89.

NOTE ON THE PERIODICAL CHANGES OF ORBIT, UNDER CERTAIN CIRCUMSTANCES, OF A PARTICLE ACTED ON BY A CENTRAL FORCE, AND ON VECTORIAL COORDINATES, ETC., TOGETHER WITH A NEW THEORY OF THE ANALOGUES TO THE CARTESIAN OVALS IN SPACE, BEING A SEQUEL TO "ASTRONOMICAL PROLUSIONS."

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A VERY singular and previously unnoticed species of discontinuity arises when, according to the equations of motion interpreted in the ordinary manner, a particle solicited by a continuous central force *would seem* as if it ought to describe an orbit external to the force-centre. An instance of this kind, probably for the first time, presented itself in a question incidentally brought forward by myself in the paper inserted in the January number of this *Magazine,* where* I alluded, in passing, to the case of a body acted on by a central force capable of making it move in a circle exterior to the force-centre, and fell into the not unnatural error, which has since been pointed out to me, and which is obvious on a moment's reflection, of stating that on arriving at a point where the motion points to the force-centre, that is, at the point where the tangent to the circle passes through this centre, the particle would go off in a straight line on account of the motion and the force coinciding in direction. But it is clear that since the instantaneous area $\frac{1}{2}\rho^2\frac{d\theta}{dt}$ remains finite at such point, it cannot abruptly become zero; the radial velocity becoming infinite, does not entitle us to reject the transverse part which remains finite; thus the radius vector ρ will continue to revolve in the same direction as before it reached the tangential point; it will therefore swing off to another curve, so that the true orbit will possess an inflexion at that point. The new curve, it may easily be proved, will be a circle equal to the former, and related to it in the manner following: let us suppose O to be the force-centre and two tangents drawn from O to meet the original circle in A and B, so that the line AB divides the circle into two unequal

[* p. 538, above.]

segments, and that the particle has been travelling, say in the upper segment, from A to B; draw the angle BOC equal to the angle AOB, and in it place a circle equal to the former, touching the part OB, OC in B and C; then the particle will describe the *lower* segment of this new circle; and so in like manner, after reaching C, will undergo a new inflexion at that point and pass on to a new circle touching OC and OD, the latter inclined to the former at the same angle as OC to OB and OB to OA. Thus, if we repeat the angular sector AOB indefinitely, and in each such sector place equal circles touching the rays of the sector, and call their upper and lower segments P, Q respectively, the particle will describe the successive arcs $P_1, Q_2, P_2, Q_3, P_3, \ldots$ *ad infinitum.* If the sectorial angle be an even aliquot part of 360°, the complete orbit will be a single anautotomic broken curve returning into itself, as, for instance, if the sector be 90° the orbit will be $P_1, Q_2, P_2, Q_2, P_3, Q_3, P_4, Q_4, P_1, Q_2, \ldots$. If the angle be an odd aliquot part of the same, the orbit will be a line returning into but crossing itself as many times as there are circles, so that in fact the whole of each circle will be described in a complete period, namely, the upper and lower segments alternately in the first half period, and the lower and upper in the second half thereof, the *period* being double the time of a revolution if the latter is defined as the interval between the body leaving and returning to any the same point. Thus, for example, if the angular sector be 72°, the orbit will be

$$P_1Q_2P_2Q_3P_3Q_4P_4Q_5P_5Q_1P_2Q_2P_3Q_3P_4Q_4P_5Q_5P_1 \text{ \&c.}$$

In like manner, if the angle included between the tangents be any commensurable part of 360°, as $\frac{m}{n}$ 360°, where m and n are integers prime to one another, the orbit will be a closed one containing mn alternate segments, or mn entire circles, according as n is even or odd. By taking n even and giving m any arbitrary odd value, a waving line will be produced forming an original and, I think, elegant pattern for a *circular lace border.* For this purpose $\frac{m}{n}$ should not be too small, in order that the disproportion between the alternate circular segments and the ratio of the border to the interior may not be so great as to offend the eye; and m not too great, in order that the traces of the pattern may not become too complicated.

I conjecture that $[m=3;\ n=64]$ and $[m=5;\ n=128]$, producing respectively 3 and 5 twists, and 5 or 6 and 6 or 7 flexures within a quadrant of each twist, would be eligible systems for the purpose. In general n ought to be even, and m a large moderate odd integer.

