93.

ON THE MOTION OF A RIGID BODY ACTED ON BY NO EXTERNAL FORCES.

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[Cf. p. 602.]

As conveying an image of the motion of a rigid body acted on by no forces, Poinsot's well-known method of representation, whether by a rolling ellipsoid or a shifting cone, labours under an obvious imperfection; the *time* is not put in evidence by it. Thus when the ellipsoid, with which alone I intend here to deal, is employed, it is true that the proportional value of the velocity of rotation about the instantaneous axis is geometrically measured by the radius vector drawn from the fixed point to the invariable tangent plane, and so by a process of summation the time of passing from one position to another may be considered as inferentially determined; but there is nothing to convey to the senses, or to the mind's eye, a notion of the effect of this summation, and thus the relation of the most important element—the time—to the position of a free revolving body remains unexpressed. I shall begin with showing how by a slight addition to Poinsot's ideal kinematical apparatus this defect may be completely removed, and the time between successive positions conceived to register itself mechanically. As the property upon which this depends readily lends itself to a geometrical form of proof, I shall, in the first instance, follow that mode of investigation, as being the more germane to the matter in hand, reserving to a later point in the memoir the analytical demonstration; that is to say, assuming Poinsot's ellipsoid, and the law which connects the velocity with the position of the body, I shall show how the time may be, as it were, mechanically extracted and summed.

It will be well, then, in the first instance to recall some simple properties of confocal ellipsoids which I shall have occasion to employ. If parallel tangent planes be drawn to a system of confocal ellipsoids, it is well known (see Dr Salmon's great work on *Surfaces*, Art. 202, 1st edition, or Art. 184, 2nd edition) that the points of contact lie in a plane curve, and that this curve is an equilateral hyperbola. Since a concentric sphere with an infinite radius belongs to the system of confocal ellipsoids supposed, it follows that the point of intersection of the perpendicular from the centre of the ellipsoid upon the tangent planes with the plane at infinity, is a point in this curve, or, in other words, such perpendicular is contained in the plane of the hyperbola, and is an asymptote to the latter. The above is all that is required to establish the dynamical theorems necessary for my immediate purpose.

The revolving body being assumed to have moments of inertia A, B, C about the principal axes, the ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1,$$

rigidly connected with the body, and which may be termed its kinematical exponent, is supposed to have its centre fixed, and to turn with a purely rolling motion upon a plane in contact with it which contains the constant impulsive couple L, capable at each moment of time in any position into which the body has turned, of communicating to it from rest the motion which it then actually possesses. If we suppose that the angular velocity of rotation is always equal to LRP, where P is the length of the perpendicular distance of the fixed centre from the tangent plane, and R is the length of the radius vector drawn from it to the point of contact, the path and velocity of the motion of the body in rigid connexion with the ellipsoid is completely represented; this is Poinsot's theorem stated in its complete form.

To fix the ideas, let us consider the invariable plane to be horizontal; if we were to apply a second plane parallel to the former fixed one, and also touching the ellipsoid, this would in no respect affect the motion—the ellipsoid might be made to roll between the two planes instead of rolling upon the under one alone; but if we were arbitrarily to alter the form of the upper part of the surface, the motion of rolling would in general be no longer possible; the only motion that could take place would be that of swinging round the vertical axis perpendicular to the two planes. In order that the ellipsoid may be able to roll as well as to swing, a certain geometrical condition must be satisfied, namely, the plane passing through the radius vector from the centre O to R, the point of contact with the given plane, and through the vertical perpendicular in question POp, must contain the point of contact r of the upper surface with the upper plane; for then, and then only, the rotation about OR may be resolved into two rotations about Or, Op respectively, and the ellipsoid whilst it rolls about OR, will be swinging round Op [or it may obviously at the same time be rolling and swinging (the latter in unequal degrees) upon each of the parallel tangent planes]; if this condition were not fulfilled, the ellipsoid, in the act of rolling upon the lower plane according to the direction of its motion, would either quit the upper one or tend to force it upwards; but as the upper, like the lower plane is supposed to be at a fixed distance from the centre, this tendency would be resisted, and thus the supposed motion of rolling upon the lower plane without quitting the contact with the upper one could not be realized.

The condition that OR, POp, Or shall lie on one plane, we have seen will be fulfilled if the upper surface be a portion of an ellipsoid confocal with the lower one, and in that case the body may remain continually in contact with both planes whilst it rolls on the lower one; and we have thus a complete solution of the kinematical problem of determining what form must be given to the upper part of a body, the lower portion of whose surface is ellipsoidal, in order that it may be able to roll as well as swing between, and in contact with, two parallel fixed planes.

Call, then, the squared semi-axes of the lower surface a^2, b^2, c^2, and those of the upper one $a^2-\lambda$, $b^2-\lambda$, $c^2-\lambda$, and let us proceed to calculate the respective values of the two rotations about Op, Or equivalent to the single rotation LPR about OR.

In PO, RO produced set off OP_1, OR_1 equal to OP, OR, and draw R_1r' parallel to Op, and rp perpendicular to Op, and make $Or=r$, $Op=p$; then by virtue of what has been remarked above, r, R_1 lie in a hyperbola, of which OpP_1 is an asymptote, and the rotation about the instantaneous axis OR is represented by $L.P.OR_1$, and may be resolved into $L.P.Or'$ about Or' and $L.P.r'R_1$ about Op.

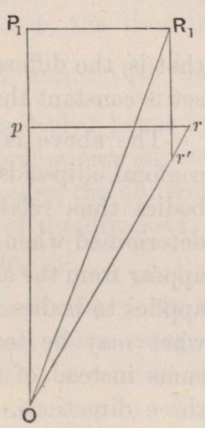

Fig. 1.

But $$L.P.Or' = L.P.r.\frac{Or'}{Or} = L.r.P.\frac{P_1R_1}{pr}$$
$$= L.r.P.\frac{Op}{OP_1} = L.r.p,$$

and $$L.P.R_1r' = L.P(OP_1 - P_1R_1 \cot r'Op)$$
$$= L.P\left(OP_1 - P_1R_1\frac{Op}{pr}\right)$$
$$= L.P\left(P - p\frac{p}{P}\right)$$
$$= L(P^2-p^2) = L\lambda;$$

for if α, β, γ be the angles which OP, Op make with the axes of the ellipsoid,

$$P^2 = a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2,$$
$$p^2 = (a^2-\lambda)(\cos\alpha)^2 + (b^2-\lambda)(\cos\beta)^2 + (c^2-\lambda)(\cos\gamma)^2,$$
$$P^2 - p^2 = \lambda\{(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2\} = \lambda.$$

Observing, then, that the motion has been resolved into a variable rotation Lpr about Or, and a uniform rotation $L\lambda$ about Op, and that accordingly the motion of a free body whose moments of inertia are as $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$

differs only by the uniform rotation $L\lambda$ from that of another one whose moments of inertia are as $\frac{1}{a^2-\lambda}, \frac{1}{b^2-\lambda}, \frac{1}{c^2-\lambda}$; we derive the following theorem :—

If the reciprocals of each of the moments of inertia of any number of rigid bodies $B, B_1, B_2, B_3, \ldots$ differ from one another by constant quantities, say those of the second, third, fourth, &c., from those of the first by $\lambda_1, \lambda_2, \lambda_3, \ldots$, and these bodies be arranged with their corresponding principal axes parallel and be set in motion by an impulsive couple L given in magnitude and direction, then, after the lapse of any interval of time t, the principal axes of all the bodies will remain equally inclined to the axis of the given couple, and moreover the parallelism of the axes may be restored by turning $B_1, B_2, B_3, \ldots$ about the axis of the couple through angles proportional to the time, namely, $L\lambda_1 t$, $L\lambda_2 t$, $L\lambda_3 t, \ldots$ respectively.

It may be further noticed that if, at any moment of time, ω, ω_1 are the angular velocities of B, B_1 about their respective instantaneous axes,

$$\begin{aligned}\omega^2-\omega_1^2 &= L^2(P^2 . R^2 - p^2 . r^2)\\ &= L^2\{P^2(R^2-P^2)-p^2(r^2-p^2)\} + L^2(P^4-p^4)\\ &= L^2\lambda(P^2+p^2),\end{aligned}$$

that is, the difference between the squared velocities of any two bodies of the set is constant throughout the motion.

The above is a theory of rigid bodies whose kinematical exponents are confocal ellipsoids, and it has been shown that the motion of the whole set of bodies thus related, both as regards position and velocity, is completely determined when we know the motion of any one of them. It will hereafter appear from the analytical treatment of the subject that an analogous theorem applies to bodies whose kinematical exponents, instead of being confocal, are what may be termed contrafocal ellipsoids; ellipsoids, that is to say, the sums instead of the differences of whose squared axes are the same in all three directions.

