92.

NOTE ON THE PROPERTIES OF THE TEST OPERATORS WHICH OCCUR IN THE CALCULUS OF INVARIANTS, THEIR DERIVATIVES, ANALOGUES, AND LAWS OF COMBINATION; WITH AN INCIDENTAL APPLICATION TO THE DEVELOPMENT IN A MACLAURINIAN SERIES OF ANY POWER OF THE LOGARITHM OF AN AUGMENTED VARIABLE.

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SUPPOSE ϕ_1 denotes any algebraical function of the two sets of elements,

$$\left(a, b, c, \ldots, \frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots\right).$$

Let $\psi *$ in general signify the process of operating with ψ upon all that follows†.

Suppose $\phi_1 * \phi_1 = \phi_2$, where the operating elements $\frac{d}{da}, \frac{d}{db}, \ldots$ of course can only operate upon the operands $a, b, c, \ldots$ in the second ϕ. In like manner, let

$$\phi_1 * \phi_1 * \phi_1 = \phi_1 * \phi_2 = \phi_3,$$

and in general

$$(\phi_1 *)^{n-1} \phi_1 = \phi_n.$$

† The symbol of an operator consists of two parts, the *corpus* or quantity, and the *asterisk* or sign of operation. Thus a simple extensor operator has one of the extensors for its corpus; a compound extensor operator has any algebraical function of any number of extensors for its corpus. The operator which represents the combined effect of two or more operators following each other in any specified order may be termed their resultant; the theorems in the text amount to saying that the resultant of any number of simple or compound extensor operators is independent of the order in which its components occur, and is equivalent to some third compound extensor operator. One great problem to be solved is to determine the corpus of a resultant in terms of the corpora of its two components. This is done in the text for the simple case where each component corpus is a simple power of one of the extensors. To attain clearness of conception, the first condition is *language*, the second *language*, the third *language*—Protean speech—the child and parent of thought.

It will follow from this that

$$\phi_1 * \phi_1 * = (\phi_1^2 + \phi_2)*,$$

$$\phi_1 * \phi_1 * \phi_1 * = (\phi_1^3 + 2\phi_1\phi_2 + \phi_3) * \dagger;$$

and in the general case there will be found no great difficulty in obtaining the following theorem,

$$(\phi_1 *)^i = \Pi i \text{ . coefficient of } t^i \text{ in } e^T,$$

where $$T = \phi_1 t + \phi_2 \frac{t^2}{1 \cdot 2} + \phi_3 \frac{t^3}{1 \cdot 2 \cdot 3} + \ldots, \qquad \text{(A)}$$

a relation which may be expressed by means of the identity ‡

$$e^{t\phi_1 *} = (e^T)* \S, \qquad \text{(A) } bis$$

which important equation has been previously noticed by Professor Cayley under a somewhat less general form.

With the exception of noticing that $(\phi_1 *)^r$ and $(\phi_1 *)^s$ are commutable symbols by virtue of their definition, that is, that

$$(\phi_1 *)^r (\phi_1 *)^s = (\phi_1 *)^s (\phi_1 *)^r,$$

I am not at present aware that this theory of derivation when the form of ϕ is left undetermined presents much that is remarkable. Very different, however, is the case when we proceed to give to ϕ the particular form in which it enters into the calculus of invariants: a most surprising and unexpected system of relations then springs up between the various orders of operators; and a vast and inexhaustible theory opens out before us, of which I want leisure to be able to do more than briefly notice one or two salient features.

† So more generally if ϕ, ψ be any two functions of $a, b, c, \ldots \frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$ we have¶

$$\phi * \psi * = (\phi\psi) * + [\phi * \psi]*,$$

and similarly $$\psi * \phi * = (\psi\phi) * + [\psi * \phi]*.$$

Hence if two operators $\phi *$, $\psi *$ are commutable, so, in respect to the symbol of operation $*$, are the two operants ϕ, ψ.

