx. a_r, a_{r+1} is a succession of simple elements, and A_r, A_{r+1} of quadratic elements.

$$\left.\begin{matrix} a_r \\ A_r \end{matrix}\right\} \text{ is an } \textit{associated} \text{ couple of elements;}$$

$$\left.\begin{matrix} a_r\ a_{r+1} \\ A_rA_{r+1} \end{matrix}\right\} \text{ is an associated couple of } \textit{successions.}$$

A succession may contain a permanence or a variation of signs, and will be termed for brevity a permanence or variation, as the case may be. Each succession in an associated couple may be respectively a *permanence* or a *variation*. Thus an associated couple may consist of two permanences or two variations, or a superior permanence and inferior variation, or an inferior permanence and superior variation; these may be denoted respectively by the symbols pP, vV, pV, vP, and termed *double* permanences, *double* variations, permanence variations, variation permanences. The meaning of the simple symbols p, v, P, V speaks for itself.

* In chap. 2 of part 2 of the *Arithmetica Universalis*, entitled "De Formâ Æquationis."

† The substance of this lecture was communicated to the Mathematical Society of London (Professor De Morgan in the Chair), June 19, 1865.

Newton's rule in its complete form may be stated as follows:—On writing the complete series of quadratic under the complete series of simple elements of fx in their natural order, the number of double permanences in the associated series, or pair of progressions so formed, is a superior limit to the number of negative roots, and the number of variation permanences in the same is a superior limit to the number of positive roots in fx.

Thus the number of negative roots = or $< \Sigma pP$ } This is the Complete Rule as
„ „ „ „ positive roots = or $< \Sigma vP$ } given in other terms by Newton.

The rule for negative roots is deducible from that for positive, by changing x into $-x$.

As a corollary, the total number of real roots = or $< \Sigma pP + \Sigma vP$, that is, = or $< \Sigma P$.

Hence, the number of imaginary roots

$$= \text{or} > n - \Sigma P, \text{ that is, } = \text{or} > \Sigma V.$$

This is Newton's incomplete rule, or *first part* of complete rule, the rule as stated by every author whom the lecturer has consulted except Newton himself*.

By a group of negative signs, or a negative group, if we understand a sequence of negative signs, with no positive sign intervening, this incomplete rule may be stated otherwise, as follows:—

The number of imaginary roots of an algebraic function cannot be less than the number of negative groups in the complete series of its quadratic elements.

Arithmetical illustrations:—

Relation of Newton's complete rule to rule of Descartes. Newton's "Imaginary positives," "imaginary negatives" equivalent to ΣpV, ΣvV.

Newton's complete rule may be made to undergo its first step of generalization as follows:—

Let the two series of simple and quadratic elements of $f(x+\lambda)$ be formed, and the double permanences due to this transformation, say $\Sigma pP(\lambda)$, or more briefly, $pP(\lambda)$, be called the number of double permanences *proper* to λ, and in like manner $pP(\mu)$ the number of the same *proper* to μ.

* In some cases this rule appears to give very little information. For example, If we take the equation $x^3+3qx+r=0$, its immediate application will only show that if all the roots are real $q<0$. If, however, we make $y=x^{\lambda}$ and apply the criteria to the transformed equation in y, giving λ successive values between 1 and ∞, the rule will in fact lead (though by a difficult process) to the true discriminative criterion; this example serves to show that there is a deeper significance seated in the Newtonian criteria than does at first sight appear.

[N.B. $pP(0)$ becomes the notation for what has been termed above ΣpP.]

Then we have the theorem following:—

Supposing $\mu > \lambda$,

$$pP(\mu) = \text{or} > pP(\lambda),$$

or more exactly,

$$pP(\mu) - pP(\lambda) = (\mu, \lambda) + 2k,$$

where (μ, λ) denotes the number of real roots included between μ and λ, and k is zero or any positive integer.

Statement of this theorem in general terms.

This is to Newton's what Fourier's is to Descartes's.

Fourier's theorem recalled; it may be stated as follows:—

Form the *simple* elements appertaining to $f(x+\lambda)$ and to $f(x+\mu)$; then

$p(\mu) - p(\lambda) = (\mu, \lambda) + 2k$, briefly $p(\mu) - p(\lambda) =$ or $> (\mu, \lambda)$; as a consequence

$p(0) - p(-\infty) = p(0)$ for $p(-\infty) = 0$; hence permanences in fx = superior limit to number of negative roots;

$p(\infty) - p(0) = v(0)$ for $p(\infty) = n$; hence variations in fx = superior limit to number of positive roots.

So from the new theorem, briefly No. 1 Theorem (presently to be established in a more general form), namely,

$$pP(\mu) - pP(\lambda) = \text{ or } > (\mu, \lambda),$$

since

$$pP(0) - pP(-\infty) = pP(0); \text{ for } p(-\infty) = 0, \ pP(-\infty) = 0,$$

we obtain

$$pP(0) = \text{or} > (0, -\infty).$$

On examination it will be found that $P(+\infty) = P(-\infty) = n$ or $(n-2)$ according as the second quadratic element in $P(0)$ is positive or negative*. Thus

$$pP(\infty) - pP(0) = n - pP(0) = vV(0) + vP(0) + pV(0) = \text{or} > (\infty, 0),$$

or else

$$pP(\infty) - pP(0) = n - 2 - pP(0) = vV(0) + vP(0) + pV(0) - 2 = \text{or} > (\infty, 0),$$

which is not what is wanted; but by changing x into $-x$, and thereby commuting the variations and permanences of the simple elements of $f(x)$ one

* If $fx = (a_0, a_1, a_2, \ldots a_n \between x, 1)^n$, and if $B = a_1^2 - a_0 a_2$, and $h = \pm\infty$, the series of quadratic elements (when in each term only the highest power of h is retained) will be found to be a^2; Bh^2; Bh^4; ... Bh^{2n-2}; a^2h^{2n}.

into the other, the rule for the negative roots gives the rule required, namely, $vP(0) =$ or $> (\infty, 0)$ *.