If the angle between the tangents to the circle from the force-centre be not an aliquot or commensurable part of 360°, the orbit will be a non-reentrant curve intersecting itself an infinite number of times. Similar or analogous conclusions are of course applicable to every case where the orbit, *seemingly* indicated by the equations of motion, is an oval, or, more generally, any curve to which tangents admit of being drawn from the force-centre,—a self-evident (now that it is stated) but none the less a very surprising feature in the *mathematical* theory of central forces. I say *mathematical*; for it ought in fairness to be observed that since it is impossible to conceive a force of infinite magnitude resulting from the attraction of a finite mass, the question involves not so much a discussion of any real phenomenon, as of the principles of interpretation applicable to an extreme case, valueless as to the establishment of a distinct independent conclusion, although not without latent importance as a safeguard against errors which might flow from the adoption of an erroneous mode of interpretation*.

It may readily be found that the velocity at any point of the orbit must be that due to infinity (otherwise a different and much more complicated curve would result), and then with the usual notation the differential polar equation to the curve becomes

$$\left(\frac{d\theta}{dr}\right)^2 = C\frac{(r^2-k^2)^2}{\sqrt{\{r^4 - r^2(r^2-k^2)^2\}}},$$

which is easily seen to be true of any arc of a circle. The phenomenon to be noticed is, that when $\frac{d\theta}{dr} = 0$, since t is the real independent variable, θ does not attain a maximum or minimum, for it is $\frac{dr}{dt}$ becoming infinite, not $\frac{d\theta}{dt}$ becoming zero, which accounts for $\frac{d\theta}{dr}$ vanishing; accordingly $\frac{d\theta}{dr}$ in passing through zero must be taken with a change of sign, which accounts for the discontinuity of the orbit regarded as a geometrical curve. This change of sign in the radical is very analogous to what happens when we calculate the potential of a spherical shell, and trace its value as the attracted point continuously receding from the centre passes from within to without the shell.

As connected with this subject of motion in a circle, I may mention that Mr Crofton has pointed out to me that my theorem concerning a homogeneous circular plate whose molecules attract according to the inverse fifth power of the distance, namely that its resultant attraction is capable of

* If we accept the very reasonable axiom that no law of force is admissible which would involve the consequence of a *finite* mass exerting an infinite attraction at a finite distance, we can find an *à priori* limit to the negative exponent of the power of the distance which can *possibly* express any law of force in nature. If my memory serves me aright, a distinguished rising French analyst, in contravention of this axiom, has assumed, for the purpose of explaining certain optical phenomena, a law of force according to some very high inverse power of the distance transcending such limit. It will be seen below that the inverse fifth power is inadmissible on this ground, and is capable of leading to irreconcileable contradictions.

making a particle move in any circle cutting the plate orthogonally, admits of being established upon my own principles without calculating, as I have done, the law of the attraction (Astronomical Prolusions*, *Phil. Mag.* Jan. 1866, p. 73); for the whole plate may be shown to be its own inverse in respect to any such orthogonal dividing circle; that is, the two parts into which it is divided by the plate will be inverses to each other in respect to the orthogonal circle, and consequently conjointly will serve to make a particle move in a segment of such circle exterior to the plate†.

Mr Crofton has also made a partial extension of the theorem to the case of a plate of the form of either one of a conjugate pair of Cartesian ovals, in a remarkable paper on the theory of these curves, lately read before the London Mathematical Society. In the "Prolusions" I raised the question of determining the force at a focus required to make a body move in such oval. This may easily be solved by aid of vectorial coordinates; and as it seems desirable to place on record the tangential affections of a curve expressed in terms of such coordinates, which I am not aware has hitherto been done, I subjoin the investigation for the purpose. The results will be seen to be of great use in simplifying the solution of the important problem of determining the most general motion of a body attracted to two or more fixed centres, a problem to which I purpose hereafter to return.

If F, G be two foci, c their distance from one another, f, g their distances from any point in a curve, ds the element of the arc at the point (f, g), θ, η the angles which ds makes with f and g, we have

$$\cos\theta = \frac{df}{ds}, \quad \cos\eta = \frac{dg}{ds}.$$

Call $g + f = u$, $g - f = v$, and let ω be the angle between f and g. Then

$$(\cos\theta)^2 + (\cos\eta)^2 + (\cos\omega)^2 - 2\cos\omega \,.\, \cos\theta \,.\, \cos\eta - 1 = 0‡.$$

Hence $$(ds)^2 = \frac{df^2 + dg^2 - 2\cos\omega \,.\, df \,.\, dg}{(\sin\omega)^2}.$$

[* p. 539, above.]