The force of the bracket explains itself. This wonderful symbol has the faculty of extending itself without ambiguity to every possible development, however new, of mathematical language. It is susceptible only of a metaphysical definition as signifying the exercise, with regard to its content, of that faculty of the human mind whereby a multitude is capable of being regarded as an individual, or a complex as a monad. In a word, it is the symbol of individuality and unification.

[‡ Cf. p. 608, below.]

§ Thus, for example, let ϕ_1 represent $x\frac{d}{dx}$, then $\phi_2, \phi_3, \ldots$ will be all equal to ϕ_1; accordingly $T = (e^t - 1)\phi_1$, and the formula in the text becomes

$$e^{\left(tx\frac{d}{dx}\right)*} = \left(e^{(e^t-1)x\frac{d}{dx}}\right)*,$$

a remarkable formula of expansion. [¶ Cf. p. 610, below.]

Let $E_1 = a\dfrac{d}{db} + 2b\dfrac{d}{dc} + 3c\dfrac{d}{dd} + \ldots + a'\dfrac{d}{db'} + 2b'\dfrac{d}{dc'} + \ldots + a''\dfrac{d}{db''} + \ldots,$

that is, $$= \Sigma\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd}\ldots\right).$$

Then if I be any function of the coefficients $a, b, c, \ldots; a', b', c', \ldots$ in the algebraic forms $(a, b, c, \ldots)(x, 1)^p$; $(a', b', c', \ldots)(x, 1)^p \ldots$, and I_1 be what I becomes when we substitute for $a, b, c, \ldots; a', b', c', \ldots$, the values which these coefficients assume when $x + h$ is written in place of h, it is, or ought to be, well known that

$$I_1 = I + E_1{*}Ih + (E_1{*})^2 I\frac{h^2}{1.2} + (E_1{*})^3 I\frac{h^3}{1.2.3} + \ldots.$$

Here $$E_1{*}E_1 = 2\Sigma\left(a\frac{d}{dc} + 3b\frac{d}{dd} + 6c\frac{d}{de} + \ldots\right),$$

$$E_1{*}E_1{*}E_1 = 2.3\Sigma\left(a\frac{d}{dd} + 6b\frac{d}{de} + \ldots\right),$$

$$\ldots = \ldots;$$

it will therefore become convenient slightly to depart from the notation applied to the general form ϕ, and to write

$$E_1 = \Sigma\left(a\frac{d}{db} + 2b\frac{d}{dc} + \ldots\right),$$

$$E_2 = \Sigma\left(a\frac{d}{dc} + 3b\frac{d}{dd} + \ldots\right),$$

....

$$E_n = \Sigma\left(a\frac{d}{da_n} + (n+1)\,b\frac{d}{db_n} + \frac{1}{2}(n+1)(n+2)\,c\frac{d}{dc_n} + \ldots\right);$$

where $a_n, b_n, c_n, \ldots$ are used to express the elements n steps more advanced than $a, b, c, \ldots$ respectively; we have then by the general theorem

$$e^{tE_1*} = (e^T)*, \tag{B}$$

where T now takes the form

$$E_1t + E_2t^2 + E_3t^3 + \ldots$$

I propose to give to the E series of operators the general name of Extensor Operators, or simply Extensors.

The first remarkable, I may say marvellous, property of these extensors is, that they form a sort of closed group; that is, any two algebraical functions whatever of the extensors regarded as algebraic functions of the quantities $a, b, c, \ldots; \dfrac{d}{db}, \dfrac{d}{dc}, \ldots$ being used as new operators and applied in succession to the same operand, the result is the same as if *some single third algebraical function* of the extensors had operated alone on this operand. The second great fact is, that the order in which the above-described operations take place is indifferent, that is, that the two operators above described are commutable; in other words, we have always

$$\left.\begin{aligned}&\theta(E_1, E_2, E_3, \ldots)* \psi(E_1, E_2, E_3, \ldots)*\\ &= \Omega(E_1, E_2, E_3, \ldots)*\\ &= \psi(E_1, E_2, E_3, \ldots)* \theta(E_1, E_2, E_3, \ldots)*\end{aligned}\right\}. \qquad (C)$$

