

ON THE MULTIPLICATION OF PARTIAL  
DIFFERENTIAL OPERATORS.[*Philosophical Magazine*, xxxiii. (1867), pp. 48—55.]

IN the last Number of the *Magazine*\* I explained the sense in which I employ the term operator as distinguished from an operand, the distinction being somewhat analogous to the grammatical one between a verb and a noun; for as a combination of the predicate and copula gives rise to a verb which has independent laws of inflexion and regimen, so an operator is a new species of quantity which, springing from the union of an operand and the symbol of operation, becomes amenable to its own proper laws of functional action and subjection †. I found it convenient also to refer to an operator as an *energized operand* ‡. At the outset of the paper a proposition was stated inadvertently, regarding any energized function of a set of variables and their corresponding elementary operators, in too general terms. Such function remaining unrestricted in regard to the principal letters  $x, y, z, \dots$  should have been limited to be a *linear* quantic in regard to the elementary operators  $\delta_x, \delta_y, \delta_z, \dots$ . If  $\phi$  be any such function, the proposition in question, thanks to the happy introduction of the star symbol, may without any auxiliary definition of the derivatives  $\phi_2, \phi_3, \dots$  employed in the preceding paper, be stated as follows, with perfect freedom from any shade of ambiguity,

$$e^{t\phi*} = [e^{(e^{t\phi*} - 1)\phi}]*,$$

[\* p. 567, above.]

† Thus an operator forms a new *part of speech* in algebra. It may be well to notice in this place, in order to prevent error arising hereafter, that the process of energization must in general be indicated, not by the mere apposition of an asterisk, but of brackets and asterisk. Thus, although  $P$  turned into an operator may be correctly designated by  $P_*$ ,  $P_*P$  similarly energized will be represented by  $[P_*P]_*$ , and not by  $P_*P_*$ .

Conversely, denenergization will consist in the abstraction of an asterisk and brackets, and not of the former merely. Thus  $P_*P_*$  denenergized is not  $P_*P$  but  $P^2 + P_*P$ , because  $P_*P_*$  is  $[P^2 + P_*P]_*$ ; whereas  $P_*P_*$  divided by  $*$ , a term employed in the sequel in a footnote, is simply  $P_*P$ , so that star division, or destellation as it may be termed, is not to be confounded with denenergization.

‡ Or I might have used the word *vitalized* to convey the same idea,—the operator being the operand endowed with power of action, but none the less for that capable of being acted upon, calling to mind the relation between dead and living matter. So denenergization might be termed *amortization*, a word which exists in the language.

which theorem ( $t$  being an arbitrary parameter) contains the general rule for expanding  $(\phi^*)^n$  in terms of the quantities

$$[\phi^*\phi]^* ; [\phi^*\phi^*\phi]^* ; [\phi^*\phi^*\phi^*\phi]^* , \&c. \dagger$$

† Thus, for example, when

$$\phi = a\delta_b + 2b\delta_c + 3c\delta_d + \dots,$$

the theorem in the text easily enables us to see that

$$e^{(e^{x\phi^* - 1})\phi} * F(a, b, c, d, \dots) = F(a, b + ax, c + 2bx + ax^2, d + 3cx + 3bx^2 + ax^3, \dots),$$

which, as remarked by Mr G. De Morgan and others at the Mathematical Society, may be regarded as a transformation and generalization of the fundamental law of development in Arbogast's theory, sometimes called by the name of Arbogast's first or unreduced method. The identification with the method in question merely requires the supposition that  $F(a, b, c, d, \dots)$  should become a function exclusively of a single one of the letters within the parentheses; but of course we must write the left-hand side of the equation under the unreduced form

$$e^{x\phi^*} F(a, b, c, d \dots).$$

The proof, as noticed by my distinguished mathematical friend Mr Samuel Roberts, of the generalized theorem is virtually implied in the method by which I established long ago the partial differential equations of the invariants to any system of forms; that is, it follows from the observation that the effect upon  $F$  of altering  $x$  into  $x + \delta x$  leaving  $a, b, c, \dots$  unaltered is the same as the effect of leaving  $x$  unaltered and altering  $b, c, d, \dots$  into