No. 1 theorem may be transformed or, rather, otherwise stated as follows:—

The simple elements of $f(x+\lambda)$ are in fact

$$\frac{f^{(n)}\lambda}{1.2\ldots n};\ \frac{1}{n}\frac{f^{(n-1)}\lambda}{1.2\ldots(n-1)};\ \frac{1.2}{n(n-1)}\frac{f^{(n-2)}\lambda}{1.2\ldots(n-2)};\ldots;\ \frac{1}{n}\frac{f'\lambda}{1};\ f\lambda.$$

It will not affect the successions either in this series itself or the derived series of quadratic elements if we reject the common factor $\frac{1}{\Pi(n)}$ from each term. It then becomes

$$f^n\lambda;\ f^{n-1}\lambda;\ 1.2f^{n-2}\lambda;\ 1.2.3f^{n-3}\lambda;\ 1.2.3.4f^{n-4}\lambda;\ \ldots;\ 1.2\ldots nf\lambda;\ \text{etc.} \quad \text{(A)}$$

and similarly rejecting the positive factors 1^2, $(1.2)^2$, $(1.2.3)^2$, etc., from the second, third, fourth derived terms respectively, the derived series may be written

$$G_n\lambda;\ G_{n-1}\lambda;\ G_{n-2}\lambda;\ \ldots\ldots\ldots;\ G\lambda, \qquad \text{(B)}$$

where in general

$$G_r\lambda = (f^r\lambda)^2 - \gamma_r f^{r-1}\lambda . f^{r+1}\lambda,$$

γ_r denoting the fraction $\dfrac{n-r+1}{n-r}$.

Theorem No. 2 stated. If γ_r in the above association of series $\begin{Bmatrix}A\\B\end{Bmatrix}$ be subject to satisfy the equation in differences $2-\gamma_r = \dfrac{1}{\gamma_{r+1}}$, provided γ_r remains always positive from $r=n$ to $r=1$ inclusive †, and if we call the number of double permanences in $\begin{Bmatrix}A\\B\end{Bmatrix}$, $pP(\lambda)$, then it is still true that

$$pP(\mu) - pP(\lambda) = (\mu, \lambda) + 2k.$$

The theorem will be subsequently simplified by integrating the above equation, and will be shown to include theorem No. 1.

* So in general $vP(\lambda) - vP(\mu) = (\mu, \lambda) + 2k'$, where k' is zero or a positive integer. We have thus a second theorem as general as the first, and the two will give different limits unless $k = k'$, that is, unless $P(\mu) = P(\lambda)$, for $2(k-k') = P(\mu) - P(\lambda)$. There is nothing corresponding to this in Fourier's theorem; for the two inequalities

$$p(\mu) - p(\lambda) = \text{ or } < (\mu, \lambda), \quad v(\lambda) - v(\mu) = \text{ or } < (\mu, \lambda)$$

constitute not distinct but identical assertions.

† As regards the necessity of the extreme limits n and 1, observe that unless γ_n were made positive the product of $\begin{bmatrix}f^{r+1}\epsilon & f^r\epsilon\\ G_{r+1}\epsilon & G_r\epsilon\end{bmatrix}$ (see C, p. 502) would not follow the sign of ϵ for the case of $r=n-1$, $G_{n-1}=0$; and unless γ_1 were made positive, $f''.f$ would not be positive (see p. 504) when $G_1=0$; consequently the three final associated pairs of signs as x increases might pass through the successive phases

$$\begin{matrix}+&-&-\\+&+&+\end{matrix}\ \Bigg|\ \begin{matrix}+&-&-\\+&0&+\end{matrix}\ \Bigg|\ \begin{matrix}+&-&-\\+&-&+\end{matrix}\ \Bigg|,$$

and thus a double permanence would be *lost* in the ascending transit.

A theorem No. 3 exists derived from an order of considerations into which this Lecture will not enter. It gives much greater generality to the value of γ_r by the introduction of a second arbitrary parameter into the criteria: see footnote *, p. 508.

I proceed to establish theorem No. 2 by a method precisely analogous to that used in establishing Fourier's simpler one.

For brevity, by f^r understand $f^r x$; by $f^r(\pm\epsilon)$, understand $f^r(x\pm\epsilon)$; and so by G_r, $G_r(\pm\epsilon)$ understand $G_r x$, $G_r(x\pm\epsilon)$.

ϵ will denote an infinitesimal.

$$\left.\begin{array}{l}\text{When } f^r=0,\ f^r\epsilon=\epsilon f^{r+1}.\\ \text{When } f^r=0,\ f^{r+1}=0\ldots f^{r+i-1}=0, \text{ then } f^r\epsilon=\dfrac{\epsilon^i}{1.2\ldots i}.f^{r+i}.\end{array}\right\}$$

When $G_r=0$, $G_r(\epsilon)=\epsilon\dfrac{d}{dx}G_r$.