Thus, for example,

$$E_\mu{}^i * E_\mu{}^j * = F(E_\mu, E_{2\mu})* = E_\mu{}^j * E_\mu{}^i *,$$

where, writing $m = \dfrac{\Pi(2\mu)}{(\Pi\mu)^2}$; F represents the *quasi*-hypergeometric series,

$$\left.\begin{aligned}&E_\mu{}^{i+j} + i\,.\,j\,.\,mE_\mu{}^{i+j-2}\,.\,E_{2\mu} + \frac{i(i-1)j(j-1)}{1\,.\,2}m^2E_\mu{}^{i+j-4}\,.\,E_{2\mu}{}^2\\ &\quad + \frac{i(i-1)(i-2)j(j-1)(j-2)}{1\,.\,2\,.\,3}m^3\,.\,E_\mu{}^{i+j-6}\,.\,E_{2\mu}{}^3 + \ldots\end{aligned}\right\}, \quad (D)$$

and $E_\mu{}^i * E_\nu{}^j *$ will be expressible under a form *quam proximè* analogous. My immediate intention in this brief notice being merely to call attention to the surprising properties of these functions, I shall conclude with adding a slight extension of theorem (B) above given, namely,

$$e^{tE_1*} = (e^T)*.$$

This may be regarded as a particular case of a more general theorem which I have discovered, namely,

$$E_1{}^j * e^{tE_1*} = e^{tE_1*}\,E_1{}^j * = \left\{\left(\frac{dT}{dt}\right)^j e^T\right\}*,$$

a theorem which, with a simple change in the coefficients of T, may be extended to the still more general form $E_\omega{}^j * e^{tE_\omega*}$, so as to give a simple solution of the equation

$$X* = (E_\omega *)^i E_\omega{}^J *,$$

where X is a form to be determined as an algebraical function of E_ω, $E_{2\omega}$, $E_{3\omega}$, &c. ....

The cardinal problem to be solved in the theory of extensors is the determination of Ω in formula (C), where ψ and θ are any given functional forms.

In the further development of this theory, it will probably be found expedient to suppose the number of the elements, $a, b, c, \ldots j, k, l$, to become finite, which will limit the number of the derived extensors, and to study the mutual reactions of the correlated series of extensors (with their derivatives), which we may characterize respectively as the E and H series, where

$$E_1 = \Sigma\left(a\frac{d}{db} + 2b\frac{d}{dc} + \ldots\right),$$

$$H_1 = \Sigma\left(l\frac{d}{dk} + 2k\frac{d}{dl} + \ldots\right).$$

Either of the above two primitive forms (as it is the imperishable glory of Professor Cayley to have discovered†) is sufficient in itself for testing the nature of every invariant satisfying the necessary and obvious condition of weight, and for deducing the complete form of a covariant from either of its extreme terms; which latter consideration affords, I think, a sufficient ground for the name (of some kind or another so much needed) *Extensors*, which I propose to give to these too-long-suffered-to-remain anonymous test operators and their derivatives.

Postscript.

Since the above was sent to press it has occurred independently to Professor Cayley, to whom I had communicated a sketch of the theory, and to myself, that the general conclusions contained in the text above would remain valid for a much more general class of operants than those there defined; and there can be little or no doubt that such is the case for all operants *lineo-linear* in a set of elements $a, b, c, \ldots$, and their præ-reciprocals $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$. Moreover a material improvement in the nomenclature has suggested itself, which I proceed to explain. It is most important in this theory to be able to distinguish between the corpus or root of an operator viewed as a function and the operator itself, and to be in possession of a single name for the former. Accordingly, in conformity with the general terminology of the new algebra, I propose to substitute the name of Protractor for Extensor to signify the *operator*, so as to be able to use the word Protractant to signify the *corpus*. Also I shall give the analogous names of Pertractant‡ and Pertractor—the former to the lineo-linear function above referred to, the latter to this function *energized*, that is, converted into an operator by the addition of the asterisk $*$, the symbol of operative power.