By turning an ellipse through 90° round its centre we obtain a contrafocal ellipse; and contrafocal ellipsoids will be those all of whose principal sections are contrafocal.

To every infinite series of confocal ellipsoids there will correspond another such series, each ellipsoid of one series being contrafocal to each of the other, and it may very easily be seen that no two ellipsoids taken respectively out of the two opposite series can be obtained from each other by a mere change of place, as is the case with contrafocal ellipses; so in the instance of binary covariants and contravariants, any such can be converted into each other by the simple interchange of x, y with y, $-x$, but no such or similar commutability exists between covariants and contravariants of the ternary species.

It may be here convenient to notice that the kinematical exponent (or momental ellipsoid) of a given uniform ellipsoid is not the ellipsoid itself, but the *reciprocal* of the contrafocal ellipsoid whose squared semi-axes are $\lambda - a^2$, $\lambda - b^2$, $\lambda - c^2$, where $\lambda = a^2 + b^2 + c^2$.

It is now clear how the time of passage from one position to another is susceptible of mechanical measurement. Let the upper part of Poinsot's ellipsoid, whose semi-axes are a, b, c, be pared away until it assumes the form of a segment of an ellipsoid whose squared semi-axes are $a^2 - \lambda$, $b^2 - \lambda$, $c^2 - \lambda$; let the lower surface be in contact with a rough plane absolutely fixed, whilst its upper surface is so with a parallel *plate* not absolutely fixed, but capable of turning round an axis perpendicular to the two planes, and which if produced would pass through the centre of the ellipsoid. Then, when by the hand or any mechanical contrivance the body is made to spin like a sort of top upon the lower plane, it will also spin upon the *plate* above, and at the same time by the friction drive it round the vertical axis; the angle of rotation round this axis will give the exact measure of the time which the *free* body ideally associated with the ellipsoid would occupy in passing from one position to another. If this angle (which of course may be made to register itself by the motion of a hand upon a fixed dial-plate immediately over the rotating one which carries the index) be called ϕ, the time in question will be $\frac{\phi}{L\lambda}$, where it is particularly deserving of notice that the denominator $L\lambda$ is independent of the initial position of the body; hence by supposing the plane and rotating-plate to be capable by a preliminary adjustment of being shifted to any required distance from one another, the ellipsoid may be started from any position we please, and the value of the divisions of the dial-plate which register the time will remain invariable.

The greater the value of λ which measures the degree of divergency of the two juxtaposed surfaces, the larger will be the divisions representing a given quantity of time; and there is no impediment to λ receiving its maximum value, which is the square of the least semi-axis (say c). The upper confocal surface then degenerates into a curve or hoop resting upon and driving before it the rotating-plate. This gives precision to the form to be assigned to the upper surface. Again, as regards the lower surface, whose form involves two parameters, namely, the ratios of the three axes, it will hereafter appear that we may without any loss of generality reduce it to depend upon a single parameter by assuming the reciprocal of the square of one of its axes equal to the sum of the reciprocals of the squares of the remaining two.

Hence with a single series of ellipsoids every possible kind of motion of a free rigid body may be completely represented both as regards time and place. Each ellipsoid with its confocal hoop may be regarded as complete in form, the former being imagined to consist of segments capable of being separated at will, so as to expose in succession each part as it is wanted of the interior hoop; and by an apparatus mechanically executable the motion may be followed without any break throughout the whole of one or any number of periods of revolution of the instantaneous axis.

Thus, then, the time of rotation of a free body may be kinematically determined. It may also, and even more simply, be measured off by direct observation of the time which a uniform ellipsoid spinning with its centre fixed upon an indefinitely rough plane occupies in passing from one position to another. To establish this somewhat remarkable law, let us consider the general case when the moments of inertia of the rolling ellipsoid have any values A, B, C. The resultant of the pressure and friction which coerce the ellipsoid to follow its actual path is a force always meeting the axis of instantaneous rotation, and giving rise therefore to an impressed couple whose axis is perpendicular to the former one. This being the case, and the ellipsoid subject to no other external force, its *vis viva* will be constant for just the same reason as the *vis viva* is so in the case of a system of particles connected in any manner, as by strings, whether elastic or inelastic, dragging each other along one or more surfaces, and acted on by no other forces except the reactions exerted by such surface or surfaces.

To render this perfectly clear, let v_1, v_2, v_3 denote the angular velocities of the rotating body about its principal axes; λ, μ, ν the angles between these axes and the instantaneous axis; J the magnitude of the couple produced by a force meeting the axis of rotation; then, by Euler's equations, we have

$$A\frac{dv_1}{dt} - (B-C)\,v_2v_3 = J\cos\lambda,$$

$$B\frac{dv_2}{dt} - (C-A)\,v_1v_3 = J\cos\mu,$$

$$C\frac{dv_3}{dt} - (A-B)\,v_1v_2 = J\cos\nu;$$

also $$v_1\cos\lambda + v_2\cos\mu + v_3\cos\nu = 0.$$

Hence $$Av_1dv_1 + Bv_2dv_2 + Cv_3dv_3 = 0,$$

and $$Av_1^2 + Bv_2^2 + Cv_3^2 = K,$$

a constant, as was to be proved.

In the case actually under consideration, if ω_1, ω_2, ω_3 are the angular velocities of the associated free body, and τ the time corresponding to t, so that dt, $d\tau$ are the intervals of time of the rolling and the free body undergoing the same infinitesimal angular displacement of position, we have

$$v_1 = \rho\omega_1, \quad v_2 = \rho\omega_2, \quad v_3 = \rho\omega_3,$$

and $$dt = \frac{d\tau}{\rho}.$$

Hence $$\rho^2 = \frac{K}{A\omega_1^2 + B\omega_2^2 + C\omega_3^2};$$

so that using the notation in ordinary use for the motion of a free body,

$$dt = \frac{d\tau}{\rho} = \frac{\omega d\omega \sqrt{(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)}}{\sqrt{\{(\omega^2 - e_1)(\omega^2 - e_2)(\omega^2 - e_3)\}}},$$

and thus the time t of the rolling ellipsoid is known as an elliptic function in terms of ω^2.

Furthermore, by the well-known equations of *vis viva* and conservation of areas applied to the free body whose kinematical exponent is the ellipsoid with semi-axes a, b, c, that is, whose moments of inertia may be denoted by $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, we have

$$\frac{\omega_1^2}{a^2} + \frac{\omega_2^2}{b^2} + \frac{\omega_3^2}{c^2} = M,$$

$$\frac{\omega_1^2}{a^4} + \frac{\omega_2^2}{b^4} + \frac{\omega_3^2}{c^4} = L^2.$$

Consequently if A, B, C are respectively representable by

$$\frac{\lambda}{a^4} + \frac{\mu}{a^2}, \quad \frac{\lambda}{b^4} + \frac{\mu}{b^2}, \quad \frac{\lambda}{c^4} + \frac{\mu}{c^2},$$

the multiplicator of $\omega d\omega$ in the numerator of the expression above given for dt, becomes a constant, namely, $\lambda L^2 + \mu M$. But this is the case when the density of the ellipsoid is uniform; for then

$$A : B : C :: b^2 + c^2 : c^2 + a^2 : a^2 + b^2,$$

and the determinant

$$\begin{vmatrix} \frac{1}{a^2}, & \frac{1}{b^2}, & \frac{1}{c^2} \\ \frac{1}{a^4}, & \frac{1}{b^4}, & \frac{1}{c^4} \\ b^2 + c^2, & c^2 + a^2, & a^2 + b^2 \end{vmatrix}, \text{ that is } \frac{1}{a^4 b^4 c^4} \Sigma (a^2 - b^2)(a^2 + b^2),$$

vanishes; in fact it is easily seen that

$$b^2 + c^2 = -\frac{a^2 b^2 c^2}{a^4} + \frac{b^2c^2 + c^2a^2 + a^2b^2}{a^2},$$

$$c^2 + a^2 = -\frac{a^2 b^2 c^2}{b^4} + \frac{b^2c^2 + c^2a^2 + a^2b^2}{b^2},$$

$$a^2 + b^2 = -\frac{a^2 b^2 c^2}{c^4} + \frac{b^2c^2 + c^2a^2 + a^2b^2}{c^2}.$$

Hence any uniform ellipsoid, with its centre fixed, compelled by friction to roll on a rough horizontal plane will move precisely like a free body with properly assigned moments of inertia acted on by no external forces, as was to be proved. We see from what has been shown above that a uniform ellipsoid whose semi-axes are a, b, c, and which rolls on a rough horizontal plane, will keep pace with the motion of a uniform free ellipsoid, provided that the moments of inertia of the latter are in the ratios of $\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}$, that is, provided its axes are in the proportions of

$$\sqrt{\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}\right)} : \sqrt{\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{1}{b^2}\right)} : \sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)},$$

and thus the relative rate of motion of the rolling ellipsoid will not be affected if an interior ellipsoid whose axes are in the proportions above written is entirely removed or its density altered in any ratio. The internal ellipsoid will in fact move precisely as if it were free and detached from the surrounding crust, and might be annihilated without affecting the motion of the latter, in analogy with the well-known fact that any weight at the centre of oscillation of a compound pendulum may be abstracted without affecting its motion.

