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ON THE MOTION OF A RIGID BODY MOVING FREELY ABOUT A FIXED POINT.

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[Cf. p. 577.]

(*Abstract.*)

THE nature of the present brief memoir will be best conveyed by my giving a succinct account of the principal results which it embodies, in the order in which they occur. The direct solution, in its present form, of the important problem of the motion of a rigid body acted on by no external forces, originating in the admirable labours of Euler, has received the last degree of finish and completeness of which it is susceptible from the powerful analysis of Jacobi; in one sense, therefore, it may be said that the discussion is closed and the question at an end. Notwithstanding this, in the mode of conceiving and representing the general character of the motion, there are certain circumstances which merit attention, and which may be expressed without reference to the formulæ in which the analytical solution is contained.

Poinsot's method of representing the motion by means of his so-called "central ellipsoid" has passed into the every-day language of geometers, and may be assumed to be familiar to all. The centre of this ellipsoid is supposed to be stationary at the point round which any given solid body is turning; its form is determined when the principal moments of inertia of that body are given, and it is supposed accurately to roll without sliding on a fixed plane whose position depends on the initial circumstances of the motion. The associated free body is conceived as being carried along by the ellipsoid, so that its path in space, its continuous succession of changes of position, is thereby completely represented; but no image is thus presented to the mind of the time in which the change of position is effected. I show how this defect in the representation may be remedied, and the time, like the law of displacement, reduced to observation by a slight modification of the apparatus of the central ellipsoid or representative nucleus, as it will

for the moment be more convenient to call it. To steady the ideas, imagine the fixed invariable plane of contact with the nucleus to be horizontal and situated under it; now conceive a portion of its upper surface, say the upper half, to be pared away until it assumes the form of a semi-ellipsoid confocal to the original surface, and that an indefinitely rough plate always remaining horizontal, but capable of turning in its own plane round a vertical axis, which, if produced, would pass through the centre of the ellipsoid, is placed in contact with this upper portion; as the nucleus is made to roll with the under part of its surface upon the fixed plane below, the friction between the upper surface and the plate will cause the latter to rotate round its axis, for the nucleus will not only roll upon the plate above, but at the same time have a swinging motion round the vertical, which will be communicated to the plate. I prove, by an easy application of the theory of confocal ellipsoids, that the time of the free body passing from one position to another will be in a constant ratio to this motion of rotation, which may be measured off upon an absolutely fixed dial face immediately over the rotating-plate; and furthermore I show that the relation between the angular divisions of this dial and the time depends only upon the spinning force which may be supposed to set the free body originally in motion, so that it will hold the same, at whatever distance, by a preliminary adjustment, the rotating-plate may be supposed to be set from the fixed horizontal plane.

Thus, then, we may realize a complete *kinematical* image of all the circumstances of the motion of a free rotating body, and reduce to a purely mechanical measurement the determination of an element hitherto unrepresented, but in reality the most important of all, namely, the *time*.

I then proceed to point out a very singular and hitherto unnoticed *dynamical* relation between the free rotating body and the ellipsoidal top, as I shall now prefer to call Poinsot's central ellipsoid, because I imagine it set spinning like a top upon the invariable plane in contact with it and left to roll of its own accord, the friction between it and the plane being supposed adequate to prevent all sliding. I start with supposing that the density of the top follows any law whatever, and call its principal moments of inertia A, B, C, its semi-axes a, b, c, the relations between these six quantities being left arbitrary.

It is easy to establish that, if a rotating body be acted on by any forces which always meet the axis about which it is at any instant turning, the *vis viva* will remain unaffected by their action; this will be the case in the present instance with the pressure and friction of the invariable plane, the only forces concerned, as we may either leave gravity out of account altogether or suppose the centre of gravity of the top to be at the centre of the ellipsoid, which will come to the same thing. By aid of this principle, conjoined with the two conditions to which the angular velocities of the associated free body are known to be subject, it is easy to infer that the velocities of this body and its representative top will, throughout the motion, remain in a constant ratio, or, if we please, equal to one another, provided that

$$A : B : C :: \frac{1}{a^2}+\frac{\lambda}{a^4} : \frac{1}{b^2}+\frac{\lambda}{b^4} : \frac{1}{c^2}+\frac{\lambda}{c^4},$$

where λ is an arbitrary constant. If, now, we revert to the natural supposition of the top being of uniform density, it is well known that

$$A : B : C :: b^2+c^2 : c^2+a^2 : a^2+b^2,$$

and these ratios may be identified with those above (although this would not at the first blush be supposed to be the case) by giving a suitable value to the arbitrary constant λ.