We thus start with a pertractant P_1 which is *energized* into a pertractor, P_1*; with this latter we continue to operate any number of times upon the original pertractant, and obtain a succession of new derived pertractants, into which it appears at present to be convenient, for the sake of uniformity, to introduce the numerical divisors 2, 3, 4, ..., so that we may define P_{n+1}, the nth *derivative* pertractant, as equal to $\frac{(P*)^n P}{\Pi(n+1)}$.

† But this magnificent discovery, whereby the determination of the number of fundamental invariants to a binary quantic of a given degree is reduced to a problem in the partition of numbers, it is but justice to M. Hermite to state, took its rise in that great analyst's discovery of the octodecimal invariant of the binary quintic. So long as the existence of this fourth invariant to that form was unsuspected, it must have remained impossible to conjecture the sufficiency of the single partial differential equation-test.

‡ Thus the "*Universal Mixed Concomitant*" $x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\ldots$ is of the genus Pertractant.

We thus obtain a series of pertractants, P_1, P_2, P_3, ..., which may be termed the primitive and prime derivative pertractants of the family.

Again, we may form any algebraical function of the primitive and its prime derivatives, and such function may be termed a compound derivative of the family; this in its turn, by the addition of the symbol of operative power, may be energized into a pertractive operator, which, containing only a single asterisk, is to be regarded as a simple or single derived pertractor, although its corpus is a compound derivative.

The first leading proposition of the theory is, that all operators so formed are *commutable*, so that, being subject to the laws of algebraical operation, they may themselves be made the subjects of algebraical functions. The second great proposition is, that any such function of one or more pertractors is reducible to the form of a *single pertractor*, that is, is an energized function of the prime pertractants P_1, P_2, P_3, ...

The theorems that have been stated concerning protractants and protractors will continue to subsist for the much more general class of pertractors and pertractants. Thus, for example, theorem (D) in the text above, when we take $\mu = 1$, becomes

$$E_1{}^i * E_1{}^j * = E_1{}^{i+j} + i\,.\,jE_1{}^{i+j-2}(2E_2) + \frac{i(i-1)j(j-1)}{1\,.\,2}E_1{}^{i+j-4}(2E_2)^2 + \ldots.$$

Mr Cayley verifies this theorem when for E_1, the leading protractant, we substitute P_1, a pertractant, as follows. Take only a single element x and its symbolical reciprocal $\dfrac{d}{dx}$, so that $P_1 = x\dfrac{d}{dx}$; then

$$P_2 = \tfrac{1}{2}P_1, \text{ and } P_1{}^i\,.\,P_1{}^j = \left\{x^i\left(\frac{d}{dx}\right)^i\right\} * \left\{x^j\left(\frac{d}{dx}\right)^j\right\},$$

is easily seen to be

$$P_1{}^{i+j} + \frac{i\cdot j}{1}P_1{}^{i+j-1} + \frac{i(i-1)j(j-1)}{1\,.\,2}P_1{}^{i+j-2} + \ldots$$

$$= P_1{}^{i+j} + \frac{i\cdot j}{1}P_1{}^{i+j-2}(2P_2) + \frac{i(i-1)j(j-1)}{1\,.\,2}P_1{}^{i+j-4}(2P_2)^2 + \ldots,$$

as before.

But I find that the theory admits of a still further and most important extension. Thus far we have been dealing with operants and operators derived from a single one of the former. But we may easily form a set of two or more, say k pertractants, that is, functions lineo-linear in $a, b, c, \ldots$; $\dfrac{d}{da}, \dfrac{d}{db}, \dfrac{d}{dc}, \ldots$ commutable *inter se*†; these being *energized* into operators