But
$$(f^r)^2-\gamma_r(f^{r-1})(f^{r+1})=0$$

and
$$\frac{d}{dx}G_r=(2-\gamma_r)f^rf^{r+1}-\gamma_rf^{r-1}f^{r+2}$$
$$=(2-\gamma_r)\frac{f^r}{f^{r+1}}\left\{(f^{r+1})^2-\frac{\gamma_r}{2-\gamma_r}.\frac{f^{r-1}.f^{r+1}}{f^r}f^{r+2}\right\}$$
$$=\frac{f^r}{\gamma_{r+1}f^{r+1}}\left\{(f^{r+1})^2-\gamma_{r+1}f^rf^{r+2}\right\},$$

or
$$G_r(\epsilon)=\frac{\epsilon}{\gamma_{r+1}}.\frac{f^r}{f^{r+1}}G_{r+1}{}^*. \qquad \text{(C)}$$

Similarly, when
$$G_r=0,\quad G_{r+1}=0\ \ldots\ G_{r+i-1}=0,$$
it may be proved that
$$G_r(\epsilon)=\frac{\epsilon^i}{\Pi(i)\,\gamma_{r+1}\gamma_{r+2}\cdots\gamma_{r+i}}\frac{f^r}{f^{r+i}}G_{r+i}{}^\dagger.$$

* Consequently if we write $\begin{bmatrix}f^{r+1}\epsilon & f^r\epsilon\\ G_{r+1}\epsilon & G_r\epsilon\end{bmatrix}$ the product of these four quantities obeys the sign of ϵ.

† For example, If $G_r=0$ and $G_{r+1}=0$, we have

$$\frac{dG_r}{dx}=0, \text{ and } \frac{d^2G_r}{dx^2}=(2-2\gamma_r)f^rf^{r+2}+(2-\gamma_r)(f^{r+1})^2-\gamma_rf^{r-1}f^{r+3}$$
$$=(2-2\gamma_r+2\gamma_{r+1}-\gamma_r\cdot\gamma_{r+1})f^rf^{r+2}-\gamma_rf^{r-1}.f^{r+3}.$$

But
$$2-2\gamma_r+2\gamma_{r+1}-\gamma_r\cdot\gamma_{r+1}=3-2\gamma_r=\frac{2}{\gamma_{r+1}}-1=\frac{1}{\gamma_{r+1}\cdot\gamma_{r+2}}.$$

Thus
$$\frac{d^2G_r}{dx^2}=\frac{1}{\gamma_{r+1}\cdot\gamma_{r+2}}\left\{f^rf^{r+2}-\gamma_r\gamma_{r+1}\gamma_{r+2}f^{r-1}.f^{r+3}\right\}$$
$$=\frac{f^r}{\gamma_{r+1}\gamma_{r+2}f^{r+2}}\left\{(f^{r+2})^2-\gamma_{r+2}f^{r+1}.f^{r+3}\right\},$$

and consequently
$$G_r(\epsilon)=\frac{f^r}{\gamma_{r+1}\gamma_{r+2}f^{r+2}}\cdot G_{r+2}\frac{\epsilon^2}{1.2}.$$

We can now trace the law of the change in the number of double permanences in the associated pair of series,

$$\left.\begin{matrix} f^n, & f^{n-1}, & f^{n-2}, \ldots f^1, & f, \\ G_n, & G_{n-1}, & G_{n-2}, \ldots G_1, & G, \end{matrix}\right\}, \text{ where } G_r = (f^r)^2 - \gamma_r f^{r-1} \cdot f^{r+1},$$

as x increases *continuously*.

No change can take place except at the instant when one or more of the terms in the inferior or superior series, or in both, simultaneously become zero.

1°. Suppose a single term in the upper series as f^r to become zero.

Writing down the sequence f^{r+1}, f^r, f^{r-1} in conjunction with the associated terms,

$$G_{r+1}, \ G_r, \ G_{r-1},$$

G_{r+1}, G_{r-1} are seen to be necessarily positive, and G_r of the contrary sign to f^{r+1}, f^{r-1}.

The number of cases depending on the signs of f^{r+1} and f^{r-1} are four, but reducible to two essentially distinct ones as below *,

$$\left\{\begin{matrix} + & 0 & + \\ + & - & + \end{matrix}\ \middle|\ \begin{matrix} + & 0 & - \\ + & + & + \end{matrix}\right\} \qquad \text{(D)}$$

For $(x-\epsilon)$ these become respectively

$$\begin{matrix} + & - & + \\ + & - & + \end{matrix}\ \middle|\ \begin{matrix} + & - & - \\ + & + & + \end{matrix}$$

For $(x+\epsilon)$ they become

$$\begin{matrix} + & + & + \\ + & - & + \end{matrix}\ \middle|\ \begin{matrix} + & + & - \\ + & + & + \end{matrix}$$

And there is neither gain nor loss of double permanences.

2°. Suppose a single term in the lower series to become zero. From the value of G_r in terms of f^{r+1}, f^r, f^{r-1}, it follows that the two extremes of these three must have the same sign; and the signs of f^{r+1}, f^r, G_{r+1}, G_{r-1} give rise to sixteen cases reducible to the four following essentially distinct ones †:—

$$\begin{matrix} + & + & + \\ + & 0 & + \end{matrix} \qquad \begin{matrix} + & - & + \\ + & 0 & + \end{matrix} \qquad \begin{matrix} + & + & + \\ + & 0 & - \end{matrix} \qquad \begin{matrix} + & - & + \\ + & 0 & - \end{matrix}\ddagger$$

* For the simultaneous reversal of the signs of the upper line will not affect the reasoning.

† For neither the reversal of the signs in the upper nor in the lower line will affect the reasoning.

‡ The number of occurrences of the combination $\begin{matrix} + & + & + \\ + & 0 & + \end{matrix}$ as x ascends from λ to μ will, except for special cases, be the value of k, and the number of occurrences of $\begin{matrix} + & - & + \\ + & 0 & + \end{matrix}$, in like manner, the value of k'; k, k' having the meanings attributed to them at footnote *, p. 501.

Immediately *after* the transit of the root, see footnote *, p. 502, these become of the form

+ + +	+ − +	+ + +	+ − +
+ + +	+ − +	+ + −	+ − −

And immediately *before* the transit they were of the form

+ + +	+ − +	+ + +	+ − +
+ − +	+ + +	+ − −	+ + −

In the first of the four cases there is a *gain* of two double permanences in ascending from $x-\epsilon$ to $x+\epsilon$; in the other three cases there is neither gain nor loss.