$$b + a\delta x, c + 2b\delta x, d + 3c\delta x, \dots$$

Consequently

$$\frac{\delta F}{\delta x} = \phi^* F, \quad \frac{\delta^2 F}{\delta x^2} = (\phi^*)^2 F, \dots,$$

and therefore, by Maclaurin's theorem,

$$F(a, b + ax, c + 2bx + ax^2, \dots) = e^{x\phi^*} F(a, b, c, \dots).$$

In memory of the author who appears to have been the first to employ the form which I have called a Protractant, it may hereafter with propriety be termed also an Arbogastiant.

The equivalence of  $e^{x\phi^*}$  with  $[e^{(e^{x\phi^* - 1})\phi}]^*$ , when  $\phi$  represents an Arbogastiant, or rather a form slightly more general, had been previously stated, but in a much less commodious manner, by Professor Cayley in a memoir contained in *Crelle's Journal*, Vol. XLVII. p. 110. An inspection of this memoir will satisfy the reader how inarticulate was the language of algebra at the not remote epoch when Mr Cayley's paper was written, and how, for want of a distinctive abstract symbol of operativeness, she strove like one lame of speech and tongue-tied, to give intelligible expression to her ideas.

With the *star* sign the restraining ligament has been cut, and henceforth algebra, as far as yet developed, may revel in unbounded freedom of utterance. The rise of this star above the mathematical horizon marks one of the epochs of algebra. It is worth remarking how already it is beginning in its turn to assume the attributes of quantity (*vide* the concluding footnote of this paper, where it is used as a divisor); so that apparently it is destined to run the same course as Newton's fluxional symbol, which is, and of fatal necessity must have been, superseded by the lettered symbols of Leibnitz, which have now long ago, to all intents and purposes, become converted into a new species of algebraical quantity. As soon as it becomes necessary (as will probably before long be the case) to express the specific relation of the star to something which limits and discriminates its mode of application, it must in its turn develop into a third species of symbolical quantity; and so there may be in store for the future of algebra an endless procession of more and more abstract symbols of operation, each successively developing into a more and more subtle species of quantity, suggesting the analogy of successive stages of so-called imponderability in the material world.

*A propos* of Arbogastiants, it is worthy of a passing notice that if  $I$  be any invariant to the form  $(a, b, c, \dots, h, k)$ , and we write  $A$  for the Arbogastiant  $(l\delta_k + 2k\delta_h + \dots)$ , then  $\frac{(A^*)^n}{\Pi n}$  expresses

In like manner, the statement concerning the commutable operators  $\phi*$  and  $\psi*$ , made in a footnote, should have been limited to the case where those two operands,  $\phi$ ,  $\psi$ , are each of them linear quantities in regard to  $\delta_x, \delta_y, \delta_z, \dots$ . The proposition advanced guardedly in the Postscript concerning any linear functions of  $x, y, z, \dots$   $\delta_x, \delta_y, \delta_z, \dots$  ("there can be little or no doubt, &c.") I now also wish to be understood as affirming absolutely. I proceed to give a universal theorem for the multiplication of any number of operators, energized functions of  $x, y, z, \dots$ ;  $\delta_x, \delta_y, \delta_z, \dots$ , freed from all restriction as to linearity of form in respect to the latter set.

The method by which I arrived at this very general theorem was in substance identical with that embodied in the demonstration spontaneously furnished me by my ever ready correspondent Professor Cayley; and as I cannot improve upon his statement, it would be a waste of time to substitute my own words for his. Accordingly, after enunciating the theorem, I shall give the proof of it in the very words of our unrivalled Cambridge Professor, from which it will be seen that in essence this theorem consists in applying the symbolical form of Taylor's theorem to the expansion which, in itself symbolical, contains the generalization of Leibnitz's theorem, thus giving rise to a *symbolism of the second order*, a phenomenon which, it is believed, here for the first time makes its appearance in analysis.