† This imports into the subject a beautiful theory of commutable matrices. In the case of two letters we have two types of commutable pertractors, from which all the rest may be derived by the laws of pertraction stated in the text. These two fundamental systems are:—

which are made to act on the functions themselves, will give rise to $\frac{1}{2}r(r+1)$ first derivatives, which, energized in their turn, will be commutable *inter se* and with the original operators: the derivatives of the next order enjoying the same properties will be $\frac{1}{6}r(r+1)(r+2)$, and so on. Thus, as before, we obtain the *prime* pertractive derivants of various orders, with the difference that there are now several of such prime derivants belonging to each order. Any function of these gives rise to a compound-pertractive derivant, the number of which is of course unlimited; these may be energized into operators, subject *inter se* to all the laws of algebraical operation, and any function of one or more of such compound-pertractive derivators will be equivalent to some single derivator belonging to the same family. In a word, the theory may be extended from the case of *Monocephalous* to that of *Polycephalous* pertractive functions and operators and their derivatives.

I will conclude for the second time with the statement of an expansion in a series which, as far as I have been able to ascertain, is new to writers on the differential calculus, to which I was led by applying to the *operand* a^x the symbolical equation previously given in a footnote. The equation in question may be written as follows:

$$e^{\left(ta\frac{d}{da}\right)}* = \left\{(e^{(e^t-1)})^{a\frac{d}{da}}\right\}*;$$

from this I have been able to deduce by a mental calculation, the steps of

---

(1) $x\delta_x$; $y\delta_y$.

(2) $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x, y)(\delta_x, \delta_y)$; $x\delta_x + y\delta_y$.

In the case of three letters, the four following types of commutable systems present themselves:—

(1) $x\delta_x$; $y\delta_y$; $z\delta_z$.

(2) $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x, y)(\delta_x, \delta_y)$; $x\delta_x + y\delta_y$; $z\delta_z$.

(3) $ax\delta_y + by\delta_z + cz\delta_x$; $\frac{1}{a}y\delta_x + \frac{1}{b}z\delta_y + \frac{1}{c}x\delta_z$.

(4) $\begin{pmatrix} a, & b, & c \\ d, & e, & f \\ g, & h, & k \end{pmatrix}(x, y, z)(\delta_x, \delta_y, \delta_z)$; $x\delta_x + y\delta_y + z\delta_z$.

Whether the above four systems are independent, and whether they constitute an exhaustive enumeration in the case of the three letters, I have not yet had time to ascertain.

The reader will please to bear in mind that any linear function of the terms in each system, or of them and their derivatives, is commutable with those terms themselves; thus, for example, the last system but one is quite as extensive as if we included in it

$$\lambda ax\delta_y + \lambda by\delta_z + \lambda cz\delta_x + \frac{\mu}{a}y\delta_x + \frac{\mu}{b}z\delta_y + \frac{\mu}{c}x\delta_z + \lambda\mu x\delta_x + \lambda\mu y\delta_y + \lambda\mu z\delta_z,$$

in which it will be noticed that the three last terms may be obtained (to a constant factor *près*) by operating with the sum of the three first upon the sum of the three middle terms, or *vice versâ*.

which I am unable to recall, a development which would be exceedingly difficult to obtain from the method of Maclaurin's theorem. I find

$$\{-\log(1-x)\}^n = x^n + S_{n,1}\frac{x^{n+1}}{n+1} + S_{n+1,2}\frac{x^{n+2}}{(n+1)(n+2)}$$
$$+ S_{n+2,3}\frac{x^{n+3}}{(n+1)(n+2)(n+3)} + \ldots,$$

where in general $S_{i,j}$ signifies the sum of the $\frac{i(i-1)\ldots(i-j+1)}{1.2\ldots j}$ products of the combinations of the numbers 1, 2, 3, ... i, taken j and j together. This development may be easily verified inductively by aid of the identical equation

$$\frac{d}{dx}\{\log(1-x)\}^n = -\frac{n\{\log(1-x)\}^{n-1}}{1-x},$$

combined with the relation

$$S_{n+j-1,j} = S_{n+j-2,j} + jS_{n+j-2,j-1}\dagger$$
$$= S_{n+j-2,j} + jS_{n+j-3,j-1} + j(j-1)S_{n+j-4,j-2}$$
$$+ j(j-1)(j-2)S_{n+j-5,j-3} + \&c.$$

It is obvious that the coefficients of the powers of x in the above expansion must be all of them integral functions of n, and must also contain n in every term except the first; and when so expressed as integer functions of n, the result obtained on the supposition of n being a positive integer will continue to subsist for all values of n. From the first part of this statement, it follows that $S_{i,j}$ may always be expressed under the form

$$\{(i+1)i(i-1)\ldots(i-j+1)\}\,\phi_{j-1}(i),$$

where $\phi_{j-1}(i)$ is a quantic in i of the degree $j-1$.