Thus for a single vanishing of an *intermediate* term in the upper or lower series double permanences may be gained, as x continuously *increases*, but can never be lost.

The same conclusion may be also established in a similar manner when several consecutive terms of the lower series forming a group vanish simultaneously, without the associated upper terms any of them vanishing; and also when *two* or more consecutive terms in the upper series vanish, which necessitates all the associated terms in the lower series also vanishing; and the law which limits the increase may be ascertained, and such increase may be shown to be always an even number.

But these cases are singular cases, and may be met at once by the consideration (equally applicable to the proof of Fourier's simpler theorem) that if two or more functions of x depending on fx vanish simultaneously, this must be by virtue of one or more relations existing between the coefficients of fx; and by giving infinitesimal variations to the coefficients, we shall leave the criteria [the two sets of terms corresponding to given limits] virtually undisturbed, and may manage that the coincidence of the transits will no longer take effect; at the same time *in general* the character of the roots will remain unaltered as regards the number of real and imaginary ones; so that the singular cases come under the operation of the same law as the general case. There is, however, an apparent possible exception to this reasoning, namely, when fx possesses one or more groups of equal roots, in which case an infinitesimal variation in the coefficients *may* be accompanied with a change of character in the roots, such as a passage from real equal to imaginary pairs: to meet this objection without embarrassing oneself with intricate considerations as to the signs to be given to the infinitesimal variations in order to avoid liability to such change, it is better to adopt the same kind of proof as is usual with Fourier's method, which there is no difficulty in doing by aid of the expressions given above with reference to the value of $f^n(\epsilon)$, and consequently also of $f^{n+1}(\epsilon), f^{n+2}(\epsilon), \ldots f^{n+i-2}(\epsilon)$, when f^n, f^{n+1}, $\ldots f^{n+i-1}$ are all zeros, from which may easily be deduced also the special expressions for $G_n(\epsilon)$, $G_{n+1}(\epsilon)$, ... $G_{(n+i-2)}(\epsilon)$ applicable to this case; and again, as regards the case when a group of lower terms vanish without the associated upper ones so doing, by aid of the general expressions given for the G functions last above written. But time would not suffice for going into these details in a single lecture*.

We must, lastly, consider what happens when one or more of the terms at either extremity vanish.

f^n and G_n are constants, and can never vanish, and G is a square, and essentially positive.

But suppose x to become a root of fx, so that $f=0$, then the last pair of couples in the associated series immediately before the transit will be $\begin{matrix} f'; & -\epsilon f' \\ f'^2; & \epsilon^2 f'^2 \end{matrix}$, and immediately subsequent to the transit $\begin{matrix} f'; & \epsilon f' \\ f'^2; & \epsilon^2 f'^2 \end{matrix}$; thus there will be one double permanence *gained* when a simple root is passed over.

Suppose now x to become a root of the ith order of repetition, equivalent, that is, to i roots passed over, so that $f=0$, $f^1=0$, $f^2=0$, ... $f^{i-1}=0$, then the last $i+1$ superior terms of the Association become, immediately after the transit,

$$f^i; \ \frac{\epsilon}{1}f^i; \ \frac{\epsilon^2}{1.2}f^i; \ \frac{\epsilon^3}{1.2.3}f^i; \ \ldots\ldots\ldots \ \frac{\epsilon^i}{1.2\ldots i}f^i,$$

tantamount to

$$1; \ \epsilon; \ \frac{\epsilon^2}{1.2}; \ \frac{\epsilon^3}{1.2.3}; \ \ldots\ldots\ldots \ \frac{\epsilon^i}{1.2\ldots i};$$

and the lower terms associated therewith, rejecting the common factor f^i, and certain obviously superfluous positive numerical factors besides, become

$$1; \ \left(1-\frac{\gamma_{i-1}}{2}\right)\epsilon^2; \ \left(1-\frac{2\gamma_{i-2}}{3}\right)\epsilon^4; \ \ldots \left(1-\frac{(1-i)\gamma_1}{i}\right)\epsilon^{2i-2}; \ \epsilon^{2i}.$$

By hypothesis $2-\gamma_{i-1}>0$. Hence there is obviously *one* double permanence, namely at the first couple of pairs above written, corresponding to a value of x immediately greater than a multiple root, whereas for a value of x immediately less than such root, there is no double permanence at all corresponding, for the upper series will consist exclusively of variations of sign when ϵ is changed into $-\epsilon$; as regards the other terms in the lower series, they will not necessarily be all positives, unless, *in order to meet the case of* equal *roots in fx*, we determine the value of γ_r in the equation $2-\gamma_r=\dfrac{1}{\gamma_{r+1}}$, subject to the condition that γ_r shall be not only positive, but also subject to the limitation $\gamma_r<\dfrac{i+1-r}{i-r}$ for all values of i not superior to n

* See Appendix for summary of demonstration applicable to these hypotheses.

[r of course being supposed less than i]. This latter limitation, however, will eventually be seen to be included in the former*. This being the case, it follows that the number of double permanences appertaining to the associated series

$$f^n, \; f^{n-1}, \; f^{n-2}, \ldots f', \; f$$
$$G_n, \; G_{n-1}, \; G_{n-2}, \ldots G_1, \; G$$

will be increased by at least as many units as there are real roots, equal† or unequal, passed over as we ascend from λ to μ, and that the excess, if any, of the increase of such number over the number of real roots will be an even integer.