Thus, then, Poinsot's central ellipsoid supposed of uniform density and set spinning upon a roughened invariable plane will represent the motion of a free rotating solid, not in space only but also in time; the body and the top may be conceived as continually moving round the same axis and at the same rate at each moment of time*.

The problem of the top is completed in the memoir by applying the general Eulerian equations to determine the friction and pressure, a process which involves some rather onerous but successfully executed algebraical calculations.

I next proceed to account analytically for the kinematical theory established at the outset of the memoir, and in doing so am necessarily led to give greater completeness to it, and at the same time an extension to the existing theory of confocal surfaces of the second order, by introducing the complementary notion of surfaces that I call contrafocal to one another: confocal ellipsoids are those the differences between the squares of whose corresponding principal axes are all three the same; contrafocal ellipsoids I define to be those the sums of the squares of whose corresponding axes are the same. Any two bodies whose central ellipsoids are either confocal or contrafocal I term related—correlated in the one case, contrarelated in the other, and I show that the kinematical construction in question is only another rendering of the first of the propositions herein subjoined concerning bodies so related.

1st. If two correlated bodies be placed with their principal axes respectively parallel and be set spinning by the same impulsive couple, they

* Accordingly, if we conceive any body as lying wholly in the interior of the ellipsoidal top, which is its kinematical exponent, such body will move precisely as if it were free, and consequently its density may be uniformly increased or diminished in any ratio, or it may be entirely removed without affecting the law of the motion of the surrounding crust in relation to space or time.

will move so that the corresponding axes of the one and the other body will continue always equally inclined to the axis of the couple, and their original parallelism at any instant may be restored by turning one of the bodies about this last-named axis through an angle proportional to the time elapsed since the commencement of the motion. Virtually, this amounts to saying that the difference between the displacements of two correlated bodies subject to the same initial impulse is equivalent to a simple uniform motion about the invariable line.

2nd. So, in like manner, if the bodies be contrarelated, the *sum* of their displacements is equivalent to a simple uniform motion about such line.

3rd. In either case alike, the difference between the squared angular velocities of the related bodies is constant throughout the motion.

From these propositions it follows that for all practical intents and purposes the motion of any body is sufficiently represented by the motion of any other one correlated or contrarelated to it. To a spectator on the invariable plane the apparent motion of one rotating body may be made identical with that of any other *related* one by merely making the plane on which he stands turn uniformly round a perpendicular axis. It becomes natural, then, to ascertain whether there is not always some one or more simplest form or forms which may be selected out of the whole couple of infinite series of related bodies, which may conveniently be adopted as the *exemplar* or type of all the rest. Obviously, the best suited for such purpose will be a body reduced to only two dimensions, in other words an indefinitely flattened disk, provided that it be possible in all cases to find a disk correlated or contrarelated to any given solid*.

The algebraical investigation for ascertaining the existence of such disk is the same whichever species of relation is made the subject of inquiry, and leads to the construction of three quadratic equations corresponding to the respective suppositions of the original body becoming indefinitely flattened in the direction of each of its three principal axes in turn; so that for a moment it might be supposed that the number of disks fulfilling the required condition could, according to circumstances, be zero, two, four, or six. But on closer examination, and bearing in mind that negative equally with imaginary moments of inertia are inadmissible, it turns out that there are always two such disks, and no more (except in the case of two of the moments of inertia being equal when the solution becomes unique). Of these two disks, one will be correlated and the other contrarelated to the given