The theories of the free body and of the ellipsoid constrained by pressure and friction to follow its path, and which has been proved above to keep

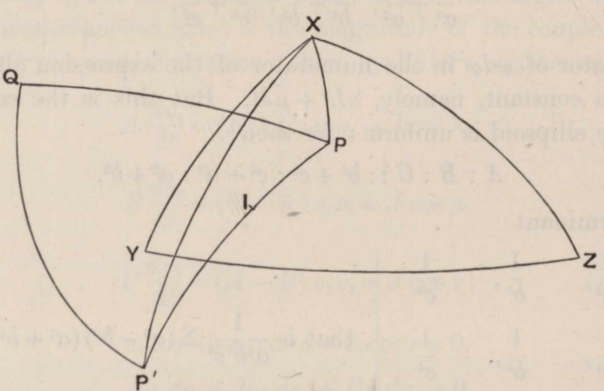

Fig. 2.

exact pace with it, are so interwoven that it would be unsatisfactory to leave the theory of the latter incomplete in any point, and I shall therefore proceed to calculate the value of the pressure and friction corresponding to any position of the rolling body. On a sphere described about the fixed point, let P and I denote the position of the instantaneous axis of rotation, and the perpendicular to the fixed plane respectively. The pole of the friction couple will be denoted by a point P' in the plane of PI distant by a quadrant from P, for its plane passes through P and through Q the pole of PI, and the pole of the pressure couple will obviously lie at Q itself. Let X, Y, Z mark in the sphere the positions of the principal axes.

Then XPP' being a quadrantal triangle,

$$\cos XP' = \sin XP \cos XPI = \frac{1}{\sin PI}(\cos XI - \cos XP \cos PI)$$

$$= \frac{1}{\sin PI}\left(\frac{\omega_1}{a^2 L} - \frac{\omega_1}{\omega}\frac{M}{L\omega}\right) = \frac{\omega_1}{L \sin PI}\left(\frac{1}{a^2} - \frac{M}{\omega^2}\right),$$

where $$\omega^2 = \omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2.$$

Again, for greater simplicity, making $\rho = 1$, that is, considering the motions of the rolling body and the free nucleus to absolutely coincide in time, we have from the Eulerian equations,

$$J \cos \lambda = (b^2 + c^2)\frac{d\omega_1}{d\tau} + (b^2 - c^2)\,\omega_2\omega_3$$

$$= \frac{c^2 - b^2}{b^2c^2}(a^2b^2 + a^2c^2 - b^2c^2)\,\omega_2\omega_3.$$

Hence if $[F]$ be the couple due to F the friction force,

$$[F] = \Sigma\,(J \cos\lambda \cos XP')$$

$$= \frac{\omega_1\omega_2\omega_3}{La^2b^2c^2 \sin PI\omega^2}\,\Sigma\,(c^2 - b^2)(a^2b^2 + a^2c^2 - b^2c^2)(\omega^2 - Ma^2)$$

$$= \frac{2(c^2 - b^2)(b^2 - a^2)(a^2 - c^2)\,\omega_1\omega_2\omega_3}{La^2b^2c^2 \sin PI}$$

$$= \frac{2(c^2 - b^2)(b^2 - a^2)(a^2 - c^2)\,\omega_1\omega_2\omega_3\omega}{a^2b^2c^2\sqrt{(L^2\omega^2 - M^2)}}.$$

And as the arm at which the friction acts, that is, the distance of the fixed centre from the point of contact between the ellipsoid and the fixed plane is $\frac{\sqrt{M}}{L} \sec PI$, that is, $\frac{\omega}{\sqrt{M}}$, we have

$$F = 2\,\frac{(c^2 - b^2)(b^2 - a^2)(a^2 - c^2)\,\omega_1\omega_2\omega_3}{a^2b^2c^2}\sqrt{\left(\frac{M}{L^2\omega^2 - M^2}\right)},$$

the mass of the ellipsoid throughout being treated as unity.

We might, in like manner, through the algorithm of spherical triangles, proceed to calculate the value of the pressure couple $[P]$ which is equal to the sum of the components $J \cos\lambda$, $J \cos\mu$, $J \cos\nu$ multiplied respectively by the sines of the perpendicular arcs dropped upon PI from X, Y, Z. But it will be obtained more expeditiously in its simplest form by first calculating J itself, the value of the entire couple, and then using the equation

$$[P]^2 = J^2 - [F]^2.$$

For brevity, in place of a^2, b^2, c^2, ω^2, L^2 write f, g, h, Ω, Λ respectively, and let $f+g+h=p$, $fg+gh+hf=q$, $fgh=r$. Then

$$\frac{\Omega_1}{f^2}+\frac{\Omega_2}{g^2}+\frac{\Omega_3}{h^2}=\Lambda,$$

$$\frac{\Omega_1}{f}+\frac{\Omega_2}{g}+\frac{\Omega_3}{h}=M,$$

$$\Omega_1+\Omega_2+\Omega_3=\Omega.$$

So that if $(f-g)(g-h)(h-f)=\zeta$, and $\kappa=\dfrac{fgh}{\zeta}$,

$$\Omega_1=\kappa\frac{h-g}{h^2g^2}\{\Omega-(g+h)M+gh\Lambda\},$$

$$\Omega_2=\kappa\frac{g-f}{f^2h^2}\{\Omega-(h+f)M+hf\Lambda\},$$

$$\Omega_3=\kappa\frac{h-g}{g^2f^2}\{\Omega-(f+g)M+fg\Lambda\}.$$

Hence
$$(\Lambda\Omega-M^2)[F]^2=-4\Omega^4+(8pM-4q\Lambda)\Omega^3$$
$$+\{(4p^2+4q)M^2-(4pq+12r)M\Lambda+4pr\Lambda^2\}\Omega^2$$
$$+\{(4pq-4r)M^3-(4q^2+4pr)M^2\Lambda+8qrM\Lambda^2-4r^2\Lambda^3\}\Omega.$$

Also from the Eulerian equations,

$$J^2=\Sigma\frac{(g-h)^2}{g^2h^2}(fg+fh-gh)^2\,\Omega_2\Omega_3$$

$$=\frac{1}{\zeta}\Sigma\{(g-h)(fg+fh-gh)^2N_1\},$$

where
$$N_1=\Omega^2-(2f+g+h)M\Omega+(fg+fh)\Lambda\Omega+(f^2+fg+fh+gh)M^2$$
$$-\{f^2(g+h)+2fgh\}M\Lambda+f^2gh\Lambda^2.$$

But

$$\Sigma(g-h)(fg+fh-gh)^2=4\Sigma(g^3h^2-g^2h^3)-4(fg+fh+gh)\Sigma(g^2h-h^2g)=0,$$

$$\Sigma(g-h)(fg+fh-gh)^2(2f+g+h)=\Sigma(fg-fh)(fg+fh-gh)^2$$
$$=4\Sigma(fg-fh)g^2h^2=-4fgh\zeta,$$

$$\Sigma(g-h)(fg+fh-gh)^2(fg+fh)=-\Sigma gh(g-h)(fg+fh-gh)^2$$
$$=-(f^2g^2+f^2h^2+g^2h^2)\Sigma gh(g-h)+2fgh\Sigma gh(g^2-h^2)$$
$$=\{(f^2g^2+f^2h^2+g^2h^2)-2fgh(f+g+h)\}\zeta,$$

$$\Sigma(g-h)(fg+fh-gh)^2(f^2+fg+fh+gh)=\Sigma f^2(g-h)(fg+fh-gh)^2$$
$$=\Sigma(fg+fh+gh)^2f^2(g-h)=-(fg+fh+gh)^2\zeta,$$

$$\Sigma(g-h)(fg+fh+gh)^2\{f^2(g+h)+2fgh\}=\Sigma(f^2g^2-f^2h^2)(fg+fh-gh)^2$$
$$=-4(fg+fh)\Sigma gh(f^2g^2-f^2h^2)=-4fgh(fg+fh+gh)\zeta,$$

$$\Sigma(g-h)(fg+fh-gh+gh)^2f^2gh=fgh\Sigma(fg-fh)(fg+fh-gh)^2=-4f^2g^2h^2\zeta.$$