To determine the value of γ_r, make $\gamma_r = \frac{U_r}{U_{r+1}}$, then

$$2 - \frac{U_r}{U_{r+1}} = \frac{U_{r+2}}{U_{r+1}},$$

or $$U_{r+2} - 2U_{r+1} + U_r = 0,$$

of which the general solution is $U_r = A + B(r-1)$; so that

$$\gamma_r = \frac{A + B(r-1)}{A + Br}.$$

If we write $A = n$, $B = -1$, we obtain

$$\gamma_{n-1} = \frac{2}{1}, \quad \gamma_{n-2} = \frac{3}{2}, \; \ldots\ldots\ldots \; \gamma_2 = \frac{n-1}{n-2}, \quad \gamma_1 = \frac{n}{n-1}.$$

These are the values of γ_r in the theorem No. 1, and they satisfy the *two* conditions above stated. For (1) γ_r is positive, and (2)

$$\gamma_r = \frac{n-r+1}{n-r} < \frac{i-r+1}{i-r}$$

for all values of $i < n$‡.

Hence theorem No. 1 is contained in theorem No. 2, and Newton's theorem is contained in theorem No. 1, so that it is a corollary in the

* It is shown at p. 507, that $\gamma_r = \frac{\nu + r - 1}{\nu + r}$, ν having any value not intermediate between $-n$ and 0. When ν is positive $\gamma_r < 1 < \frac{i+1-r}{i-r}$.

Again, when ν is negative let $\nu = -\nu'$, then $\nu' =$ or $> n =$ or $> i$.

Hence $\gamma_r = \frac{\nu' - r + 1}{\nu' - r} =$ or $< \frac{i+1-r}{i-r}$, and the required condition is still satisfied. The sole exception is when $\nu = -n$ and $i = n$, that is, when the original Newtonian criteria are those employed, and the equation has all its roots equal to one another, for which case see footnote ‡, below.

† Each root repeated m times counts as m roots.

‡ The degree of fx being n, i is necessarily less than n, unless all the roots are equal, a case which may be considered excluded, as then all the Newtonian criteria become zero.

second order of derivation from our theorem No. 2. The general value of γ_r is $\dfrac{\nu+(r-1)}{\nu+r}$, ν being any real quantity whatever not *intermediate* between 0 and $-n$. To obtain theorem No. 1 we must make $\nu=-n$. In that case $\gamma_n=\dfrac{1}{0}$ and accordingly when $G_{n-1}=0$, $G_{n-1}(\epsilon)$ by the formula at p. 502 should vanish. It will easily be found that for this peculiar value $G_{n-1}(x)$ becomes independent of x, in fact a constant*.

Theorem No. 2 may also be stated as follows:—

If $$c_0+c_1x+c_2x^2+\ldots\ldots+c_nx^n=f(x),$$

and ν be any real quantity not *intermediate* between 0 and $-n$, and if

$$c_0,\ \frac{c_1}{\nu},\ \frac{1.2c_2}{\nu(\nu+1)},\ \frac{1.2.3c_3}{\nu(\nu+1)(\nu+2)},\ \ldots\ldots\ldots$$

say, $$\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ldots,$$

be taken as the simple elements of fx, and

$$\alpha_0{}^2,\ \alpha_1{}^2-\alpha_0\alpha_2,\ \alpha_2{}^2-\alpha_1\alpha_3,\ \ldots\ldots\ \alpha^2{}_{n-1}-\alpha_{n-2}\alpha_n,\ \alpha_n{}^2,$$

say, $$A_0,\quad A_1,\quad A_2\ \ldots\ldots\ldots\ldots\ldots\ A_{n-1},\quad A_n,$$

as the quadratic elements of the same; and if we understand by the (cA) association the paired series

$$\begin{Bmatrix} c_0\ c_1\ \ldots\ldots\ldots\ c_n \\ A_0A_1\ldots\ldots\ldots A_n \end{Bmatrix},$$

and if $pP(\lambda)$ signifies the number of double permanences in the (cA) association corresponding to $f(x+\lambda)$, then

$$pP(\mu)-pP(\lambda)=(\mu,\lambda)+2k\dagger,$$

where k is zero or some positive integer.

When $\nu=-n$, this theorem becomes effectively identical with theorem No. 1 in its original form.

The existence of a theorem No. 3, including No. 2, and containing two arbitrary parameters, becomes apparent from the consideration that $fx=0$ will have the same *finite* roots as $Fx=0$, where

$$Fx=\epsilon x^{n+i+j}+x^ifx+\eta,$$

* Analogous observations apply to the other extreme or limiting case, namely, that where ν is made equal to zero, for then $\gamma_1=0$ so that $G_1=(f^1)^2$, and is always positive, so that when f^1 becomes 0, for this particular system of γ's, the three last couples in the pair of progressions before such transit may be $\begin{matrix}+&-&+\\+&+&+\end{matrix}$, and after transit $\begin{matrix}+&+&+\\+&+&+\end{matrix}$; and thus contrary to what usually happens (see p. 503), there may on the above hypothesis be an *extra* gain of two double permanences and loss of two variation permanences, thus increasing the values of k and k' in footnote *, p. 501, which of course does not affect the validity of the theorem.

† And of course also $vP(\lambda)-vP(\mu)=(\mu,\lambda)+2k'$, where k' is zero, or some positive integer.

ϵ, η being two infinitesimals: this brings in two parameters, which, so far as this particular method of demonstration of their existence applies, would seem to be necessarily integer, but, from the nature of the question, must obviously admit of some wider definition*.