† And equally it follows that a homogeneous plate whose molecules exert a *repulsive* force following the inverse fifth power of the distance, would serve to make a particle move in the *interior* segment of an orthogonal circle. Quære as to how the motion must be conceived to take place when the attracted or repelled particle enters or quits the plate? To fix the ideas, suppose the plate attractive. The orbit described within the plate must touch the radius, for the force becomes infinite in the direction of the radius, and must tend towards the centre without becoming convex to it, on account of the force being attractive. I do not see how these conditions can be reconciled, except by supposing the remainder of the motion to take place along the radius itself, which involves the supposition of the transverse velocity at immergence becoming instantaneously destroyed, and the same at emergence when the force is repulsive.

‡ The left-hand side of this equation, calling the directions of f, g, ds, A, B, C, is

$$\begin{vmatrix} 0, & \cos AB, & \cos AC, & 1 \\ \cos BA, & 0, & \cos BC, & 1 \\ \cos CA, & \cos CB, & 0, & 1 \\ 1, & 1 & 1, & 0 \end{vmatrix}$$

And, by trigonometry,

$$1+\cos\omega=\frac{u^2-c^2}{2fg}\,;\quad 1-\cos\omega=\frac{c^2-v^2}{2fg}\,.$$

Hence
$$(ds)^2=\frac{(du)^2\,(1-\cos\omega)-(dv)^2\,(1+\cos\omega)}{(\sin\omega)^2}$$

$$=\frac{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2}{(u^2-v^2)}\,;$$

and again,

$$\sin\theta=\sqrt{\left(\frac{(ds)^2-(df)^2}{(ds)^2}\right)}=\frac{df-dg\,.\cos\omega}{\sin\omega ds}=\frac{du\,(1-\cos\omega)-dv\,(1+\cos\omega)}{2\sin\omega ds}$$

$$=\frac{(c^2-v^2)\,du-(u^2-c^2)\,dv}{4fg\sin\omega\,ds}$$

$$=\frac{(c^2-v^2)\,du+(u^2-c^2)\,dv}{4fg\sin\omega ds}=\frac{(c^2-v^2)\,du-(c^2-u^2)\,dv}{\sqrt{[(u^2-v^2)\{(c^2-v^2)du^2+(c^2-u^2)\,dv^2\}]}}\,,$$

and similarly,

$$\sin\eta=\frac{(c^2-v^2)\,du+(c^2-u^2)\,dv}{\sqrt{[(u^2-v^2)\,\{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2\}]}}\,.$$

It is worthy of passing observation that the above expressions lead immediately to the integral of the fundamental equation in the addition of elliptic functions; for if we call p, q the two perpendiculars from the foci upon ds, we have

$$\frac{(c^2-v^2)^2\,(du)^2-(u^2-c^2)^2\,dv^2}{(c^2-v^2)\,du^2-(u^2-c^2)\,dv^2}=4fg\sin\theta\,.\sin\eta=4pq.$$

Suppose now
$$4pq=c^2-a^2.$$

Then
$$(c^2-v^2)^2\,du^2-(u^2-c^2)^2\,dv^2=(c^2-a^2)\,(c^2-v^2)\,du^2-(c^2-a^2)\,(u^2-c^2)\,dv^2,$$

or
$$\frac{du^2}{(u^2-a^2)\,(u^2-c^2)}-\frac{dv^2}{(v^2-a^2)\,(v^2-c^2)}=0.$$

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and in like manner for four lines in space A, B, C, D in spaces, the determinant

$$\begin{vmatrix} 0, & \cos AB, & \cos AC, & \cos AD, & 1 \\ \cos BA, & 0, & \cos BC, & \cos BD, & 1 \\ \cos CA, & \cos CB, & 0, & \cos CD, & 1 \\ \cos DA, & \cos DB, & \cos DC, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1 \end{vmatrix}=0.$$

This important equation is nowhere *explicitly* given in treatises on trigonometry or determinants, but is virtually included in a theorem which is to be found in Balzer, and probably elsewhere, as affirmed concerning the four sides of a wry quadrilateral; for *any* four lines in space which meet in a point being given, a wry quadrilateral may be formed with sides parallel respectively to the same. The above equation enables us to express the element of a curve in space in terms of vectorial coordinates and their differentials.