Hence $$J^2 = \{4rM + (q^2 - 4pr)\Lambda\}\Omega - (qM - 2r\Lambda)^2,$$

and $$(\Lambda\Omega - M^2) J^2 = \{4rM\Lambda + (q^2 - 4pr)\Lambda^2\}\Omega^2$$
$$- \{4rM^3 + (2q^2 - 4pr) M^2\Lambda - 4qrM\Lambda^2 + 4r^2\Lambda^3\}\Omega + M^2(qM - 2r\Lambda)^2.$$

Hence $$(\Lambda\Omega - M^2)[P]^2 = (\Lambda\Omega - M^2) J^2 - (\Lambda\Omega - M^2)[F]^2$$
$$= 4\Omega^4 - (8pM^2 - 4q\Lambda)\Omega^3 + \{(4p^2 + 4q) M^2 - (4pq + 8r) M\Lambda + q^2\Lambda^2\}\Omega^2$$
$$- \{4pqM^3 - (2q^2 - 8pr) M^2\Lambda + 4qrM\Lambda^2\}\Omega + M^2(qM - 2r\Lambda)^2.$$

Hence $$[P] = \frac{2\Omega^2 - (2pM - q\Lambda)\Omega + M(qM - 2r\Lambda)}{\sqrt{(\Lambda\Omega - M^2)}};$$

and as the arm at which this couple acts is

$$\sqrt{\left(\frac{\omega^2}{M} - \frac{M}{L^2}\right)}, \text{ that is } \sqrt{\left(\frac{\Lambda\Omega - M^2}{M\Lambda}\right)},$$

the pressure $$P = \frac{2\Omega^2 - (2pM - q\Lambda)\Omega + (qM^2 - 2rM\Lambda)}{\Lambda\Omega - M^2}.$$

If we call the constant perpendicular from the centre and the radius vector to the point of contact h and l respectively, and substitute for $\frac{\Omega}{\Lambda}$, $\frac{M}{\Lambda}$ their respective values h^2l^2, h^2, we may express $\frac{F}{P}$ as a function of h, l, and making this a maximum in respect to l, the least sufficient value of the coefficient of friction necessary to ensure rolling may be deduced in terms of the quantities $\frac{a}{h}$, $\frac{b}{h}$, $\frac{c}{h}$.

Also if θ denote the angle between the axis of the couple J and the pole of the plane PI, we have

$$(\cos\theta)^2 = \frac{[P]^2}{J^2} = \frac{\{2h^4l^4 - (2ph^2 - q) h^2l^2 + (qh^4 - 2rh^2)\}^2}{(h^2l^2 - h^4)\{(4rh^2 + q^2 - 4pr) h^2l^2 - (qh^2 - 2r)^2\}},$$

or $$\cos\theta = \frac{h\{2h^2l^4 - (2ph^2 - q) l^2 + (qh^2 - 2r)\}}{\sqrt{(l^2 - h^2)}\sqrt{\{(4rh^2 + q^2 - 4pr) h^2l^2 - (qh^2 - 2r)^2\}}}.$$

It has been already seen how, by the method of confocal ellipsoids, the number of constants entering into the question of the rotation of a rigid body about its centre of gravity has virtually been reduced by a unit; to render this important theory complete, and to give it the fullest extension of which it is capable, a corresponding dynamical theory of contrafocal ellipsoids remains to be developed, and might undoubtedly be discussed by analogous geometrical methods; but it will be found more expedient to take up the subject afresh from a purely analytical point of view, and then the theory will present itself in all its completeness under a single aspect.

Calling α, β, γ the angles which the invariable axis makes with the principal axes of the rotating body, we have the well-known equations

$$\cos\alpha = \frac{A\omega_1}{L}, \quad \cos\beta = \frac{B\omega_2}{L}, \quad \cos\gamma = \frac{C\omega_3}{L},$$

(immediate deductions from the self-obvious principle of the constancy of the couple competent at any instant to communicate to the rotating body the motion it is then actually endued with, conjoined with the geometrical property of the principal axes that the moment in respect to any one of them of the momenta of the particles of the body due to rotation about either of the other two is zero).

Consequently from the principle of *vis viva*, that is, from the equation

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = M,$$

in addition to the equation

$$(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1, \qquad (1)$$

we have the equation

$$\frac{(\cos\alpha)^2}{A} + \frac{(\cos\beta)^2}{B} + \frac{(\cos\gamma)^2}{C} = \frac{M}{L^2}, \qquad (2)$$

and the Eulerian system of equations*,

$$A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 = 0, \quad B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 = 0, \quad C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 = 0,$$

* To make this paper complete within itself so as to come within the comprehension of those who have no previous knowledge of the special problem which it treats, it seems desirable to indicate an elementary method of obtaining these ofttimes herein quoted equations.

1. Suppose no external forces in operation. Consider the effects of the three partial velocities ω_1, ω_2, ω_3 in succession as if the others were non-existent.

Referring to fig. 3, ω_1 tends to produce no motion about OY or OZ in the time dt, because the moments of the centrifugal forces about these axes, quantitatively represented by Σmzx, Σmxy respectively, are each zero by virtue of the geometrical definition of the principal axes.

Thus to each partial velocity in the time dt is due only a motion of rotation about its own axis. Hence if $d\gamma$ is the variation in γ due to ω_1,

$$d\gamma = ZZ' \cos YZI = \omega_1 dt \frac{\cos\beta}{\sin\gamma},$$

or

$$d\cos\gamma = \omega_1 \cos\beta\, dt.$$

Similarly as regards the variation of $\cos\gamma$ due to ω_2,

$$d\cos\gamma = -\omega_2\cos\alpha\, dt.$$

Hence the total variation $d\cos\gamma = (\omega_1\cos\beta - \omega_2\cos\alpha)\,dt$,

that is

$$\frac{C}{L}d\omega_3 = \left(\frac{B\omega_2\omega_1}{L} - \frac{A\omega_1\omega_2}{L}\right)dt,$$

or

$$d\omega_3 = \frac{B-A}{C}\omega_2\omega_1 dt,$$

with analogous equations for $d\omega_2$, $d\omega_1$.

becomes
$$\left.\begin{aligned}\frac{d\cos\alpha}{dt}-L\left(\frac{1}{C}-\frac{1}{B}\right)\cos\beta\cos\gamma=0,\\ \frac{d\cos\beta}{dt}-L\left(\frac{1}{A}-\frac{1}{C}\right)\cos\gamma\cos\alpha=0,\\ \frac{d\cos\gamma}{dt}-L\left(\frac{1}{B}-\frac{1}{A}\right)\cos\alpha\cos\beta=0.\end{aligned}\right\}\quad(3)$$

The above equations suffice to express the relations of the angles which the invariable line in space makes with fixed lines in the moving body to one another and to the time: to complete the solution it will be sufficient to express in terms of the time, or of any quantity dependent on the time, the position of any of the planes drawn through a principal axis and the invariable line.

The letters X, Y, Z, I retaining their previous signification, let ZZ' represent the infinitesimal angular displacement of Z due to the rotation ω_1 about X in the time dt.

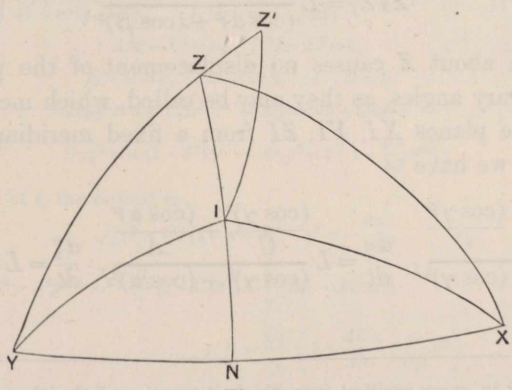

Fig. 3.

When the impressed couples about OX, OY, OZ respectively are L, M, N the variations in the angular velocities due to them being
$$\frac{Ldt}{A},\ \frac{Mdt}{B},\ \frac{Ndt}{C},$$
these quantities must be added to the values of $d\omega_1$, $d\omega_2$, $d\omega_3$ indicated above. We have thus the equations in question.

It may be as well also here to indicate in the fewest words the rationale of the ellipsoidal representation of the motion.