The labours of all preceding writers on this subject have been confined exclusively to the *imperfect* form of Newton's theorem; nor previously to the lecturer's communication of his Trilogy of Algebraical researches to the Royal Society of this year was anything more made out of it than to show that if *any* negative terms occur in the quadratic elements, there must be *some* imaginary roots†; but this becomes immediately apparent if we consider f a homogeneous function of the nth degree in x and y; for then the number of imaginary roots of $\frac{x}{y}$ or $\frac{y}{x}$ in $f(x, y)$ cannot be fewer than in

* If we use theorem No. 2, j may be made zero (it will be found) without any loss of generality; in fact if it be made greater than zero, the result obtained will be included in that obtained by making it equal to zero. On the other hand, if we use theorem No. 1, retaining for i, j their general values, the result obtained will be the same as that which flows from the use of theorem No. 2, with $j=0$, except that it will be specialized by ν being restricted to integer values only. By aid of the method above indicated, we may substitute as the values of $a_0, a_1, a_2, \ldots a_n$ in p. 507, [see *Erratum*, p. 513]

$$c_0;\ \frac{i+1}{\nu}c_1;\ \frac{(i+1)(i+2)}{\nu(\nu+1)}c_2;\ \frac{(i+1)(i+2)(i+3)}{\nu(\nu+1)(\nu+2)}c_3;\ \ldots\ldots$$

provided $\nu =$ or $> i$ or $\nu =$ or $< -n$ and $i =$ or > 0. The particular form of demonstration indicated requires i to be integer; but this restriction I have great reason to believe, indeed have scarcely a doubt, is unnecessary and may be neglected.

$$1;\ \frac{\nu}{i+1};\ \frac{\nu(\nu+1)}{(i+1)(i+2)};\ \ldots\ldots$$

may be termed a divestible system of coefficients; such system is a generalization of the ordinary binomial system

$$1;\ n;\ \frac{n(n-1)}{2};\ \ldots\ldots$$

If we call $\alpha, \beta, \gamma \ldots$ any system of the like nature, and form the equation

$$\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \ldots\ldots\ldots = 0,$$

it is obvious, or at least very readily proved, that the *simple* roots of this equation must all, or all but one, be imaginary. It is not unlikely that every system $\alpha, \beta, \gamma \ldots$ which satisfies the above condition and one or more other general conditions, may be employed as a *divestible system*. The particular system, involving two arbitrary parameters, above given will be found to satisfy the further condition that *all* its Newtonian non-trivial criteria (the $A_1, A_2, \ldots\ldots A_{n-1}$ of p. 488) become negative. When $n=2$ or $n=3$, the second condition implies the first; and for these cases it is easily proved that every system of quantities which satisfies the second (and therefore the first condition) forms a valid system of divestible factors. If we make $i=0$ and $\nu=1$, we learn that the equation

$$1 + x + x^2 + \ldots + x^n = 0$$

can never have more than one real root; if we make $i=0$ and $\nu=\infty$, we learn the same of the equation

$$1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots\ldots + \frac{x^n}{1.2\ldots n} = 0.$$

Newton's own rule would only, in these and such like examples, reveal the existence of a single pair of imaginary roots.

† *See Postscript.*

$\left(\frac{d}{dx}\right)^i\left(\frac{d}{dx}\right)^j f(x, y)$, i and j being *any* two integers: making $i+j=n-2$, and giving j all values from 0 to $n-2$ in succession, it will readily be seen that Newton's criteria are the quantities which respectively serve to determine whether the quadratic *functions* obtained by these various substitutions have real or imaginary roots. If any one of them is negative, one of those derivatives has imaginary roots, and therefore the primitive function $f(x, y)$ has at least one such pair*.

Newton's assertion (if it is his own assertion), that his rule will in *general* give the *actual* number, and not merely a superior limit to such number of real roots, is certainly more than questionable. For if $f'x$, $f''x$, $f'''x$, $f''''x$, $f'''''x$ have respectively not more than i_1, i_2, i_3, i_4, i_5 pairs of imaginary roots, this rule *cannot* reveal the existence of more than j pairs, where j is the *least* number of the set of numbers $i_1+1, i_2+1, i_3+2, i_4+2, i_5+3, \ldots\ldots$

Geometrical illustration.

Ortæ à Cartesio, quam Newtonus insuper auxit,
Doctrinae, en! demum, fons et origo patent.

Postscript.

Dr J. R. Young contests the accuracy of the assertion on p. 508, and claims to have demonstrated Newton's rule twenty years ago. Call the equation

$$(a, b, c, d, e, \ldots\ldots \between x, 1)^n = 0\,;$$

the derived cubics are of course to a factor *près* respectively

$$(a, b, c, d \between x, 1)^3;\ (b, c, d, e \between x, 1)^3;\ (c, d, e, f \between x, 1)^3;\ \&c.$$

* In like manner, if $[a, b, c, d]$ represent the discriminant with its sign reversed of

$$(a, b, c, d \between x, y)^3,$$

the fact of any of the quantities

$$[a, b, c, d],\quad [b, c, d, e],\quad [c, d, e, f], \&c.$$

becoming negative will imply the existence of some imaginary roots in

$$(a, b, c, d, e, f \ldots \between x, y)^n,$$

and so in general. One would be glad to know whether by aid of a complete table of the discriminants of a function of the nth order and of its successive derivatives (respectively 2, 3, ... $(n-1)$ in number), it is possible or not to ascertain the exact number of its real roots.

When n is 4 the only dubious case arising under such table is that where the discriminant of the quartic itself is positive but those of its two derived cubics and three derived quadratics each negative. In such case it remains to be ascertained whether it is true or not that the roots of the quartic cannot all be imaginary, or (which is here the same thing) must all of necessity be real.