Let  $\phi_1, \phi_2, \phi_3, \dots \phi_r$  be any functions of  $x, y, z, \dots$ ;  $\delta_x, \delta_y, \delta_z, \dots$ , capable of being developed in a series of integer powers of the latter set of variables, where it is of course understood that

$$\delta_x = \frac{d}{dx}, \quad \delta_y = \frac{d}{dy}, \quad \delta_z = \frac{d}{dz}, \dots;$$

in like manner let

$$\delta'_x = \frac{d}{d\delta_x}, \quad \delta'_y = \frac{d}{d\delta_y}, \quad \delta'_z = \frac{d}{d\delta_z}, \dots,$$

the effect of the substitution  $\left[ \begin{smallmatrix} b, c, \dots k, l \\ a, b, \dots h, k \end{smallmatrix} \right]$  performed upon  $I$ . This theorem is an easy consequence of the conjunction of the three circumstances, (1) that if  $I_{x,y}$  is what  $I$  becomes when for  $a, b, c, \dots k$  we substitute respectively  $ax+by, bx+cy, \dots kx+ly$ ,  $I_{x,y}$  will be a covariant to the form  $(a, b, c, \dots h, k, l)$ , and that consequently the last coefficient in  $I_{x,y}$  will be  $\frac{(Ax)^n}{\Pi n} I$ ; (2) that this coefficient must bear the same relation to  $l, k, \dots c, b$  as the first does to  $a, b, c, \dots k$ ; and (3) that an invariant to the form  $(l, k, \dots c, b)$  is identical with the same invariant to the form  $(b, c, \dots k, l)$ .

I think I have been informed that Leibnitz was the first to employ the method of the so-called separation of symbols: in his tract on the 'Calculus of Differences,' the poet sage of Collingwood contributed powerfully to its further development; if he should chance to cast his eyes over these pages he will, I fear, stand aghast at the Frankenstein he has thus (it may be unwittingly) played no unimportant part in bringing into existence; or, rather, I should fear, did not all the world know his perfect candour and unstinted sympathy with every form of manifestation of human intelligence.

so that in fact  $\delta'_x, \delta'_y, \delta'_z, \dots$  are abbreviated expressions for  $\delta_{\delta_x}, \delta_{\delta_y}, \delta_{\delta_z}, \dots$  or, if we please so to say, for

$$\frac{d}{d \delta_x}, \quad \frac{d}{d \delta_y}, \quad \frac{d}{d \delta_z}.$$

Let  $\delta_{x,i}, \delta'_{x,i}$  signify the operants  $\delta_x, \delta'_x$  restricted to operate exclusively on  $\phi_i$ ; finally, let

$$\Delta_{i,j} = \delta'_{x,i} \cdot \delta_{x,j} + \delta'_{y,i} \cdot \delta_{y,j} + \delta'_{z,i} \cdot \delta_{z,j} + \dots;$$

then giving to  $i, j$  all possible values subject to the inequalities  $i < j, j < n + 1$ , the following equation is true,

$$\phi_1 * \phi_2 * \phi_3 * \dots * \phi_n * = [e^{\sum \Delta_{i,j}} \phi_1 \phi_2 \phi_3 \dots \phi_n] *.$$

What follows within inverted commas is from Mr Cayley's pen.

