90.

SUPPLEMENTAL NOTE ON THE ANALOGUES IN SPACE TO THE CARTESIAN OVALS *IN PLANO*.

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To complete the theory given in the last Number of the *Magazine,* concerning the new ovals in space, I ought to notice that the focal cubic there spoken of is the circular cubic of which the four points in the circle, by means of which it is determined, are the foci. This will become evident from comparison with Dr Salmon's *Higher Plane Curves,* p. 175. My focal cubic is the locus of one set of foci of a system of conics whose axes are parallel, which pass therefore through four points lying in a circle. The axis in which the foci are taken, and which is parallel to the real asymptote, in general meets the focal curve in two points. Whenever these points come together, this parallel to the asymptote becomes a tangent; and the foci do come together for the circle itself and for the three pairs of lines which can be drawn through the four points in question. Hence the focal cubic not only passes through the centre of the circle and through the intersections of the three pairs of lines just spoken of, but at each of these four points is parallel to the real asymptote, that is, to the line bisecting one of the angles in which the diagonals cross. It has also two circular points at infinity. All these conditions are fulfilled by one of Dr Salmon's pair of circular cubics, of which the four points in question are the foci. These curves are therefore identical; or, to express the same idea more fully, the *two* conjugate circular cubics, of which four points in a circle are the foci, together constitute the *complete* locus of the foci of the system of conics which can be drawn through those four points*. It is interesting, moreover, to notice that the spherical curve

* Hence, as shown by Dr Salmon, the focal cubic consists of an oval and a serpentine branch. The two associated focal cubics, the same eminent author has shown, may be regarded as the locus of the intersections of similar conics having for their respective pairs of foci the two pairs of points which make up the given set of four foci; but their simpler geometrical definition, as the complete locus of the foci of the conics drawn through the four given points, appears to have escaped observation.

which is the intersection of any two right cones with parallel axes, and which is necessarily contained also in a third right cone fulfilling the same condition, may be regarded as the *inverse* of any plane section of the spindle or *tore* formed by the revolution of a circle about an axis cutting it, in respect to either point of intersection of the spindle with its axis. The spherical curve in question is of course no other than the so-called pair of twisted Cartesian ovals; and its focal curve may be any of Dr Salmon's circular cubics of the first kind, that is, one whose four real foci lie in a circle. Finally, if a double-curvature (that is, twisted) Cartesian is given, we may define its focal curve very simply as *one of the two circular cubics of which the points in which it is intersected by the plane passing through the axes of its containing right cones are the four real foci.* The Cartesian itself is contained in a sphere, in a paraboloid of revolution, in three right cones with parallel axes*, and also in three surfaces of revolution produced by the rotation of cardioids (with their triple foci lying respectively at the points of inflexion of the focal curve) about the stationary tangents†.

If the two parabolas drawn through the four given points lying in a circle which serve to determine the focal curve be

$$x^2 + 2ex + 2fy = 0, \quad y^2 + 2gx + 2hy = 0,$$

I find that the equation to the focal curve which is the locus of the foci lying in the y axis of the conics drawn through the four foci is

$$h(x-e)(x^2+y^2) + (fh - ke + kx)^2 - 2(fh - ke + kx)hy - h^2x^2 = 0.$$

* So remarkable is this property of the three cones, that, at the risk of tedious reiteration, I think it desirable to present it under the same vivid form in which it strikes my own mind. If any two indefinite straight lines cross, they may be regarded as representing a couple of right cones generated respectively by the revolution of the lines about the two bisectors of the angle which they form. Imagine now a quadrangle inscribed in a circle; its diagonals and pairs of opposite sides produced indefinitely will represent three couples of right cones. This triad of couples may be resolved into a couple of triads, the cones of each triad having their axes parallel *inter se* and perpendicular to those of the other triad; the three cones of each triad respectively will have a *common* intersection, the two intersections being consociated twisted Cartesians whose focal cubics are respectively the two consociated circular cubics of which the angles of the quadrangle are the common foci. Moreover each such twisted Cartesian is a spherical curve lying in the sphere of which the circle circumscribed about the quadrangle is a great circle. The verification of these laws of intersection might be used to form the subject of a new and instructive *plate* for students of ordinary *descriptive geometry*.

† It is interesting to trace the change of form in the general double-curvature Cartesians in regard to the real and imaginary. For this purpose conceive a sphere penetrated by a conical bodkin of indefinite length; when the point of the bodkin just pricks the sphere externally, the curve consists of a single point; as the bodkin is pushed in, the curve becomes a single oval; when the point of the bodkin again meets the sphere internally, the curve will consist of an oval and a conjugate point; then two ovals are formed; then when the bodkin and sphere touch, of an oval and a conjugate point; then of a single oval; and after the bodkin again touches the sphere, in the other side, of a single point; and finally the curve returns wholly into the *limbo* of the imaginary, whence it originally issued. There is apparently nothing analogous to this in the geometrical genesis of the plane Cartesian ovals.