Furthermore, if we suppose

$$\phi_{j-1}(i) = \frac{\psi_{j-1}(i)}{2^\alpha . 3^\beta . 5^\gamma . 7^\delta . 11^\epsilon \ldots p^{\phi(p)}\ldots},$$

p being any prime number, and ψ a function of i of the degree $(j-1)$ all whose coefficients are integer, and (consistently with this being the case) as small as they can be made, there is no difficulty in obtaining the value of $\phi(p)$ under the following form,

$$\phi(p) = \Sigma_{\mu=\infty}^{\mu=0} E\frac{j}{(p-1)p^\mu}\ddagger,$$

† The equation in differences $S_{n,j} = S_{n-1,j} + jS_{n-1,j-1}$ gives an easy algorithm for calculating $S_{n,j}$, and shows *à priori* that it is divisible by $(n+1)n\ldots(n-j+1)$.

‡ Consequently $\phi(p)$, the exponent of p, is always less than $\frac{pj}{(p-1)^2}$, and *à fortiori* than $\frac{j}{p-2}$.

where, as usual, the symbol E signifies that *only* the integer part is to be preserved of the number upon which it acts. The value of the coefficient of i^{j-1} in $\phi_{j-1}(i)$ is easily ascertained to be $\frac{1}{(1.2.3\ldots j)\,2^j}$, and consequently the coefficient of i^{j-1} in ψ is always an odd number, the number of times that 2 is contained in this denominator being

$$\Sigma_{\mu=\infty}^{\mu=0} E\frac{j}{2^\mu}.$$

The maximum prime in the denominator of the fraction which expresses $\phi_{j-1}(i)$ enters always as a simple factor, because, as we know by M. Bertrand's theorem, there is always a prime number included between $q+1$ and $2q+2$. Consequently, supposing j to be $2q$ or $2q+1$, since there exists a prime number p greater than $q+1$, and not greater than $2q+1$, this prime number will appear in the denominator of $S_{n,j}$ with the exponent $E\left[\frac{2q \text{ or } 2q+1}{p-1}\right]$, that is, *unity*.

Conversely, if by any means not founded on the above theorem we could ascertain this fact, we should be in possession of an entirely new proof of that celebrated theorem. It is perhaps also worthy of a passing notice, that $(-)^j.\phi_{(j-1)}(j-1)$ may easily be proved to be equal to the coefficient of t^j in $\log\log(1+t)-\log t$†.

I have calculated the values of $S_{i,1}$, $S_{i,2}$, $S_{i,3}$, $S_{i,4}$, which are as follows:

$$\frac{(i+1)\,i}{2},\quad \frac{(i+1)\,i\,(i-1)}{2^3.3}(3i+2),\quad \frac{(i+1)\,i\,(i-1)\,(i-2)}{2^4.3}\,i\,(i+1),$$

$$\frac{(i+1)\,i\,(i-1)\,(i-2)\,(i-3)}{2^4.3^2.5}(15i^3-15i^2+10i-8)\,‡.$$

† And more generally if

$$S_{n+j-1,j}=\{(n+j)(n+j-1)\ldots n\}\,(C_jn^{j-1}+C_{j-1}n^{j-2}+\ldots+C_1),$$

$$(-)^jC_\omega=\frac{C_1}{\Pi\omega}.\text{ coefficient of } t^j \text{ in } \left(\log\frac{\{\log(1+t)\}}{t}\right)^\omega.$$

‡ In his great and most useful work on the *Calculus* (p. 264), Professor De Morgan has applied Arbogast's method to the expansion of $\{\log(1+x)\}^n$, and worked out his results completely as far as the coefficients of x^4 inclusive. His $\frac{C}{n}$, $\frac{E}{n}$, $\frac{F}{n}$, when $i-1$, $i-2$, $i-3$ are substituted in these quotients for n, become identical with the *non-trivial*, or so to say outstanding factors in my $S_{i,2}$; $S_{i,3}$; $S_{i,4}$ respectively.