It seems desirable to show *à priori* that when the roots of

$$fx = x^n + npx^{n-1} + \tfrac{1}{2}n(n-1)qx^{n-2} + \&c.$$

are all real, the criteria p^2-q, q^2-pr, &c. are necessarily all positive. This endoscopic method of proof I have not yet been able to complete; but I have noticed that if $\alpha, \beta, \gamma, \delta \ldots$ are the roots of fx, and

$$\frac{fx}{(x-\alpha)(x-\beta)} = x^{n-1} + \lambda x^{n-2} + \mu x^{n-3} + \ldots,$$

the following somewhat curious relations obtain, namely,

$$p^2 - q = \Sigma[(\alpha-\beta)^2(\lambda^2-\mu)];\quad q^2 - pr = \Sigma[(\alpha-\beta)^2(\mu^2-\lambda\nu)], \&c.$$

It is with these cubics that Dr Young, in his argument (if it may be called so) exclusively deals. Omitting merely superfluous observations his ratiocination runs as follows:—"If any of these limiting cubics indicate imaginary roots when submitted to the criteria, such indications will imply imaginary roots in the proposed equation. But several indications, apparently distinct, may offer themselves in those equations which, upon closer examination, may be found to be necessarily dependent or concurrent. Distinct imaginary pairs can of course be inferred only from independent and non-concurring conditions. We have therefore to inquire how these are to be discovered in the above series of equations. And first we may remark that since only one imaginary pair can enter into a cubic equation, it follows that whether the criterion of imaginary roots is satisfied by the three leading terms of any of the above cubics or by the three final terms, or simultaneously by both sets of three, one imaginary pair, and one only, is implied. Hence when both sets of three terms, furnished by any cubic, fulfil the proposed conditions, these conditions, though really independent, that is, not necessarily implied one in the other, nevertheless necessarily concur in indicating the same thing. Thus only a single imaginary pair can be inferred from any one of the limiting cubics, whether the criterion is satisfied for one set of three terms or for the two consecutively.

. . . .

"If the first set of three (that is, the leading terms in the first set) satisfy the criterion, we can immediately infer the existence of one imaginary pair. If the next set (the final terms of the same cubic) also satisfy it, the preceding condition merely recurs, and supplies no additional information. In this case the following set of three, the leading terms of the next cubic..., furnish the same concurring condition..., and so on, until we arrive at a set of three terms for which the condition fails, thus putting a stop to the series of concurring indications, and preparing the way for new and distinct conditions altogether unconnected with the former. As soon as the criterion again holds, the condition being thus entirely independent of, and unconnected with, the former, must imply another and distinct imaginary pair, and so on to the end of the series." Dr Young subsequently goes on to observe that the criteria of the successive cubics (discarding repetitions) are identical with those of the original equation, which he says, "we now know to be so connected together, that if, when proceeding from one set of three terms in the equation to the three next in order, the consecutive criteria both have place, the recurrence is to be regarded merely as a second indication of the same thing—the existence of a single imaginary pair; and that as soon as the condition fails, preparation is made for a new and independent indication, and so on, until all the sets of three have been examined. Hence the indications that are really non-concurrent, and consequently the number of imaginary roots inferrible from them, may be noted" according to a rule "which is virtually the same as that of Newton."

This is the sum and substance, in a simplified form, of Dr Young's so-called proof. "It is such stuff as dreams are made of," and, culminating as it does in a palpable *petitio principii*, does not need a detailed refutation at the hands of the author of this lecture. It is not by such vague rhetorical processes, but by quite a different kind of mental toil, that the truths of science are to be won, or a way opened to the inner recesses of the reason.

Appendix on the singular cases referred to in the text foregoing, see p. 504.

It may easily be shown that there are only four hypotheses admissible concerning vanishing f's and G's.

1°. f^r may vanish, but not the adjacent terms nor the associated term G_r.

2°. G_r may vanish, but not the adjacent terms nor the associated term f^r.

3°. $G_r, G_{r+1}, \ldots G_{r+i-1}$ (i being greater than unity) may vanish *without any of* the associated terms vanishing.

4°. $f^r, f^{r+1}, \ldots f^{r+i-1}$ (i being greater than unity) may vanish, and *consequently* also $G_r, G_{r+1}, \ldots G_{r+i-1}$ *all* vanish.

Hypotheses (1) and (2), which are the only cases that can happen in general, have been discussed in the text preceding.

Now consider the 3rd hypothesis.

1st. Suppose i any even number (say 4), so that

$$G_r = 0,\ G_{r+1} = 0,\ G_{r+2} = 0,\ G_{r+3} = 0,$$

then $$f^{r-1},\ f^{r+1},\ f^{r+3} \text{ have the same sign } inter\ se,$$

and $$f^r,\ f^{r+2},\ f^{r+4} \text{ have the same sign } inter\ se,$$

and there will be four essentially distinct cases as below, representing the partial association

$$\begin{Bmatrix} f^{r-1}, & f^r, & f^{r+1}, & f^{r+2}, & f^{r+3}, & f^{r+4} \\ G_{r-1}, & G_r, & G_{r+1}, & G_{r+2}, & G_{r+3}, & G_{r+4} \end{Bmatrix}$$

at the moment of x taking the critical value which causes the G's to vanish, namely—

$$\begin{array}{cccccc|cccccc|cccccc|cccccc|} + & + & + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & -. & + & - & + & - & + & - \\ + & 0 & 0 & 0 & 0 & + & + & 0 & 0 & 0 & 0 & - & + & 0 & 0 & 0 & 0 & +. & + & 0 & 0 & 0 & 0 & - \end{array}$$

Immediately *after* the passage of the critical value these signs become

$$\begin{array}{cccccc|cccccc|cccccc|cccccc|} + & + & + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & - & + & - & + & - \\ + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & + & + & - & + & - & + & - \end{array}$$

as appears from the expression for $G_r(\epsilon)$ in terms of G_{r+i}, f^r, f^{r+i} given at p. 502; and immediately *before* the passage these partial associations are of the form

$$\begin{array}{cccccc|cccccc|cccccc|cccccc|} + & + & + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & + & - & + & + & + & - & + & - & + & - & + & + & + & + & + & + & + & + & + & + & + & - \end{array}$$

There will thus be a gain of 4 double permanences in passing upwards through the critical values in the two first cases, and neither gain nor loss in the two last.