The integral, therefore, of this equation must express the fact that u and v are, or may be regarded as, the sum and difference of the distances of two fixed points distant c apart from any point in a fixed straight line, the product of whose distances from those points is $c^2 - a^2$, or also, if we please, as the sum and difference of the distances of two fixed points distant a apart from any point in a fixed straight line the product of whose distances from the points is $a^2 - c^2$.

Thus, parting from the first construction, if we write $y + \lambda x = L$ as the equation to the straight line, the origin being taken midway between the two points, and the axis of x coincident with the line joining them, we obtain

$$c^2 - a^2 = \frac{L^2 - \lambda^2 \frac{c^2}{4}}{1 + \lambda^2},$$

or $$L^2 = \frac{c^2}{4}\lambda^2 + (c^2 - a^2)(1 + \lambda^2);$$

we have also $$u^2 = y^2 + \frac{c^2}{4} - cx + x^2; \quad v^2 = y^2 + \frac{c^2}{4} + cx + x^2;$$

so that $$x = \frac{v^2 - u^2}{2c}, \quad y^2 = u^2 - v^2 - \frac{c^2}{2} - \frac{(v^2 - u^2)^2}{2c^2},$$

and that the required integral will be

$$\sqrt{\left((u^2 - v^2) - \frac{c^2}{2} - \frac{(v^2 - u^2)^2}{2c^2}\right)} + \lambda \frac{v^2 - u^2}{2c}$$
$$+ \sqrt{\left(\frac{c^2}{4}\lambda^2 + (c^2 - a^2)(1 + \lambda^2)\right)} = 0,$$

which, completely rationalized, will lead to an equation of the eighth degree in u, v, and quadratic in λ^2.

A similar rational equation in u, v, μ^2 can be obtained by interchanging a and c with one another, and λ with μ; and as each equation represents the complete integral, μ^2 will necessarily be a *linear function* of λ^2 when each is regarded as a function of u, v. This linear relation we can establish *à priori*; for we have

$$y + \lambda x = \sqrt{\left(\frac{c^2}{4}\lambda^2 + (c^2 - a^2)(1 + \lambda^2)\right)},$$

$$y + \mu x = \sqrt{\left(\frac{a^2}{4}\mu^2 + (a^2 - c^2)(1 + \mu^2)\right)}.$$

Hence making $x = 0$, we have

$$(5a^2 - 4c^2)\mu^2 - (5c^2 - 4a^2)\lambda^2 + 8(a^2 - c^2) = 0.$$

If we are content to leave the integral irrational in λ or μ respectively, then it presents itself under the form of a biquadratic rational equation in u and v.

Combining the above construction of the integral with the well-known one through spherical triangles, we obtain an interesting geometrical theorem, namely, that if from a given spherical lune two arcs be cut off by an arc of constant length, their *sines* may always be represented by the sum and difference of the distances of two fixed points from a variable point in a fixed straight line; and moreover there will be two systems of such line and associated points.

Besides the general integral, we have also the singular ones given by

$$u=a \text{ or } v=a, \text{ or } u=c \text{ or } v=c,$$

indicating the familiar proposition that the product of the focal distances from the tangents of an ellipse or hyperbola are constant; $u=a$ and $v=c$ will correspond to an ellipse and hyperbola, of which the foci in the one are the vertices of the other, and *vice versâ*. If from any external point we draw a pair of tangents to either of these curves, $\frac{du}{dv}$, that is $\frac{df+dg}{df-dg}$, and therefore $\frac{df}{dg}$, will have the same value at each point of contact; so that if α, α' and β, β' be the angles which the tangents respectively make with the focal distances of the points of contact, we have $\frac{\cos\alpha}{\cos\alpha'}=\frac{\cos\beta}{\cos\beta'}$ and also $\alpha'-\alpha$ the same in absolute magnitude as $\beta'-\beta$, from which it is easy to infer $\alpha=\beta$, $\alpha'=\beta'$, showing that the tangents to an ellipse or hyperbola make equal angles with the focal distances at the points of contact, as is also known from the theory of confocal conics.