I have since calculated the same factors for $S_{i,5}$, $S_{i,6}$ corresponding to Professor De Morgan's $\frac{G}{n}$, $\frac{H}{n}$, when n is replaced by $i-4$, $i-5$ respectively. The calculations are rather laborious, extending in the latter case to 8 places of digits; but comparatively very small numbers appear in the final expressions. For $S_{i,5}$ I find the outstanding factor takes the exceedingly simple form

$$\frac{i\,(i+1)(3i^2-i-6)}{2^8.3^2.5},$$

and for $S_{i,6}$ the form

$$\frac{63i^5-315i^3+224i^2+140i-96}{2^{10}.3^3.5.7}.$$

I think there can be little doubt that the outstanding factor in $S_{i,j}$ becomes more liable to

The following observation from Professor Cayley will be found interesting:—

"In the case of two variables, if

$$P_1 = (ax+by)\frac{d}{dx} + (cx+dy)\frac{d}{dy},$$

then in the notation of matrices,

$$P_1 = \begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix} (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_2 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^2 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_3 = \tfrac{1}{6}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right);$$

whence also $P_1 * P_2 = P_2 * P_1 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right) = 3P_3,$

which accords with your theorem,

$$E_1 * E_2 * = E_2 * E_1 * = E_1 E_2 * + 3E_3 *."$$

I have taken the liberty of writing in the above $\frac{d}{dx}, \frac{d}{dy}$ for δ_x, δ_y, and P_1 for δ in the original. It will be useful to bear in mind that in any operator such as $E_1 *$ or $E_2 *$, *the asterisk forms an integral part* of the symbol†. Thus $E_1 * E_2 *$, if we choose, may be written under the form of $E_1 *$ multiplied by $E_2 *$, that is, $(E_1 *) \times (E_2 *)$, where the cross is the sign of ordinary algebraical multiplication.

decomposition into algebraical factors in proportion as the number $j+1$ becomes more separable into *numerical* factors, that is, in proportion as $j+1$ contains a smaller number of *distinct* prime factors. For this reason I purpose calculating $S_{i,7}$, $S_{i,8}$ against the appearance of the next Number of the *Magazine*. The nature of the roots, as regards being real or imaginary in the equation $S_{i,j}=0$, is also probably well deserving of study. It is worthy of notice that in each of the irreducible factors of $S_{i,j}$ for the values of j above considered, the coefficients are composed exclusively of the prime factors which enter into $j+1$. It is hardly necessary to observe that the quantities $\frac{S_{n+j-1,j}}{(n+1)(n+2)\dots(n+j)}$, when expressed in a rational integral form, are the coefficients of the powers of x in the series for $[\log(1+x)]^n$, when n is regarded no longer as a positive integer, but as an arbitrary variable.

† The operant, sign of operation, and operand form a triad somewhat analogous to the subject, copula, and predicate of the logicians; and as in the admirable new school of philosophical grammar the copula is for certain purposes incorporated with the predicate, so *ex converso* in this system the sign of operation is taken up by the operant; but, herein advantageously differing from the practice of the grammarians alluded to, the combination assumes a distinct name from its leading element and is styled an operator.

I ought to mention that my information in this matter is derived from the statements which have appeared in the public prints, and not from a direct study of that wonderful manual of the quintessence of grammar so unpretendingly ushered into the world as a *primer*, but which, whatever name it goes by, can hardly fail to bring about a philosophical revival of the intellect of the rising generation of Englishmen. I wait for a favourable opportunity of leisure to address the full energies of my mind to the invigorating and congenial task of mastering its subtle differentiations and profound arduous abstractions.