"Write

$$\xi = \delta_x, \quad \eta = \delta_y,$$

$$A = (x, y)^\alpha (\xi, \eta)^\alpha,$$

namely,  $A$ , any function of degrees  $\alpha, \alpha$ ; and so

$$B = (x, y)^\beta (\xi, \eta)^\beta, \text{ \&c.,}$$

and

$$A_{12} = (x_1, y_1)^\alpha (\xi_2, \eta_2)^\alpha, \text{ \&c.,}$$

but all suffixes are to be ultimately rejected. Then

$$\begin{aligned} B * A * &= (x_2, y_2)^\beta (\xi + \xi_1, \eta + \eta_1)^\beta (x_1, y_1)^\alpha (\xi, \eta)^\alpha * \\ &= e^{\xi \delta_{\xi_1} + \eta \delta_{\eta_1}} (x_2, y_2)^\beta (\xi_1, \eta_1)^\beta (x_1, y_1)^\alpha (\xi, \eta)^\alpha * \\ &= e^{\Delta_{01}} B_{21} A_{10} * \text{ if } \Delta_{01} = \xi \delta_{\xi_1} + \eta \delta_{\eta_1}. \end{aligned}$$

Similarly,

$$\begin{aligned} C * B * A * &= (x_3, y_3)^\gamma (\xi + \xi_1 + \xi_2, \eta + \eta_1 + \eta_2)^\gamma (x_2, y_2)^\beta (\xi + \xi_1, \eta + \eta_1)^\beta \\ &\hspace{15em} (x_1, y_1)^\alpha (\xi, \eta)^\alpha * \\ &= e^{(\xi_1 + \xi) \delta_{\xi_2} + (\eta + \eta_1) \delta_{\eta_2}} (x_3, y_3)^\gamma (\xi_2, \eta_2)^\gamma \dots \dots \dots \\ &= e^{\Delta_{12} + \Delta_{02}} C_{32} e^{\Delta_{01}} B_{21} A_{10} * \\ &= e^{\Delta_{12} + \Delta_{02} + \Delta_{01}} C_{32} B_{21} A_{10}, \text{ and so on.} \end{aligned}$$

This seems the easiest proof of your general theorem."

The reader of sufficient intelligence to understand the theorem itself will have no difficulty in supplying the few missing links between my statement and the above demonstration of it. I will content myself with appending a single example to illustrate its meaning and mode of application.

Let  $P$  be any *lineo-linear* function of  $x, y, z, \dots$ ;  $\delta_x, \delta_y, \delta_z, \dots$ ; and in general let

$$(P *)^{n-1} P = P_n.$$

Let it be proposed to expand

$$P^i * P^j *.$$

Here, calling

$$P^i = \phi_1, \quad P^j = \phi_2, \\ \Delta_{1,2} = \delta'_{x,1} \delta_{x,2} + \delta'_{y,1} \delta_{y,2} + \dots,$$

it is easily seen† that

$$\Delta_{1,2}(\phi_1 \phi_2) = i \cdot j P^{i-1} \cdot P^{j-1} \cdot P * P \\ = ij P^{i+j-2} \cdot P_2, \\ (\Delta_{1,2})^2(\phi_1 \phi_2) = i(i-1) P^{i-2} \cdot j(j-1) P^{j-2} \cdot P_2^2, \\ \dots\dots\dots$$

Hence

$$P^i * P^j * = P^{i+j} * + ij P^{i+j-2} \cdot P_2 * + \frac{ij(i-1)(j-1)}{1 \cdot 2} P^{i+j-4} \cdot P_2^2 * + \&c.,$$

agreeably to the theorem given for protractors, and stated subsequently to hold good for pertractors in the previous paper,  $P_2$  here denoting what was called  $2P_2$  in the passages referred to.

This theorem, it should be observed, remains true when  $P$ , remaining a linear quantic in  $\delta_x, \delta_y, \delta_z, \dots$  is any function whatever of  $x, y, z$ .

