88.

ON AN IMPROVED FORM OF STATEMENT OF THE NEW RULE FOR THE SEPARATION OF THE ROOTS OF AN ALGEBRAICAL EQUATION, WITH A POSTSCRIPT CONTAINING A NEW THEOREM.

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MY new rule (of which the demonstration will be found in a paper by the late lamented Mr Purkiss in the last Number of the *Cambridge, Oxford, and Dublin Mathematical Messenger*) for separating the roots of an algebraical equation, I mean the rule which bears to Newton's rule generalized the same relation as Fourier's to Descartes's, is susceptible of a certain slight improvement as regards the mode of statement, which appears to me deserving of notice.

If we suppose $fx=0$ to be the equation in the theorem as originally stated, I have employed the double progression

$$\begin{matrix} fx, & f_1x, & f_2x, & \dots & f_nx, \\ Gx, & G_1x, & G_2x, & \dots & G_nx, \end{matrix}$$

where f_rx means $\left(\frac{d}{dx}\right)^r fx$, and G_rx means $(f_rx)^2-\gamma_r f_{r-1}x \,.\, f_{r+1}x$, γ_r being a known function of r involving an arbitrary parameter, confined between limits of which one is dependent on n.

In applying the theorem, it becomes necessary to count the number of compound successions for which, on writing a given value α for x, $f_r \,.\, f_{r+1}$ and $G_r \,.\, G_{r+1}$ are both simultaneously positive, and also the number of the same for which $f_r \,.\, f_{r+1}$, and $G_r \,.\, G_{r+1}$ are simultaneously negative and positive respectively, the succession $\begin{array}{|c|c|} f_r & f_{r+1} \\ G_r & G_{r+1} \end{array}$ in the first case constituting what I have called a double permanence, and in the other case a variation-permanence. This latter is of course to be distinguished from a permanence-variation, which corresponds to the supposition of f_r, f_{r+1} bearing like, and G_r, G_{r+1} unlike signs—there being in fact four kinds of succession, namely, double permanences, variation-permanences, permanence-variations, and double variations.

If the enunciation of the theorem can be made to refer to double variations and double permanences exclusively, it is evident that something will have been gained in point of simplicity of statement*; and this can easily be effected in the manner following.

Let

$$H_r x = (f_r x)^3 - \gamma_r \cdot f_{r-1}x \cdot f_r x \cdot f_{r+1}x,$$

so that

$$H_r x = f_r x \cdot G_r x, \quad H_{r+1}x = f_{r+1}x \cdot G_{r+1}x;$$

then, when $f_r x$, $f_{r+1}x$ have the same sign, the nature of the succession $H_r x$, $H_{r+1}x$ will evidently be the same as that of $G_r x$, $G_{r+1}x$; but when $f_r x$, $f_{r+1}x$ have unlike signs, the nature of the succession $H_r x$, $H_{r+1}x$ will be contrary to that of $G_r x$, $G_{r+1}x$.

Accordingly when $\begin{array}{|c|c|}\hline f_r & f_{r+1} \\ G_r & G_{r+1} \\ \hline\end{array}$ constitutes a double permanence, $\begin{array}{|c|c|}\hline f_r & f_{r+1} \\ H_r & H_{r+1} \\ \hline\end{array}$ will also constitute a double permanence; but when $\begin{array}{|c|c|}\hline f_r & f_{r+1} \\ G_r & G_{r+1} \\ \hline\end{array}$ constitutes a variation-permanence $\begin{array}{|c|c|}\hline f_r & f_{r+1} \\ H_r & H_{r+1} \\ \hline\end{array}$ will constitute a variation-variation, that is, a double variation.

If, then, we take for our double progression

$$\left\{\begin{array}{lllll} fx, & f_1x, & f_2x, & \ldots & f_nx, \\ Hx, & H_1x, & H_2x, & \ldots & H_nx, \end{array}\right\}$$

the rule, or rather the independent pair of rules referred to, will take the following simplified form.

Supposing a, b to be any two real quantities in ascending order of magnitude, on substituting for x first a and then b, in the simultaneous progressions above written, double permanences (in passing from a to b) may be gained, but cannot be lost: double variations may be lost, but cannot be gained. And the number of real roots included between a and b will either be equal

* Moreover, so stated the theorem becomes more closely analogous to Fourier's. It may not be unreasonable to imagine that a third progression may remain to be invented such that the number of *triple* permanences and *triple* variations of sign in the three combined may afford a new superior limit, and so on *ad infinitum*; but this of course is at present a matter of pure conjecture.

or inferior to the number of double permanences so gained, and also equal or inferior to the number of double variations so lost—the difference, if there be any in either case, being some even number. The value of γ_r is $\frac{\nu+r-1}{\nu+r}$, where ν is limited not to fall within the limits 0 and $-n$. By ascertaining the gain of double permanences and the loss of double variations consequent on the replacement of a by b, we are furnished with two *independent* superior limits to the number of real roots included between a and b.

Postscript.

It often happens that the pursuit of the beautiful and appropriate, or, as it may be otherwise expressed, the endeavour after the perfect, is rewarded with a new insight into the true. So it is in the present instance; for the substitution of the H for the G series, devised solely for the purpose of giving greater clearness to the enunciation of a known theorem, leads to a supplemental theorem which combines with and lends additional completeness and harmony to the original one.