A, B, C being the principal moments of inertia, and $Ax^2+By^2+Cz^2=1$ the equation to the ellipsoid, the relation
$$\omega_1:\omega_2:\omega_3::A\cos\alpha:B\cos\beta:C\cos\gamma$$
shows that the invariable line coincides in direction with the pedal to the radius vector drawn in the direction of the instantaneous axis.

2. Consequently the length of such pedal being
$$\frac{(\cos\alpha)^2}{A}+\frac{(\cos\beta)^2}{B}+\frac{(\cos\gamma)^2}{C},$$
which is constant, a plane drawn at that constant length perpendicular to the invariable line

Then $$ZIZ' = ZZ' \frac{\sin IZZ'}{\sin IZ'} = ZZ' \frac{\cos NX}{\sin IZ}.$$

But $$\frac{\cos NX}{\cos NY} = \frac{\cos IX}{\cos IY} = \frac{\cos\alpha}{\cos\beta},$$

or $$\cos NX = \frac{\cos\alpha}{\sqrt{\{(\cos\alpha)^2 + (\cos\beta)^2\}}},$$

and $$\sin IZ = \sqrt{\{1 - (\cos\gamma)^2\}} = \sqrt{\{(\cos\alpha)^2 + (\cos\beta)^2\}}.$$

Hence $$ZIZ' = L \frac{\frac{(\cos\alpha)^2}{A} dt}{(\cos\alpha)^2 + (\cos\beta)^2}.$$

Similarly, if ZIZ_1 be the angular displacement of the plane ZI measured in the same direction as before,

$$ZIZ_1 = L \frac{\frac{(\cos\beta)^2}{B} dt}{(\cos\alpha)^2 + (\cos\beta)^2},$$

and the rotation about Z causes no displacement of the plane in question. Hence if the horary angles, as they may be called, which measure the angular deviations of the planes XI, YI, ZI from a fixed meridian plane through I be called ξ, η, ζ, we have

$$\frac{d\xi}{dt} = L \frac{\frac{(\cos\beta)^2}{B} + \frac{(\cos\gamma)^2}{C}}{(\cos\beta)^2 + (\cos\gamma)^2}, \quad \frac{d\eta}{dt} = L \frac{\frac{(\cos\gamma)^2}{C} + \frac{(\cos\alpha)^2}{A}}{(\cos\gamma)^2 + (\cos\alpha)^2}, \quad \frac{d\zeta}{dt} = L \frac{\frac{(\cos\alpha)^2}{A} + \frac{(\cos\beta)^2}{B}}{(\cos\alpha)^2 + (\cos\beta)^2} \text{*} \quad (4).$$

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touches the ellipsoid in every position into which it turns, and therefore the ellipsoid with its centre fixed rolls on such plane. This proves the identity of the two motions *quâ* space.

3. The moment of inertia in respect to the instantaneous axis being represented by the inverse squared length of the radius vector of the ellipsoid in the direction of that axis, the square root of the *vis viva* (a constant) is proportional to the angular velocity divided by the radius vector drawn to the point of contact, so that the former is proportional to the latter; this completes the representation by expressing through means of the ellipsoid the relation of the motion of the associated free body to time, or at all events it gives the law from which that relation may be extracted.

The above contains the whole sum, pith, and substance of Poinsot's ellipsoidal mode of representation.

* By combining this with the system of equations previously found, both η and ζ may readily be obtained under the form of elliptic functions of the third kind in terms of $\cos\gamma$, but $\eta - \zeta$ or the angle I in the quadrantal spherical triangle XIY of fig. 3 will also be expressible as a function of α, β, and therefore of γ. The comparison of the forms of $\eta - \zeta$ given by the two methods respectively, leads therefore to a theorem in elliptic functions; Professor Cayley has worked this out, and finds that it is the well-known theorem which expresses the dependence between two elliptic functions of the third order, the product of whose parameters is equal to the square of the modulus. I subjoin an extract from his letter, in which I have only introduced some slight changes in the lettering:—

If, now, preserving L constant we replace $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, M by

$$\frac{1}{A}-\lambda\,;\ \frac{1}{B}-\lambda\,;\ \frac{1}{C}-\lambda\,;\ M-\lambda L^2,$$

the equations (1), (2), (3) remain unaltered, and the right-hand sides of equations (4) become each of them simply altered by the addition of the term $-L\lambda$, which may be expressed by saying that the difference between

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"Writing
$$Ap^2+Bq^2+Cr^2=M,$$
$$A^2p^2+B^2q^2+C^2r^2=L^2,$$

your theorem is
$$\int\frac{M-Ap^2}{L^2-A^2p^2}dt-\int\frac{M-Bq^2}{L^2-B^2q^2}dt=\cos^{-1}(\ldots\ldots)$$
$$=\tan^{-1}(\ldots\ldots),$$

where
$$dt=-\frac{Cdr}{(A-B)\,pq}.$$

Whence expressing everything in terms of r, this is
$$\int\frac{F+Gr^2}{(1+nr^2)\sqrt{(R)}}dr-\int\frac{F_1+G_1r^2}{(1+n_1r^2)\sqrt{(R)}}dr=\tan^{-1}(\ldots\ldots);$$

write for shortness,
$$AM-L^2=a,\quad BM-L^2=b,$$
$$B-C=\alpha,\quad C-A=\beta,\quad A-B=\gamma.$$

Then we have
$$B\gamma q^2=a+C\beta r^2\,;\quad -A\gamma p^2=b-C\alpha r^2\,;$$

or if $r^2=-\frac{a}{C\beta}\theta^2$,
$$B\gamma q^2=a(1-\theta^2)\,;\quad -A\gamma p^2=b\left(1+\frac{a\alpha}{b\beta}\theta^2\right);$$

so that using θ instead of r, the radical is
$$\sqrt{\{(1-\theta^2)(1-\kappa^2\theta^2)\}},\quad \kappa^2=-\frac{a\alpha}{b\beta},$$

$$L^2-A^2p^2=L^2+\frac{A}{\gamma}(b-C\alpha r^2)=\frac{1}{\gamma}(Ba-AC\alpha r^2)$$
$$=\frac{Ba}{\gamma}\left(1+\frac{AC\alpha}{Ba}\cdot\frac{a}{C\beta}\theta^2\right)$$
$$=\frac{Ba}{\gamma}\left(1+\frac{A\alpha}{B\beta}\theta^2\right),$$

$$L^2-B^2q^2=L^2-\frac{B}{\gamma}(a+C\beta r^2)=\frac{1}{\gamma}(-Ab-BC\beta r^2)$$
$$=-\frac{Ab}{\gamma}\left(1-\frac{BC\beta}{Ab}\cdot\frac{a}{C\beta}\theta^2\right)$$
$$=-\frac{AB}{\gamma}\left(1-\frac{Ba}{Ab}\theta^2\right).$$

So that the form is
$$\int d\theta\frac{F+G\theta^2}{(1+n\theta^2)\sqrt{(\Theta)}}-\int d\theta\frac{F_1+G_1\theta^2}{(1+n_1\theta^2)\sqrt{(\Theta)}}=\ldots\ldots,$$

where
$$n=\frac{A\alpha}{B\beta},\quad n_1=-\frac{Ba}{Ab},\quad \kappa^2=-\frac{a\alpha}{b\beta},$$

and thus
$$nn_1=-\frac{a\alpha}{b\beta}=\kappa^2,$$

so that the relation is the known one between the two forms
$$\int\frac{d\theta}{(1+n\theta^2)\sqrt{(\Theta)}}\ \text{and}\ \int\frac{d\theta}{\left(1+\frac{\kappa^2}{n}\theta^2\right)\sqrt{(\Theta)}},$$

with reciprocal parameters."

the displacements at any moment of time of two bodies whose kinematical exponents are confocal ellipsoids, is equivalent to a displacement round the invariable line proportional to the time elapsed since the positions were coincident or parallel, as previously found by geometrical reasoning.

Again, if we replace $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, M by

$$\lambda - \frac{1}{A};\quad \lambda - \frac{1}{B};\quad \lambda - \frac{1}{C};\quad \lambda L^2 - M,$$

the equations (1), (2), (3) will remain unaltered, provided we write $180 - \alpha$; $180 - \beta$; $180 - \gamma$ in place of α, β, γ, and the equations (4) will receive an augmentation of $L\lambda$ on their right-hand sides, but remain otherwise unaltered, provided we substitute $-\xi, -\eta, -\zeta$ for ξ, η, ζ. Or again, we may state the same result without substituting for the angles of inclinations their supplements, but leaving them unaltered if we change the sign of L; showing that if two bodies whose kinematical exponents or momental ellipsoids are contrafocal, be set in a parallel position at rest, and are acted on by two equal and coaxial but contrary impulsive couples, their principal axes will continue throughout the motion to make equal but contrary angles with the invariable line, and will admit of being brought back to a position of parallelism by means of a rotatory displacement about the invariable line proportional to the time. Thus, leaving out of consideration this displacement, *correlated* solid bodies (as those may be termed whose kinematical exponents are confocal ellipsoids) may be made to move equally and similarly, and *contrarelated* ones (as we may term those whose kinematical exponents are contrafocal ellipsoids) equally and contrarily without the action of any external force. It will eventually be seen that there is a practical advantage in considering L as retaining the same sign in both cases, and throwing the contrariety of motion in the second case upon the change of the inclinations α, β, γ into their supplements.