Let us agree to employ  $(i), (j)$  as *umbræ*, such that  $(i)^n, (j)^n$  shall denote the factorial quantities

$$i(i-1) \dots (i-n+1); \quad j(j-1) \dots (j-n+1)$$

for all values of  $n$ ; then we may express the above theorem under the subjoined condensed form, which will be useful for the better understanding of the sequel,

$$P^i * P^j * = [e^{\frac{(i)(j) P * P}{P^2}} \cdot P^{i+j}] *.$$

Suppose now that we wish to obtain the product of three factors,

$$(P *)^i, \quad (P *)^j, \quad (P *)^k.$$

Call  $e^{\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}}$ , for the sake of brevity,  $E$ . The first term in the expansion of  $E$  is unity. The second is  $\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}$ , which, applied to  $P^i \cdot P^j \cdot P^k$ , gives

$$\{ij + ik + jk\} P^{i+j+k-2} \cdot (P * P).$$

The third term is

$$\frac{1}{2} (\Delta_{1,2}^2 + \Delta_{1,3}^2 + \Delta_{2,3}^2 + 2\Delta_{1,2} \cdot \Delta_{1,3} + 2\Delta_{1,3} \cdot \Delta_{2,3} + 2\Delta_{1,2} \cdot \Delta_{2,3});$$

the effect of the application of the first three quantities within the above parentheses is to introduce terms whose sum is

$$\frac{1}{2} \{ (i^2 - i)(j^2 - j) + (i^2 - i)(k^2 - k) + (j^2 - j)(k^2 - k) \} P^{i+j+k-4} (P * P)^2;$$

† Thus, for example, to fix the ideas, observe that

$$\Sigma (\delta'_{x,1} \cdot \delta_{x,2}) (ax\delta_y + by\delta_z) (cy\delta_z + dz\delta_t) = ax \cdot c\delta_z + by \cdot d\delta_t = (ax\delta_y + by\delta_z) * (cy\delta_z + dz\delta_t).$$

So again,

$$(\delta'_{y,1} \cdot \delta_{y,2})^2 (x\delta_y)^i (y\delta_z)^j = i(i-1) \cdot j(j-1) \cdot (x\delta_y)^{i-2} (y\delta_z)^{j-2} \cdot x^2 (\delta_z)^2 \\ = i(i-1) \cdot j(j-1) (x\delta_y)^{i-2} (y\delta_z)^{j-2} (x\delta_y * y\delta_z)^2.$$

the effect of the fourth and fifth quantities is to introduce terms whose sum is

$$\{(i^2 - i)jk + ij(k^2 - k)\} P^{i+j+k-4} \cdot (P*P)^2;$$

and the effect of the sixth term is to introduce the terms

$$ik(j^2 - j)P^{i+j+k-4}(P*)^2 + ijk \cdot P^{i+j+k-3} \cdot P*P*P;$$

giving altogether for the complete sum

$$[\frac{1}{2}\{(i)(j) + (i)(k) + (j)(k)\}^2 P^{i+j+k-4} P_2^2 + ijk P^{i+j+k-3} \cdot P_3 + \dots]*.$$

And in general the effect of the term  $(\Delta_{1,2} + \Delta_{2,3} + \Delta_{1,3})^r$  in the expansion of  $E$  will be to introduce terms containing all the quantities of the form

$$\frac{(P*P)^\beta}{P^{2\beta}} \cdot \frac{(P*P*P)^\gamma}{P^{3\gamma}} \cdot P^{i+j+k}$$

that can be got consistent with the satisfaction of the equation in integers  $2\beta + 3\gamma = r$ . The upshot of the calculation is that

$$P^{i*}P^{j*}P^{k*} = \{e^{\sum \left\{ \frac{(i)(j)P*P}{P^2} + \frac{(i)(j)(k)P*P*P}{P^3} \right\}} P^{i+j+k}\} *;$$

where it is of course to be understood that  $(i), (j), (k)$  are mere *umbræ*, subject to the law above stated for conversion of their powers into factorials of actual quantities.