2nd. If i is any odd number, say 3, there will still be 4 cases, for which respectively the signs after transit will be

$$\begin{array}{ccccc|ccccc|ccccc|ccccc} + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & + & - & + & - & + \\ + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & + & - & + & - & - \end{array}$$

and before transit,

$$\begin{array}{ccccc|ccccc|ccccc|ccccc} + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & + & - & + & - & + \\ + & - & + & - & + & + & - & + & - & - & + & + & + & + & + & + & + & + & + & - \end{array}$$

showing a gain in the upward passage of 4, 2, 0, 0 double permanences respectively.

Lastly, let us consider the 4th hypothesis.

1st. Suppose i is any even number (say 4), and that

$$f^r = 0,\ f^{r+1} = 0,\ f^{r+2} = 0,\ f^{r+3} = 0,$$

then, after the transit, the f and G associated series from $r+4$ to $r-1$ become, writing for brevity's sake ϕ for f^{r+4} and ψ for f^{r-1},

$$\phi;\quad \epsilon\phi;\quad \frac{\epsilon^2}{1\,.\,2}\phi;\ \frac{\epsilon^3}{1\,.\,2\,.\,3}\phi;\ \frac{\epsilon^4}{1\,.\,2\,.\,3\,.\,4}\phi;\quad \psi;$$

$$\phi^2;\ k_1\epsilon^2\phi^2;\ k_2\epsilon^4\phi^2;\quad k_3\epsilon^6\phi^2;\quad -\frac{\epsilon^3}{1\,.\,2\,.\,3}\gamma_r\phi\psi;\ \psi^2;$$

where γ_r is positive, and, as follows from what is proved in footnote *, p. 502, k_1, k_2, k_3 also will be all positive. Thus there will be essentially only two cases, according as ϕ, ψ have the same or contrary signs; after transit, the association will be of the form

$$\left|\begin{array}{cccccc} + & + & + & + & + & + \\ + & + & + & + & - & + \end{array}\right| \text{ or } \left|\begin{array}{cccccc} + & + & + & + & + & - \\ + & + & + & + & + & + \end{array}\right|$$

and before transit (changing ϵ into $(-\epsilon)$) of the form

$$\left|\begin{array}{cccccc} + & - & + & - & + & + \\ + & + & + & + & + & + \end{array}\right| \text{ or } \left|\begin{array}{cccccc} + & - & + & - & + & - \\ + & + & + & + & - & + \end{array}\right|$$

showing a gain of 2 double permanences, or else of 4 such.

2nd. If i is any odd number, as 3, the associated series corresponding to the values of r, from $r+3$ to $r-1$, using ϕ to denote f^{r+3}, will be

$$\phi;\quad \epsilon\phi;\quad \frac{\epsilon^2}{1.2}\phi;\quad \frac{\epsilon^3}{1.2.3}\phi;\quad \psi;$$

$$\phi^2;\ k_1\epsilon^2\phi^2;\ k_2\epsilon^2\phi^2;\ -\frac{\epsilon^2}{1.2}\gamma_r\phi\psi;\ \psi^2,$$

and thus the signs after transit will be

$$\left|\begin{matrix} + & + & + & + & + \\ + & + & + & - & + \end{matrix}\right| \quad \text{or} \quad \left|\begin{matrix} + & + & + & + & - \\ + & + & + & + & + \end{matrix}\right|$$

and before transit

$$\left|\begin{matrix} + & - & + & - & + \\ + & + & + & - & + \end{matrix}\right| \quad \text{or} \quad \left|\begin{matrix} + & - & + & - & - \\ + & + & + & + & + \end{matrix}\right|$$

showing a gain of 2 double permanences on either supposition in the ascent from $x-\epsilon$ to $x+\epsilon$. And so in general from the 3rd and 4th (the two singular) hypotheses, whatever the value of i may be, an even number of double permanences may be gained in passing upwards through a critical value of x, but none can ever be lost*: this completes the demonstration.

* The number of double permanences gained on the 3rd hypothesis will be in the 4 subcases respectively $2i$, $2i$, 0, 0 if $2i$ consecutive G's vanish, and $2i+2$, $2i$, 0, 0 if $2i+1$ of them vanish; on the 4th hypothesis, in the 2 subcases the respective numbers gained will be $2i-2$, $2i$ if $2i$ consecutive f's vanish, and $2i$, $2i$ if $2i+1$ of them vanish: this statement requires a slight modification for the particular form of the theorem No. 2 corresponding to $\nu=0$. It may be noticed that in an exhaustive study of this theorem two sorts of singularities occur separately or in combination, namely, those arising out of the form of the equation, and those imparted to the criteria by giving critical values to the limited arbitrary parameter.

ERRATUM.

Theorem 3 is stated erroneously in footnote (*), p. 508. Correctly stated it furnishes a wider generalization of Newton's own theorem than can be obtained directly from the theorem 1 or 2 of the text, but not a generalization of those theorems themselves. It should be as follows:—

If
$$fx = a_0 + \frac{\nu}{i+1}a_1x + \frac{\nu(\nu+1)}{(i+1)(i+2)}a_2x^2 + \&c.,$$

and we form the pair of progressions

$$a_0;\quad \nu a_1;\quad \nu^2 a_2;\quad \nu^3 a_3;\quad \ldots\ldots$$
$$a_0^2;\ a_1^2-a_0a_2;\ a_2^2-a_1a_3;\ a_3^2-a_2a_4;\ \ldots\ldots$$

where $\nu =$ or $> i$ or else $=$ or $< -n$,

then if i is a positive *integer* the number of double permanences is a limit to the number of the negative roots, and the number of variation permanences to that of the positive roots in fx. *Possibly* this theorem continues to subsist when i is *any* positive quantity.