Thus the motion of a body is arithmetically given when that of any other of the series of those to whose kinematical exponents its own is either confocal or contrafocal has been determined.

Alike for the two cases of con- and contra-focalism it will be convenient to disregard this uniform motion of rotation, treating it in the light merely of a correction*, so that the motions of all the bodies contained in either one series may be considered in regard to themselves as *coincident*, and as *supplemental* (in a sense that explains itself) in regard to the motions of the bodies belonging to the other series. I shall now show as a corollary from the

* The apparent motions of any two correlated or contrarelated bodies to two spectators standing respectively on the invariable plane of each may be made identical or similar, provided a certain uniform angular velocity be imparted to one of these planes.

above proposition that, with the above understanding, the motion of any rigid body may (subject to an unimportant exception that will be stated in its proper place) be made identical with that of one real indefinitely flattened disk, and supplemental to that of another. The case of a disk, it will be noticed, is that in which one of the principal moments of inertia becomes equal to the sum of the other two; in general these moments of inertia must not only be positive, but each must be not greater than the sum of the other two, as is the case with the lengths of the sides of a triangle; in the extreme case, when the body is reduced to but two dimensions, the greatest becomes equal to the sum of the other two, and conversely, when this is so, the body can only be of the form of a flat disk; the above is obvious when it is remembered that the moments of inertia are the sums of the three intrinsically positive quantities Σmx^2, Σmy^2, Σmz^2 taken two and two together. So also it is well to notice that the modular quantity $\frac{M}{L^2}$ in equation (2) is not absolutely arbitrary, but besides being essentially positive, is conditioned to lie between the least and greatest of the quantities $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, since otherwise the quantities $(\cos\alpha)^2$, $(\cos\beta)^2$, $(\cos\gamma)^2$ in equations (1) and (2) could not all remain positive, and consequently such equations would not correspond to any real case of motion.

Let A, B, C be arranged in order of magnitude, and suppose

$$\frac{1}{A_1}=\frac{1}{A}-\frac{1}{\mu},\quad \frac{1}{B_1}=\frac{1}{B}-\frac{1}{\mu},\quad \frac{1}{C_1}=\frac{1}{C}-\frac{1}{\mu},\quad \frac{M_1}{L^2}=\frac{M}{L^2}-\frac{1}{\mu},$$

and let μ be so determined as to make one of the quantities A_1, B_1, C_1 equal to the sum of the other two. Then

(1) Any imaginary value of μ must be neglected.

(2) Any value of μ which makes A_1, B_1, C_1 of different algebraical signs must be neglected.

(3) If μ, being real, makes A_1, B_1, C_1 all positive, these quantities will correspond to the moments of a real disk whose representative ellipsoid is confocal to that of the body whose moments of inertia A, B, C are given.

(4) If μ, being real, makes A_1, B_1, C_1 all negative, by taking $-A_1$, $-B_1$, $-C_1$, that is, the reciprocals of $\frac{1}{\mu}-\frac{1}{A}$, $\frac{1}{\mu}-\frac{1}{B}$, $\frac{1}{\mu}-\frac{1}{C}$ as the new moments of inertia, we evidently shall have obtained a reduction to a disk of the supplemental or contrafocal kind.

In case (3) $M-\frac{L^2}{\mu}$, and in case (4) $\frac{L^2}{\mu}-M$ is to be substituted for M, so that the necessary condition of $\frac{L^2}{M}$ being intermediate between the greatest and least of the quantities A, B, C will continue to be fulfilled in the disk by $\frac{L^2}{M_1}$ remaining intermediate between the greatest and least of the quantities A_1, B_1, C_1.

Suppose $A_1 + B_1 = C_1$, then

$$\frac{A}{A-\mu} + \frac{B}{B-\mu} = \frac{C}{C-\mu},$$

or
$$(A + B - C)\mu^2 + AB\mu + ABC = 0.$$

The determinant (that is, negative discriminant) of this equation is

$$AB(AB - CA - CB + C^2) \text{ or } AB(A - C)(B - C),$$

so that if C is the least or greatest moment of inertia, μ will have real values, but will be unreal if C is the mean moment of inertia.

Suppose now that $A_1 + B_1 = C_1$ for one value of μ, to find the values A', B', C' corresponding to the conjugate disk, we obtain from the above equation in μ, by substituting A_1, B_1, C_1 for A, B, C,

$$2A_1B_1\mu - A_1B_1C_1 = 0, \text{ or } \mu = -\frac{C_1}{2},$$

and accordingly

$$\frac{1}{A'} : \frac{1}{B'} :: \frac{1}{A_1} - \frac{2}{A_1+B_1} : \frac{1}{B_1} - \frac{2}{A_1+B_1} :: \frac{B_1 - A_1}{2A_1} : \frac{A_1 - B_1}{2B_1} :: \frac{1}{A_1} : -\frac{1}{B_1}.$$

Hence if A_1, B_1 have the same signs, A', B' have opposite signs, and *vice versâ*, if A_1, B_1 have opposite signs A', B', and therefore A', B', C' have all the same signs for $C' = A' + B'$.

Consequently one and one only of each of the two solutions for disks drawn perpendicular respectively to the extreme principal axes, makes the three moments of inertia all of the same sign, and consequently each such solution leads either to a direct or supplemental reduction to the disk form.

Now, suppose that A, B, C being all of the same signs, A has become equal to $B + C$, so that the equation in μ becomes

$$2B\mu^2 + 2AB\mu + ABC = 0,$$

or
$$\mu^2 - A\mu + \frac{AC}{2} = 0.$$

Let μ_1, μ' be the two values of μ from this equation, so that

$$\frac{1}{A_1} = \frac{1}{A} - \frac{1}{\mu_1}, \quad \frac{1}{B_1} = \frac{1}{B} - \frac{1}{\mu_1}, \quad \frac{1}{C_1} = \frac{1}{C} - \frac{1}{\mu_1},$$

$$\frac{1}{A'} = \frac{1}{A} - \frac{1}{\mu'}, \quad \frac{1}{B'} = \frac{1}{B} - \frac{1}{\mu'}, \quad \frac{1}{C'} = \frac{1}{C} - \frac{1}{\mu'},$$

and
$$A_1 + B_1 = C_1, \quad A' + B' = C'.$$

Then $$\frac{1}{A_1}+\frac{1}{A'}=\frac{2}{A}-\left(\frac{1}{\mu_1}+\frac{1}{\mu'}\right)=\frac{2}{A}-\frac{2}{C}=2\left\{\frac{1}{B+C}-\frac{1}{C}\right\}=-\frac{2}{AC},$$

$$\frac{1}{A_1}\cdot\frac{1}{A'}=\frac{1}{A^2}-\left(\frac{1}{\mu_1}+\frac{1}{\mu'}\right)\frac{1}{A}+\frac{1}{\mu_1\mu'}$$

$$=\frac{1}{A^2}.$$

Hence if A, B, C have the positive sign, A_1 and A' are both negative, and if A, B, C have the negative sign, A_1 and A' are both positive; consequently, on the first supposition, the signs of one of the two systems A_1, B_1, C_1; A', B', C' will be all negative, and on the second supposition all positive. Hence one of the two reductions falls under case (3), that is, is proper or direct, and the other under case (4), and is improper or supplemental. As nothing in nature exists in vain, it will presently be seen that the choice which is always possible between these two modes of reduction leads to an important simplification of the cases which arise in the problem of rotation, and that there need never be any room for doubt as to which of the two sorts of reduction should be employed in any specified problem.

The case of exception to which allusion has been made in anticipation, arises when two of the moments of inertia are equal; for then, supposing A, A, C to be the original moments of inertia, the new moments of inertia will be A_1, A_1, C_1; and since C_1 cannot be zero, we can only suppose $C_1=2A_1$; and making

$$\frac{1}{A_1}=\frac{1}{A}-\frac{1}{\mu},\quad \frac{1}{C_1}=\frac{1}{C}-\frac{1}{\mu},$$

the equation in μ becomes

$$\frac{2A}{A-\mu}=\frac{C}{C-\mu},$$

or $$(2A-C)\,\mu=AC,\quad \mu=\frac{AC}{2A-C},$$

and $$A_1=\frac{A\mu}{\mu-A}=\frac{AC}{2\,(C-A)},\quad C_1=\frac{AC}{C-A};$$

so that the reduction will be proper or improper according as the unequal moment of inertia is greater or less than either of the equal ones.