* The peculiar feature in the absolute motion of a disk is, that whilst it is turning in its own plane with a variable velocity, the rate at which it turns about itself is constant, as will at once become evident from eliminating r between the two equations

$$Ap^2 + Bq^2 + (A+B)\,r^2 = M,$$
$$A^2p^2 + B^2q^2 + (A+B)^2\,r^2 = L^2.$$

body, and they will be respectively perpendicular to the axes of greatest and least moments of inertia. We have thus the choice between two methods of reduction to the type form, and this choice is not a matter of unimportance (in nature nothing exists in vain); for by means thereof the motion of any given body subject to any initial conditions can be made to depend upon either at will of the two comprehensive cases (Legendre's 1st and 3rd) to which the motion of a free rotating body is usually referred, so that the distinction between these two cases (corresponding to the two species of Polhodes on either side of the "Dividing Polhode," according to Poinsot's method of exposition) is virtually abrogated.

From the preceding theory, it follows (as also may be made to appear alike from an attentive synoptic view of the commonly received analytical formulæ as from Poinsot's theory of the associated "sliding and rolling cone") that in the problem of the motion of a free body, of whatever form and subject to whatever initial conditions one pleases, there enter but two arbitrary parameters. Calling A, B, $A+B$ the three moments of inertia of one or the other equivalent disks, L the magnitude of the impulsive couple, M the *vis viva*, these two parameters (say p, q) will be $\frac{AM}{L^2}$, $\frac{BM}{L^2}$; if to them we add a quantity $\frac{Mt}{L}$, say τ (where t is the time reckoned, as it may be, from an *intrinsic* epoch as explained in the memoir), all other elements of the motion, as the total or partial velocities and the angles, whatever they may be, selected to determine the position of the rotating body become known functions of these three quantities, p, q, τ, and may be reduced to tables of triple entry, or be graphically represented by a few charts of curves; and it should be noticed that p, the smaller of the two parameters p, q, will be always necessarily included between 0 and 1, and that the other parameter q may, by a due choice of the species of reduction adopted, be *forcibly retained* within the same limits. The five quantities $0, p, q, 1, p+q$ will then form an ascending series of magnitudes subject only to the liability of the middle term q to become equal to 1 on the one hand, or to p on the other: q becoming unity corresponds to the case of the so-called "Dividing Polhode," Legendre's 2nd case ("*cas très-remarquable*"); and q becoming equal to p is of course the case of the body itself, or its "central ellipsoid," becoming a figure of revolution, in which case the motion is practically the same as that of a uniform circular plate.

Besides these two exceptional cases, the only singular cases properly so called, the quinary scale of magnitudes just exhibited serves to indicate all the more remarkable cases (requiring or inviting particular methods of treatment) which can present themselves in the theory. These may be distinguished into special cases, which arise from any two consecutive terms becoming (to use Prof. De Morgan's expressive term) subequal, that is, differing from one another by a quantity whose square may be neglected, and double special cases, which arise when any three consecutive terms become subequal; all of which, together with peculiar subcases appertaining to the double special class, perhaps deserve more thorough examination than may have been hitherto accorded to them. I conclude the memoir with pointing out the place which this problem of Rotation appears to me to occupy in dynamical theory, as belonging to a natural and perfectly well defined group of questions, of which the motion of a body attracted to two fixed centres and the renowned problem of three bodies acted on by their mutual attractions are conspicuous instances. This group is characterised by the feature that, as regards them, equations of motion admit of being constructed, from which not only the element of time, as in ordinary mechanical problems, but also an element of absolute space is shut out; supposing the equations thus reduced by two in the number of the variables to have been integrated, Jacobi's theory of the last multiplier serves to reduce both the excluded elements to quadratures and thus to complete the solution. I notice that whilst the time may fairly be said to be eliminated, the space element may be more properly said to undergo the negative process, if it may be so called, of ab-limination; it is not introduced into and then expelled from, but prevented from ever making its appearance at all in the resolving system of differential equations. It is from the study of one of these allied but more difficult questions that the present memoir has taken its rise as a collateral inquiry and elucidatory digression.