In precisely the same manner we may integrate the general equation $F(2p, 2q)=C$, where

$$2p=\sqrt{\left(\frac{u+v}{u-v}\right)}\,\frac{(c^2-v^2)\,du+(c^2-u^2)\,dv}{\sqrt{\{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2\}}},$$

$$2q=\sqrt{\left(\frac{u-v}{u+v}\right)}\,\frac{(c^2-v^2)\,du-(c^2-u^2)\,dv}{\sqrt{\{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2\}}},$$

F being any form of function whatever; the integral will always be

$$\sqrt{\left\{(u^2-v^2)-\frac{c^2}{2}-\left(\frac{v^2-u^2}{2c^2}\right)^2\right\}}+\lambda\,\frac{v^2-u^2}{2c}+L=0,$$

where the relation between L and λ depends upon, and may be determined from, the nature of F*.

* By varying the curve to which ds refers, we may obtain innumerable classes of differential equations whose integrals can be determined. Moreover, by taking ds the element of a curve in space referred to three foci, ds can be expressed by aid of the theorem given in a previous footnote as a function of the three focal distances f, g, h and their differentials; and consequently the lengths of the perpendiculars upon it from the three foci can be expressed in like manner, and we may thus obtain integrable forms of simultaneous *binary* systems of differential equations between f, g, h.

As regards the expression for ρ, the radius of curvature in terms of vectorial coordinates, we may employ the well-known formula

$$\frac{1}{\rho}=\frac{ds^2}{\sqrt{\{(d^2x)^2+(d^2y)^2\}}},$$

where $$x=\frac{f^2+c^2-g^2}{2c}=\frac{uv+c^2}{2c},$$

$$y=\frac{2\sqrt{\left(\frac{u+c}{2}\cdot\frac{u-c}{2}\cdot\frac{c+v}{2}\cdot\frac{c-v}{2}\right)}}{c}=\frac{\sqrt{\{(u^2-c^2)(c^2-v^2)\}}}{2c};$$

so that $$2\frac{c}{\rho}=\frac{ds^2}{\sqrt{[\{d^2(uv)\}^2-\{d^2\sqrt{[(c^2-u^2)(c^2-v^2)]}\}^2]}}$$

$$=\frac{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2}{(u^2-v^2)\sqrt{[\{d^2(uv)\}^2-\{d^2\sqrt{[(c^2-u^2)(c^2-v^2)]}\}^2]}},$$

which I have not thought it necessary to reduce further. As regards the original question of determining the central force towards a focus, say F, proper to make a body move in a Cartesian oval, we have

$$-F=\frac{1}{2}\frac{d}{df}v^2=2h^2\frac{d}{df}\cdot\left(\frac{1}{2p}\right)^2,$$

where $\frac{h}{2}$ is the instantaneous area, and, if the equation to the oval be

$f-kg=m$,

$$df=kdg;\quad du=(1+k)\,dg;\quad dv=(1-k)\,dg;\quad u=\left(1+\frac{1}{k}\right)f-\frac{m}{k};$$

$$v=\left(1-\frac{1}{k}\right)f+\frac{m}{k};$$

so that $$\left(\frac{1}{2p}\right)^2=\frac{(u-v)\,\{(1+k)^2(c^2-v^2)+(1-k)^2(c^2-u^2)\}}{(u+v)\,\{(1+k)(c^2-v^2)+(1-k)(c^2-u^2)\}^2},$$

from which F may be calculated and expressed under the form $\frac{P}{f^2Q}$, where P and Q are each rational integral functions of the fourth degree in f.

It does not seem to me worth while to work out the actual values of P, Q for the general form of the oval (in algebra, as in common life, there is wisdom in knowing where to stop); but it did appear to me desirable to ascertain the *form* of the expression for the retaining force, which, it is hardly necessary to add, it would have been quite impossible to do had the ordinary system of coordinates been employed. The fact of this force being a rational function of the distance is a result not without interest; and for particular varieties of the curves belonging to the class of Cartesian ovals, it will be easy to obtain its actual value as a function of the distance.

POSTSCRIPT.

On the Curve in Space which is the Analogue to the Cartesian Ovals in plano.