A relation has been obtained geometrically in the commencement of this memoir between the squared velocities of any two dynamically equivalent bodies represented by confocal ellipsoids. To complete the theory, it is proper to find the exact nature of this relation when a given body has been reduced to a disk, whether by the direct or supplemental method.

First, in the case of direct reduction, using v_1, v_2, v_3 for the angular velocities of the disk, and ω_1, ω_2, ω_3 for those of the associated body in

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corresponding positions about the principal axes, and υ, ω for the total angular velocities of the disk and body respectively,

$$\upsilon_1 = \frac{L}{A_1}\cos\alpha,\quad \upsilon_2 = \frac{L}{B_1}\cos\beta,\quad \upsilon_3 = \frac{L}{C_1}\cos\gamma,$$

$$\omega_1 = \frac{L}{A}\cos\alpha,\quad \omega_2 = \frac{L}{B}\cos\beta,\quad \omega_3 = \frac{L}{C}\cos\gamma,$$

$$\frac{1}{A_1} = \frac{1}{A} - \lambda,\quad \frac{1}{B_1} = \frac{1}{B} - \lambda,\quad \frac{1}{C_1} = \frac{1}{C} - \lambda.$$

Hence $$\omega^2 = \Sigma\omega_1^2 = \Sigma\left(\frac{L}{A_1} + L\lambda\right)^2 (\cos\alpha)^2 = \Sigma\upsilon_1^2 + 2L^2\lambda\Sigma\frac{(\cos\alpha)^2}{A_1} + L^2\lambda^2$$

$$= \upsilon^2 + 2L^2\lambda\frac{M}{L^2} + L^2\lambda^2,$$

or $$\omega^2 - \upsilon^2 = \lambda L^2\left(\frac{2M}{L^2} + \lambda\right).$$

And again, in the case of supplemental reduction, using υ_1, υ_2, υ_3, υ for the partial and total angular velocities of the disk,

$$\upsilon_1 = -\frac{L}{A_1}\cos\alpha,\quad \upsilon_2 = -\frac{L}{B_1}\cos\beta,\quad \upsilon_3 = -\frac{L}{C_1}\cos\gamma,$$

$$\frac{1}{A'} = \lambda - \frac{1}{A},\quad \frac{1}{B'} = \lambda - \frac{1}{B},\quad \frac{1}{C'} = \lambda - \frac{1}{C},$$

$$\omega^2 = \Sigma\left(L\lambda - \frac{L}{A_1}\right)^2 (\cos\alpha)^2 = \upsilon^2 - 2L^2\lambda\frac{M}{L^2} + L^2\lambda^2,$$

or $$\omega^2 - \upsilon^2 = \lambda L^2\left(-\frac{2M}{L^2} + \lambda\right);$$

showing that in both cases alike the difference between the squared velocity of the body and that of either its representative disks is constant throughout the motion, as might also have been predicted *à priori* from the form of the elliptic function connecting the time with the squared velocity. In the case of disk motion there is a distinctive feature which is deserving of notice. In this case we have

$$A\omega_1^2 + B\omega_2^2 + (A+B)\,\omega_3^2 = M,$$

$$A^2\omega_1^2 + B^2\omega_2^2 + (A+B)^2\,\omega_3^2 = L^2.$$

Hence $$AB(\omega_1^2 + \omega_2^2) = (A+B)^2 M - L^2,$$

showing that the angular velocity with which the disk turns about a line in its own plane is constant throughout the motion, whilst the velocity about the axis perpendicular to its plane is continually varying, in the first particular agreeing with, and in the second differing from what takes place for a body of three dimensions with two of its principal moments of inertia equal.

It is easy to see how in the general case every conceivable motion of a body of any form may be tabulated and reduced to a table of treble entry, and how greatly the use of such tables may be facilitated, and seemingly distinct cases reduced to identity by aid of the twofold method of reduction above explained. Let us consider the case of a body whose principal moments of inertia are A, B, C, arranged in ascending order of magnitude.

We have seen that the quantity $\frac{M}{L^2}$ must always be intermediate between $\frac{1}{A}$ and $\frac{1}{C}$.

If the direct reduction be employed instead of $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{M}{L^2}$, we shall have

$$\frac{1}{A}-\lambda,\ \frac{1}{B}-\lambda,\ \frac{1}{C}-\lambda,\ \frac{M}{L^2}-\lambda, \text{ say } \frac{1}{A_1}, \frac{1}{B_1}, \frac{1}{C_1}, \frac{M_1}{L^2};$$

and if $\frac{M}{L^2}$ is intermediate between $\frac{1}{B}$ and $\frac{1}{C}$, $\frac{M_1}{L^2}$ will be intermediate between $\frac{1}{B_1}$ and $\frac{1}{C_1}$, where $C_1 = A_1 + B_1$.

On the other hand, if the supplemental method be employed,

$$\lambda-\frac{1}{A},\ \lambda-\frac{1}{B},\ \lambda-\frac{1}{C},\ \lambda-\frac{M}{L^2}, \text{ say } \frac{1}{C'}, \frac{1}{B'}, \frac{1}{A'}, \frac{M'}{L^2},$$

where $C' = A' + B'$ will take the place of $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}$; so that if $\frac{M}{L^2}$ is intermediate between $\frac{1}{A}$ and $\frac{1}{B}$, $\frac{M'}{L^2}$ will be intermediate between $\frac{1}{B'}$ and $\frac{1}{C'}$.

Hence by using the direct method of reduction in the case where $\frac{L^2}{M}$ is greater than B, and the supplemental method of reduction where $\frac{L^2}{M}$ is less than B, the original body can be always replaced by a disk of which A_1, B_1, $A_1 + B_1$ are the new principal moments of inertia, L the given initial impulsive couple, M the new *vis viva*, and where the ascending order of the magnitudes is $\frac{1}{A+B}, \frac{M}{L^2}, \frac{1}{B}, \frac{1}{A}$, so that $\frac{BM}{L^2}$, $\frac{AM}{L^2}$ will be both of them less than unity. This reduction being effected when the motion of the disk is known, that of the associated body is given.

Calling the two parameters $\frac{BM}{L^2}$, $\frac{AM}{L^2}$, q_2, q_1* respectively, an inspection

* Calling A, B, C the original moments of inertia, it is important to notice that we have seen that no real distinction of motion arises from $\frac{M}{L^2}$ lying between $\frac{1}{A}$ and $\frac{1}{B}$ on the one hand, or between $\frac{1}{B}$ and $\frac{1}{C}$ on the other; the so-called two kinds of polhods and Legendre's primary

of the system of equations (1, 2, 3, 4) at pp. [588—590] will show that the angles $\alpha, \beta, \gamma, \xi, \eta, \zeta, \omega$ are known, and may be registered in a table when $\frac{M}{L}t, q_1, q_2$ are given, the time t being reckoned from some determinate epoch, which must be so fixed as to be identical for the disk and the associated body*. We may assume as such epoch indifferently the moment when the axis of the disk has its maximum, or when it has its minimum inclination to the invariable line, that is, when the quantity $(\cos\gamma)^2$ in the equations

$$\left.\begin{aligned}(\cos\alpha)^2+(\cos\beta)^2+(\cos\gamma)^2=1,\\ \frac{(\cos\alpha)^2}{q_1}+\frac{(\cos\beta)^2}{q_2}+\frac{(\cos\gamma)^2}{q_1+q_2}=1,\end{aligned}\right\}*$$

attains its maximum or minimum value; the equations being linear between $(\cos\alpha)^2$, $(\cos\beta)^2$, $(\cos\gamma)^2$, say between x, y, z, the extreme values of z of course correspond to the zero values of x and y respectively.

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distinction of the problem into his cases (1) and (2) turn entirely upon this difference, but the two kinds of motion are convertible into one another (save as to the correction for the uniform displacement round the invariable line) by the theory of contra-relation. The real essential distinction of cases can only arise from particular values being assumed by q_1, q_2.

The quantities $0, q_1, q_2, 1, q_1+q_2$ are written in natural ascending order.

The two *singular* cases are (A) when $q_1=q_2$, which is the case of two equal moments, (B) when $q_2=1$, which is Legendre's *Troisième Cas*, *Cas très-remarquable*, Arts. 26, 27, corresponding to the instantaneous axis describing the so-called "separating polhod."