When $x^2+y^2=0$, this gives $(fh-ke+kx-hy)^2=0$, showing that $x=0$, $y=0$ is a *focus,* which demonstrates the focal character of each of the four fixed points.

Departing from the theory of the quasi-Cartesian ovals, if in general we take *any* four fixed points lying at the intersections of the two conics,

$$U=ax^2+by^2+2hxy+2gzx+2fzy,$$

and

$$V=\alpha x^2+\beta y^2+2\eta xy+2\gamma zx+2\phi zy,$$

the foci of the conic $U+\lambda V$ will be given by the equality

$$\begin{vmatrix} a+\lambda\alpha, & h+\lambda\eta, & g+\lambda\gamma, & 1 \\ h+\lambda\eta, & b+\lambda\beta, & f+\lambda\phi, & i \\ g+\lambda\gamma, & f+\lambda\phi, & 0, & -(x+iy) \\ 1, & i, & -(x+iy), & 0 \end{vmatrix}=0,$$

which, by equating real and imaginary parts, gives two equations between x, y, λ.

By aid of these equations λ may be expressed as a rational integral function of x, y, which we know *à priori* must be of the second degree only, since otherwise, on substituting for its value in either of the equations between x, y, λ, we should obtain an equation above the sixth degree in x, y, contrary to Chasles's theorem (we shall also see that this is the case, without having recourse to this theorem, by the reasoning below).

Let $x^2+y^2=0$, then the above equality becomes

$$(f-ig+(\phi-i\gamma)\lambda)^2=0,$$

showing that the origin, that is, any one at will and therefore all of the four fixed points, is a focus. If λ were above the second degree in x, y, the line joining this point with either of the circular points at infinity would be *always* at least a triple tangent to the focal curve; but in the case where the focal curve breaks up into two distinct subloci, we have seen that these tangents to each sublocus are simple, that is, that they are double in regard to the whole locus, wherefore λ must be always a quadratic function only of x, y. Consequently each circular point at infinity must itself be a double point upon the curve.

If now we regard the four given points as syzygetic foci of a curve (a term indispensable to give precision to the theory of foci when interpreted in the Plückerian sense), that is, if we suppose a plane curve defined by the equation

$$\lambda\sqrt{(A)}+\mu\sqrt{(B)}+\nu\sqrt{(C)}+\pi\sqrt{(D)}=0,$$

where A, B, C, D are the characteristics of the infinitesimal circles of which the four given points are the centres; and if in the norm of the linear function above written we make $\lambda\pm\mu\pm\nu\pm\pi=0$, and conjoin with this two other equations between λ, μ, ν, π, which will make the term $(x^2+y^2)^3(Lx+My)$

in the norm vanish identically, that is, the equations $L=0$, $M=0$, we shall obtain a group* of curves of the sixth degree, each possessing precisely the same geometrical characters as have been proved to be satisfied by the curve of foci of the conics $U+\lambda V$ drawn through the four fixed points, namely of having the circular points at infinity for double points, and being doubly touched† by each line joining either of them with any one of the four fixed points; and if we are at liberty to assume (which, however, requires further investigation‡) that the curve containing the foci of $U+\lambda V$ must be identical with one of the group, then this curve of foci will be defined by the equation

$$l\sqrt{(A)}+m\sqrt{(B)}+n\sqrt{(C)}+p\sqrt{(D)}=0\,;$$

whence, calling a, b, c, d the four fixed points, and F, G, H the points of intersection of the opposite sides of the quadrangle $abcd$, the signs of the square roots below written must be capable of being so assumed that the determinant

$$\begin{vmatrix} (aF)^{\frac{1}{2}}, & (bF)^{\frac{1}{2}}, & (cF)^{\frac{1}{2}}, & (dF)^{\frac{1}{2}} \\ (aG)^{\frac{1}{2}}, & (bG)^{\frac{1}{2}}, & (cG)^{\frac{1}{2}}, & (dG)^{\frac{1}{2}} \\ (aH)^{\frac{1}{2}}, & (bH)^{\frac{1}{2}}, & (cH)^{\frac{1}{2}}, & (dH)^{\frac{1}{2}} \\ 1, & 1, & 1, & 1 \end{vmatrix}$$

shall be equal to zero, constituting a remarkable theorem concerning seven points (four quite arbitrary) in a plane. If (as seems probable) the case supposed is what actually obtains, a geometrical rule must exist for determining the proper combination of signs to be employed in the above determinant; and then l, m, n, p will be proportional to the first minors of the three first lines of the matrix above written§.