By a Cartesoid we may understand a surface such that a linear relation exists between the distances of any point in it from three fixed points in a plane, and by a twisted Cartesian the intersection of two Cartesoids whose three fixed points of reference are identical. A twisted Cartesian, then, will be a curve in space whose distances from three fixed points (its foci) are connected by two linear relations: from this it is obvious that it may be conceived also as the intersection of two surfaces of revolution generated by the rotation about their lines of foci of two plane Cartesians having one focus in common, so that it will consist of a system of closed rings. If F, G, H, K be any four points in a plane, and if the areas of the triangles GHK, HKF, KFG, FGH be called F_1, G_1, H_1, K_1 respectively, and P be any point in space, it is easy to prove that

$$F_1 . PF^2 - G_1 . PG^2 + H_1 . PH^2 - K_1 . PK^2 = E,$$

where E is a sort of *geometrical invariant* independent of the position of P. Its value may be expressed by the equation

$$-16E^2 = \begin{vmatrix} 0, & FG^2, & FH^2, & FK^2 \\ GF^2, & 0, & GH^2, & GK^2 \\ HF^2, & HG^2, & 0, & HK^2 \\ KF^2, & KG^2, & KH^2, & 0 \end{vmatrix}.$$

By making P coincide with F we find

$$\pm E = FG^2 . HKF + FH^2 . GKF - FK^2 . GFH.$$

Hence, if the position of K be determined by linear coordinates, x, y, and of F, G, H by coordinates of the like kind, it is obvious that E becomes a rational quadratic function of x, y; F_1, G_1, H_1 linear functions of x, y; and K_1 independent of x, y.

Let P be any point in a twisted Cartesian whose foci are F, G, H; ρ, σ, τ the distances of P from these foci. Then we have

$$l\rho + m\sigma + n\tau + p = 0, \tag{1}$$

$$l'\rho + m'\sigma + n'\tau + p' = 0, \tag{2}$$

where l, m, n, p; l', m', n', p' are constants.

Let υ be the distance of P from K, then

$$F_1\rho^2 - G_1\sigma^2 + H_1\tau^2 - E = -K_1\upsilon^2, \tag{3}$$

and υ will be a linear function of ρ, σ, τ, provided that the values of ρ, σ in terms of τ determined from (1) and (2) make the left-hand side of (3) a perfect square.

The condition that this may happen is

$$\begin{vmatrix} F_1, & 0, & 0, & 0, & l, & l' \\ 0, & -G_1, & 0, & 0, & m, & m' \\ 0, & 0, & H_1, & 0, & n, & n' \\ 0, & 0, & 0, & -E, & p, & p' \\ l, & m, & n, & p, & 0, & 0 \\ l', & m', & n', & p', & 0, & 0 \end{vmatrix} = 0. \tag{4}$$

It is easy to see that the determinant above written consists exclusively of terms in which only *binary* combinations of F_1, G_1, H_1, E appear. Consequently equation (4) is an equation of the third degree in x, y. When this equation is satisfied, K is a focus just like F, G, H. Hence we may conclude that any given twisted Cartesian possesses an infinite number of foci, every point that lies in a certain curve of the third degree being a focus. When three foci are given there are four disposable parameters, and no more, for determining this curve, which therefore cannot be any cubic curve, but is subject to satisfy two conditions. This cubic curve of foci for the twisted Cartesian is the analogue of the three focal points appertaining to the ordinary plane Cartesian*.

We are now in a position to obtain a much simpler mode of genesis of the twisted Cartesian. If F, G, H be any three points in a right line whose distances from each of a group of points in a plane more than *two* in number are subject to two linear relations, it is easy to prove that these latter will lie in a Cartesian oval, of which F, G, H are the three foci. If then we draw any transversal in the plane of the focal cubic cutting it in three points F, G, H, and make a plane revolve about this line, each group of points in which the twisted curve is cut by this revolving plane being subject to the same two linear conditions of distance from F, G, H, they and therefore the entire twisted curve will lie in a surface generated by the revolution of a certain Cartesian oval about F, G, H. By drawing F, G, H parallel to an asymptote†, one of the points, say H, goes off to infinity, and F, G become the foci of a

* It is due to Mr Crofton to state that the idea which has led to the discovery of this property of the twisted Cartesian was suggested by the method employed by that excellent geometer for establishing the existence of the third focus for the plane ovals, as described by him in a remarkable paper on the theory of these curves read before the London Mathematical Society on the 19th instant. It is important to notice that, since the distances of the points in the twisted curve from any one of the original foci are linearly related to those from any other point L, and also from any other point M in the focal cubic, the distances from L and M are themselves linearly related.