Besides these properly called singular cases, there are what may be termed special cases arising from sequences of two or three terms in the above quinary scale becoming approximately equal, or *subequal*, in Mr De Morgan's language, which relation may be denoted by the ordinary sign of equivalence.

Thus we shall have special cases when

$$q_1\equiv 0,\ \text{or}\ q_2\equiv q_1,\ \text{or}\ 1\equiv q_2,\ \text{or}\ q_1+q_2\equiv 1,$$

and double-special cases when

$$q_2\equiv q_1\equiv 0,\quad 1\equiv q_2\equiv q_1,\quad q_1+q_2\equiv 1\equiv q_2.$$

The last of these is of course tantamount to $1\equiv q_2$ with $q_1\equiv 0$. But even this table does not exhaust all the specially notable cases; for in the first of the double-special cases which corresponds to that of a body differing little from a sphere, we may again mark off as extra-double special the case where $\frac{q_1}{q_2}\equiv 0$, and also that where $\frac{q_1}{q_2}\equiv 1$.

It does not fall in with the plan of this paper to investigate these several cases, but they are probably all of them deserving of particular examination.

* We may express the motion in terms of the parameters q_1, q_2 as follows, writing x, y, z for $\cos\alpha, \cos\beta, \cos\gamma$:

$$\left.\begin{aligned}x^2+y^2+z^2=1,\\ \frac{x^2}{q_1}+\frac{y^2}{q_2}+\frac{z^2}{q_1+q_2}=1,\end{aligned}\right\}\tag{1}$$

$$\frac{dz}{dt}=L\left(\frac{1}{A}-\frac{1}{B}\right)xy=\frac{M}{L}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)xy,\tag{2}$$

$$d\zeta=L\frac{\frac{x^2}{A}+\frac{y^2}{B}}{1-z^2}dt=\frac{M}{L}\left(\frac{1-\frac{z^2}{q_1+q_2}}{1-z^2}\right)dt.\tag{3}$$

In using such tables of treble entry, we may suppose the initial angular velocities about the principal axes to be given, from which and the known moments of inertia the quantities L and M may be calculated, and then by the direct or supplemental method of reduction the value of λ and of the two parameters q_1, q_2 in the equivalent disk, each less than unity, found. 1st. If the reduction is direct—from the given inclination of the axis of the disk to the invariable line—the time t_0 from the epoch can be found by inspection, and then the entries corresponding to $t + t_0$ will give the inclinations at the end of the time t of the principal axes to the invariable line, and the position of the node defined as the intersection of the invariable plane with the plane through the invariable line and the axis of the disk (which axis coincides with a known one of the two extreme axes of the given body), and also the total angular velocity; the corresponding position of the node and value of the total angular velocity of the original body are then known by simple arithmetical computations from the theorems above given, involving λ only for the first, and λ, L, M for the second. 2nd. If the reduction is contrary or supplemental, we have only to substitute the supplemental angles of inclination to the invariable line in determining t_0, and proceed in all other respects as before, taking the supplements of the angles given in the tables in lieu of the angles themselves. In the special case of a body with two equal moments of inertia, were not the simplicity of the motion such as to render tabulation unnecessary, a distinct set of tables of double entry would of course be employed. It is, I think, conceivable that the supposed tables of treble entry might be of some practical value in

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Hence
$$\frac{M}{L}dt = \frac{dz}{\sqrt{(Z_1 Z_2)}},$$
where
$$Z_1 = \left(1 - \frac{1}{q_2}\right) + z^2\left(\frac{1}{q_2} - \frac{1}{q_1 + q_2}\right),$$
$$Z_2 = \left(1 - \frac{1}{q_1}\right) + z^2\left(\frac{1}{q_1} - \frac{1}{q_1 + q_2}\right),$$
and
$$d\zeta = \frac{1}{q_1 + q_2}\frac{dz}{\sqrt{(Z_1 Z_2)}} + \left(1 - \frac{1}{q_1 + q_2}\right)\frac{dz}{(1 - z^2)\sqrt{(Z_1 Z_2)}}.$$
The limiting values of z correspond to $Z_1 = 0$, $Z_2 = 0$, or, which is the same thing, to the values of z when y and x are successively made zero in the equation (1).

It may be useful to the reader to be enabled to compare the above values of t and ζ in terms of z with the equivalent determination of Legendre, *Exerc. du Cal. Intég.* tome II. p. 334, namely
$$i\,dt = \frac{d\psi}{\sqrt{\{1 - c^2(\sin\psi)^2\}}},$$
$$d\phi = \frac{2\tan\mu}{1 - m\sin\beta}\cdot\left(\frac{n+1}{2}\frac{d\psi}{\sqrt{\{1 - c^2(\sin\psi)^2\}}} - \frac{d\psi}{\{1 + (\tan\mu)^2(\sin\psi)^2\}\sqrt{\{1 - c^2(\sin\psi)^2\}}}\right);$$
for this purpose it will be necessary to bear in mind that Legendre's A, B, C are not the moments of inertia themselves, but the elements out of whose binary combinations they are formed, and that his middle magnitude is not B but A; the reader will then find it necessary to trace the values of Legendre's i, W, ϵ, ψ, θ, β, m, n, μ, c by the formulæ and definitions given at pages 334, 319, 328, 315, 321, 322, 325, 319 *bis*, 333, and possibly some other which has disappeared from my notes of the Exercises, tome II.

studying by arithmetical or graphical methods the geological phenomenon of evagation of the pole of the earth regarded as a body of irregular form, and in other dynamical problems of a gyroscopical character where an exact determination of the effect of a given disturbing cause might be difficult or unattainable.

The fact that there are no essential differences in the motion of a rigid body of any form and started under any initial circumstances whatever, but such as depend upon the particular values of the two positive proper fractions q_1, q_2, enables us at once to see what are the special cases which alone can arise, and whether or no there is any real distinction to be made between the general cases of the theory. At first sight it would seem that four essential parameters enter into the question, the ratios of the initial values of the partial velocities ω_1, ω_2, ω_3, and the ratios of the constants $A : B : C$, the principal moments of inertia; but one parameter is saved by the substitution of an indefinitely flattened disk for a solid, and another by the introduction of an intrinsic epoch from which the time is reckoned, and thus a table of treble instead of quintuple entry is competent to represent every possible variety of conditions.

The problem that has been treated of in the foregoing pages is one (and possibly the simplest) instance of a well-defined class of dynamical questions subject to a peculiar method of treatment, which consists in the postponement of the determination of the absolute displacement of the moving system until after its displacement relative to a fixed line has been previously determined. The three problems which may be said to form a natural (not merely a historically connected) group, and which offer the most important illustrations of the class in question, are those of the rotation of a free body, of the motion of a particle attracted to two fixed centres of force, and the problem of three bodies. In the first and third of these, the invariable line is a line perpendicular to the invariable plane, determinable by composition of the momenta of the several elements of the system at any instant of time. In the second the invariable line is the line joining the fixed centres; and the distances of the moving point from the two fixed centres or the angles which they make with the line of centres may be expressed by equations complete within themselves, and into which the position of the plane containing the moveable point and the fixed line does not enter. So again in the problem of three bodies, without having recourse to the methods of deformation employed by Jacobi, and those who have followed in his track in treating the question, it is obvious, *à priori*, that one integral may be gained, in the sense of one less being required, by forming a system of equations from which the position of the intersection of the plane of the three bodies with the invariable plane is excluded, equivalent in effect to the so-called "elimination of the node" on Jacobi's method; in which, however, the node so called is not to be confounded with the intersection above named, but is the mutual intersection of two ideal instantaneous orbits with each other and the invariable plane.

In every ordinary dynamical problem, by a well-known simple contrivance, the *time* element may be preliminarily thrown out of the differential equations of the motion; in the class of which the three noble and celebrated questions here referred to are the conspicuous types, a certain space element is capable of being similarly left out to the end; thus the number of linear differential equations required for the determination of the remaining elements is reduced by two, and if *all* the integrals of this reduced system are capable of being found, then we know, *à priori*, by the theory of the last multiplier, how to reduce to quadratures the values of the two outstanding elements. The process whereby the space coordinate referring to absolute position is, so to say, avoided in this class of dynamical questions, is not, or at least need not be considered as, one of elimination properly so called; elimination is the act of extruding a variable from a system of equations in which it has appeared; the process to be applied in the case before us is one not of extrusion, but of exclusion *ab initio*, or as it may be rendered in a single word, of ab-limination.

I propose at an early moment to return to a consideration of the particular method of ab-limination above indicated as applicable to the problem of three bodies, in the study of which this memoir took its rise.