The above theory, very hastily sketched out under the pressure of other occupations, will serve at all events to manifest in how very imperfect and inchoate a form the theory of foci at present exists, and may serve to raise

* No two of the group can be identical; for in such case one of the focal distances could be eliminated, and the syzygy be reducible by one term, contrary to hypothesis.

† In general, if a curve is defined by a homogeneous linear relation between its distances from r, or a non-homogeneous linear relation between its distances from $r-1$ points, it will easily be seen that the line joining each such point with either circular point at infinity will be a tangent *touching* the curve in 2^{r-2} points.

‡ To the group of syzygetic curves there are only eight parameters, and these eight tangents (all double) at the four foci are common to each member of the group and to the locus of the foci of $U+\lambda V$. To complete the proof of the supposed identity of the latter with one of the former, it is necessary to show that the *number* of curves having the eight tangents in question is precisely equal to the number of systems of solutions of the equations

$$l+m+n+p=0,$$
$$L=0,\quad M=0.$$

§ Calling the determinant D, the equation $D=0$ serves to fix the allowable combinations of the doubtful signs, just as in Cardan's rule a certain equation, which the product of the two associated cube-roots in the solution is bound to satisfy, fixes their allowable combinations of values.

the question whether there is not a family of curves distinguishable by the possession of syzygetic foci, and which may be termed syzygetic or norm curves (including as elementary members of the group conics, Cartesian ovals, circular cubics, &c.), forming a distinct genus, and calling for a special and detailed examination of their focal properties*.

* In an evil hour for the cause of sound nomenclature, and by an over-hasty generalization, a property of foci properly so called was substituted for their true definition as centres of linear relationship. The temptation was great, it must be allowed; for the new definition gave a means of describing each focus *per se* without reference to the associated points: it calls to mind the analogous attempted definition of *man* as a *featherless biped*, and is open to the like kind of objections. The mischief being done, and under cover of the authority of names so great that to root it out seems now hopeless, the best remedy to apply is, as I think, that used in the text, of distinguishing the centres of linear relationship as *syzygetic* foci or foci *proper*. There is room for a grand chapter in the promised and anxiously-expected new edition of Dr Salmon's *Higher Plane Curves*, on a systematic and exhaustive development of the laws of foci proper, and the algebraical philosophy, as it may well be termed, of true focal curves, that is, curves the distances of whose points from one or more sets of fixed points are subject to linear relations. Nothing can be more curious than the study of the way in which, starting from a given set of fixed points, other foci (as in the Cartesian ovals) are found capable of replacing one or more of the given ones, constituting the theory of substitution—and then, again, how, as in the conic sections and in circular cubics, besides this faculty of mutual substitutability of foci of the same set, one set may be entirely replaced by one or more other sets, constituting the theory of plurality or distribution. Algebra cannot but gain largely by these ideas of substitution and distribution being fully worked out.

In answer to my objections to the undue extension of the term *focus*, it has been urged that a focus, as originally presenting itself in the theory of conics, is susceptible of two distinct definitions—first as a member of a syzygetic group, and again as a point whose squared distance from any point in its curve is the square of a linear function of the coordinates, —that it is legitimate to generalize the conception from either of these points of view, and that the latter leads to the definition of a focus as a point whose squared distance from any point in its curve, multiplied by a quantic, gives rise to a second quantic containing a squared linear function as a factor. But I answer to this, that the generalization is carried too far and too fast, two steps in enlargement of the original condition being taken at once to arrive at it; that the first step should be to define a focus as a point such that the squared distance in question, multiplied by a quantic, viewed as a function of the coordinates, shall be a perfect square; and that when this first step is taken, the foci so obtained are foci of a peculiar kind, and probably retain their quality as *foci* proper, or centres of linear relationship. At all events they possess the property of giving, by their junctions with the circular points at infinity, multiple tangents to the curve, according to the law stated in a previous footnote concerning such foci.

If the word *focus* is retained to signify the proper or syzygetic species, some slight modification of the word may be used to denote the genus, namely, foci which satisfy the larger definition of being points of intersection of the simple tangents to the circular points at infinity. I thought of the word *focal* for the purpose; but this is objectionable, for the reason that it would probably be found advisable to retain that word to denote the class of *curves* which possess foci proper. On the whole, the word *subfocus* seems to me best to meet the exigency of the case, and possesses the recommendation of being capable (with dialectic variations) of passing current in each of the five accepted tongues—Latin, German, French, English, and Italian, which happily at the present day may be regarded as the common property and inheritance of mathematical Europe.