† It will presently appear that there is but one real asymptote to the focal cubic.

conic; and as we may draw any other transversal parallel to the former cutting the cubic in two other points F', G', we learn that the twisted Cartesian is always expressible as the intersection of two surfaces of revolution of the second degree whose axes are parallel, and is thus a curve of only the fourth order. It follows, moreover, that the focal cubic is the locus of the foci of a family of conics in involution whose axes are parallel.

But we may still further simplify the conception of these remarkable analogues to the ovals of Descartes. One of the system of parallels last described will be the asymptote itself meeting the cubic in only one point, so that the revolving conic becomes a parabola; and again, if we draw another transversal parallel to the asymptote and touching the cubic, the two foci come together, and the conic becomes a circle. Hence *every twisted Cartesian is the intersection of a sphere and a paraboloid of revolution**.

We are now in a position to turn back upon the focal cubic itself and make it disclose its true nature; for it will be no other than one of the two curves of foci of the system of conics passing through four points which lie in a circle. The axes of such a system always retain their parallelism; and consequently there will be two separately determinable curves of foci—those, namely, which lie in one set of parallel axes, and those which lie in the other. By a general theorem of M. Chasles, the complete curve of foci is of the sixth order, and consequently each of the two in question ought to be, as we learn from the preceding theory it is, a curve of only the third degree†.

The equation of either may easily be found, and is of the form

$$x(x^2 + y^2 + A) + Bx^2 + Cxy + Dy^2 = 0,$$

to which there is only one real asymptote, namely, $x + D = 0$. This, then, is the general equation to the focal cubic to a twisted Cartesian, and shows it to belong to the class of circular cubics.

The focal cubic is or may be determined by a circle involving three constants and four points arbitrarily chosen in the circle, which, together with the three constants for fixing the plane of the circle, give ten parameters in all.

It passes through the intersections of the three pairs of opposite sides of the quadrilateral inscribed in the circle, the centre of the circle, and the two circular points at infinity; the special relations of the three intersections to the cubic await further investigation. The twisted cubic with which it is associated may be determined by means of two right cones, each involving

* Or, as is evident from the text, the intersection of two (and therefore also of *three*) right cones with parallel axes whose plane will contain the focal cubic.

† Every focal cubic to a given twisted Cartesian has thus its conjugate corresponding to another twisted Cartesian, which may be regarded as the conjugate of the first; and the mutual relations of such curves seem to *invite* further investigation.

six constants; but as the axes must be coplanar and parallel, the number of parameters is reduced from twelve to ten, thus showing that, when the focal curve is given, the associated ovals are determined (in this respect differing from the plane ovals, in which one parameter remains indeterminate when the trifocal system of points—the analogue of the focal cubic—is given). It will probably be found that when five points in the focal curve are given, thus leaving two parameters disposable, the twisted ovals drawn through any given point will cut each other orthogonally, as Mr Crofton has shown to be the case for the plane curves in his beautiful paper on the Cartesian ovals. I find that when the focal cubic is defined by means of the circle $x^2+y^2-c^2=0$, and of its intersection with the parabola $Ax^2+2ex+2fy+g=0$, its equation becomes $Aex(x^2+y^2+c^2)+(A^2-Ag)x^2-(ey-fz)^2=0$.

I have already implicitly alluded in a preceding footnote, but think it well again to call express attention, to the remarkable property of the new ovals, of giving *circular* perspective projections on the same plane for three different positions of the eye, the lines joining the eye with the centre of each projection being all three parallel to one another and perpendicular to the plane of the picture. This fact involves the truth of the elegant and probably well-known elementary geometrical proposition, that if the opposite sides of a quadrilateral inscribed in a circle be produced, the lines which bisect the acute angles thus formed will be perpendicular to one another, and respectively parallel to the two bisectors of the angles formed by the diagonals at their intersection. I must now leave to professed geometers (among whose glorious ranks I do not claim to be numbered) the further study of those wonderful twin beings, twisted Cartesians as I have called them, but which those who so think fit may of course designate more simply as ovals with the name of their originator prefixed. By supposing the vertices of the three containing cones to be brought indefinitely near to the plane of the picture, my ovals ought to revert to the Cartesian form.