The law for any number of factors is now obvious, and may be extended to the case where the factors are powers, not of one single operant  $P$ , but of different operants  $P, Q, R, S$ , subject to the sole condition of their being linear quantics in regard of  $\delta_x, \delta_y, \delta_z, \dots$ ; and it will be found that†

$$P^{i*}Q^{j*}R^{k*}S^{l*} \dots = \left\{ e^{\sum \left( \frac{(i)P*(j)Q}{PQ} + \frac{(i)P*(j)Q*(k)R}{PQR} + \frac{(i)P*(j)Q*(k)R*(l)S}{PQRS} \dots \right)} \cdot P^i Q^j R^k S^l \dots \right\} *;$$

† Thus, for the particular case when  $P = Q = R \dots$ , we have

$$P^{i_1*}P^{i_2*}P^{i_3*} \dots P^{i_\omega*} = \left\{ \left( e^{\frac{\Omega}{P*}} \right) P^{\sum i} \right\} *;$$

where  $\Omega = \{P + (i_1)P*\} \{P + (i_2)P*\} \dots \{P + (i_\omega)P*\} - P^\omega - \sum i P^{\omega-1} \cdot P*$ .

Observe that if we convene to understand by

$$\frac{A*}{L} \cdot \frac{B*}{M} \dots \frac{C}{N} \div *;$$

the expression

$$\frac{A*B* \dots *C}{LM \dots N} ,$$

then if we write

$$\frac{(i)P*}{P} = p; \quad \frac{(j)Q*}{Q} = q; \dots$$

the above theorem takes the form

$$P^{i*}Q^{j*}R^{k*} \dots = [e^{\sum \frac{pq + pqr + \dots}{*}} P^i \cdot Q^j \cdot R^k \dots] *.$$

Now suppose

$$\phi = P^\alpha \cdot P^\beta \cdot P^\gamma \dots,$$

$$\psi = Q^j \cdot Q^j \dots,$$

$$\vartheta = R^\alpha \cdot R^\alpha \dots$$

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 .....

where in the summation which gives the exponent of  $e$ , it is to be understood that the natural order of  $P, Q, R, S$  in the numerators is to be maintained.

The formula from which this result has been thus simply derived is of that fundamental character which entitles it to be regarded as a master theorem, that is, rather as a method than an ordinary formula. As already observed, it essentially consists in the union of two known theorems; but these combined and, as it were, duly adjusted and focussed, constitute together an instrument of research as unlike either of its separate elements as a telescope differs in its powers and functions from the pair of lenses out of which it has been formed. And truly the formula in question has a telescopic power in the sense of bringing the remote results of calculation close up to the mental vision.

The very first application made of this instrument, directed to the algebraical firmament, has been rewarded by the discovery of the beautifully simple and general expansion given in the text above—a result in beauty and the feeling of wonder it awakens fairly to be paralleled with the spectacle which gladdened the eyes of Galileo when for the first time he pointed *his* telescope to the skies.

and write

$$s_1 = \frac{(i) P_{\alpha^*}}{P_{\alpha}} + \frac{(i') P_{\beta^*}}{P_{\beta}} + \dots$$

$$s_2 = \frac{j) Q_{\alpha^*}}{Q_{\alpha}} + \frac{j) Q_{\beta^*}}{Q_{\beta}} + \dots$$

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then I think there can be little doubt, or, at all events, there is a strong presumption that the following ultra-general theorem holds good:—

$$\phi_* \psi_* \mathfrak{D}_* = [e^{\sum \frac{s_1 s_2 + s_1 s_2 s_3 + \dots}{*}} \cdot \phi \psi \mathfrak{D} \dots]_*.$$

If we suppose all the  $P$ 's *inter se*, all the  $Q$ 's *inter se*, &c. to coincide, the above expansion is certainly true, as may be inferred from the expansion proved in the text, conjoined with the known theorem in factorials, that  $\{(i) + (j) + \dots\}^n$  is identical with what  $(i + j + \dots)^n$  becomes when, in the development of the latter expression for any power of any element, we substitute the corresponding factorial product, that is, when in it for  $i^q, j^q, \dots$  we substitute  $(i)^q, (j)^q, \dots$

Even if on examination the above equation should turn out not to be exact, the mere statement of it will be useful in indicating the kind of expression that is applicable. According to the conservative maxim that my universally lamented friend the late Mr Buckle used to be fond of citing, in science even a wrong rule is preferable to anarchy and confusion.