At present the theory stands thus: a superior limit to the number of real roots between two limits a and b is afforded by counting, as x ascends from the one to the other, the loss of changes or gain of permanences (these two numbers are identical) in the f or Fourierian progression, and also by counting the loss of double changes, or gain of double permanences, in the f and H progressions combined: these two are distinct. We have thus the choice of three superior limits. I shall show that a fourth independent one is afforded by considering the loss of changes or gain of permanences in the single H progression, and combining it with such loss or gain in the single f progression.

We have

$$H_r x = f_r x \,.\, G_r x,$$

where

$$G_r x = (f_r x)^2 - \gamma_r (f_{r-1} x)(f_{r+1} x),$$

γ_r being essentially positive for all values of γ.

It has been proved [p. 502, above] (see Mr Purkiss's paper above referred to) that, when $G_r x = 0$,

$$G_r(x+\epsilon) = \frac{\epsilon}{\gamma_{r+1}} \,.\, \frac{f_r x}{f_{r+1} x} G_{r+1} x,$$

ϵ being an infinitesimal.

Now suppose $H_r x = 0$. This may happen in two distinct ways, namely, either when $G_r x = 0$, or when $f_r x = 0$.

1. Let $G_r x = 0$, then

$$\frac{d}{dx} H_r x = f_r x \frac{d}{dx} G_r x.$$

Hence $$H_r(x+\epsilon) = \frac{\epsilon}{\gamma_{r+1}} \cdot \frac{(f_r x)^2}{f_{r+1}x} \cdot G_{r+1}x$$
$$= \frac{\epsilon}{\gamma_{r+1}} \left(\frac{f_r x}{f_{r+1}x}\right)^2 H_{r+1}x.$$

2. Let $f_r x = 0$, then
$$\frac{dH_r x}{dx} = G_r x \,.\, f_{r+1}x\,;$$
also $\quad G_{r-1}x = (f_{r-1}x)^2$; $G_r x = -\gamma_r \,.\, f_{r-1}x \,.\, f_{r+1}x$; $G_{r+1}x = (f_{r+1}x)^2$.
Thus $H_{r-1}x$, $H_r(x+\epsilon)$, $H_{r+1}x$ are conformable in signs to
$$f_{r-1}x, \quad -f_{r-1}x \,.\, \epsilon, \quad f_{r+1}x \qquad (\alpha)$$
in this case, and in the case preceding to
$$H_{r-1}x, \quad H_{r+1}x \,.\, \epsilon, \quad H_{r+1}x. \qquad (\beta)$$

The above cases have reference to any *intermediate* H becoming zero; the final H is $(fx)^3$, and the last but one is
$$(f'x)^3 - \gamma_1 (f''x \,.\, f'x \,.\, fx);$$
and accordingly when $fx = 0$, $H_1 x$, $H(x+\epsilon)$ become of the same signs as
$$f'x;\ \epsilon f'x. \qquad (\gamma)$$

By combining the results (α), (β), (γ), and denoting by ν the number of real roots included between (a) and (b), it is easy to infer the equation
$$\nu = P - 2\phi + 2\eta,$$
where P is the number of permanences gained in passing up x from a to b in the H progression, ϕ is the collective number of times that any intermediate G vanishes at a moment when the preceding and subsequent H's have like signs, and η is the collective number of times that any intermediate f vanishes at a moment when the two adjacent f's have like signs. But if p is the number of permanences gained from the f (Fourier's series) by passing up x from a to b, we have $\nu = p - 2\eta$, where η represents the same quantity as above.

Hence $$2\nu = P + p - 2\phi\,;$$
and accordingly there emerges a new superior limit to ν, namely, $\frac{P+p}{2}$, an unlooked-for and striking conclusion.

Thus, for example, if $p = P + 2$, ν cannot be greater than $P+1$, and therefore not greater than P, because it must differ from p or P (whose sum is necessarily even) by an even number*. To make the preceding demonstra-

* And so in general, when $p - P$ is positive and not divisible by $4p$, the superior limit given by Fourier's theorem may be replaced by $\frac{p+P}{2} - 1$.

tion absolutely rigorous, it would be necessary to consider the singular cases when several consecutive terms of the H or G series vanish simultaneously, either with or without the corresponding terms of the f series vanishing too: this inquiry, which is necessarily tedious, and the result of which it is easy to anticipate, must be adjourned to a more suitable occasion.

If we call Λ the new superior limit, we have found

$$\Lambda - \nu = \phi,$$

where ϕ is the collective number of values of x included between a and b for which any function $G_r x$ vanishes, whilst $H_{r-1}x$ and $H_{r+1}x$ have like signs; but since, when $G_r x = 0$, $f_{r-1}x$ and $f_{r+1}x$ must have like signs, ϕ may be defined more simply as the number of values of x between the given limits for which simultaneously, and for any value of r, $G_r x$ vanishes whilst $G_{r-1}x$ and $G_{r+1}x$ have like signs.

This quantity ϕ, the difference between the limit and the number of roots limited, may be odd or even, and not necessarily the latter, as is the case in all existing theorems of a similar nature.

Since $\Lambda = \frac{p+P}{2}$, it follows that when $P = 0$, that is, whenever the passage from a to b leaves the number of permanences in the H series unaltered, the limit p given by Fourier's theorem may be replaced by $\frac{p}{2}$ or $\frac{p}{2} - 1$, according as p is or is not divisible by 4.