# 96.

# THOUGHTS ON INVERSE ORTHOGONAL MATRICES, SIMULTANEOUS SIGN-SUCCESSIONS, AND TESSELLATED PAVEMENTS IN TWO OR MORE COLOURS, WITH APPLICATIONS TO NEWTON'S RULE, ORNAMENTAL TILE-WORK, AND THE THEORY OF NUMBERS.

[*Philosophical Magazine*, XXXIV. (1867), pp. 461—475.]

## PART I.—*Matrices and Sign-successions.*

1. A SELF-RECIPROCAL matrix may be defined as a square array of elements of which each is proportional to its first minor. When the condition is superadded that the sum of the squares of the terms in each row or in each column, or (which comes to the same) that the complete determinant shall be equal to unity, it becomes *strictly* orthogonal; but, by an allowable extension of language, any self-reciprocal matrix may be termed orthogonal when the epithet of *strictness* is withdrawn. The general notion is that of homographic relation between each element and its first minor, that is, the relation $a + bx + c\xi + dx\xi = 0$ between the corresponding terms $x$ and $\xi$ of the matrix and its reciprocal. When $a = 0$ and $d = 0$, we have the case of orthogonalism as above defined*. When $b = 0$ and $c = 0$, so that each term in either matrix is in the inverse ratio of its first minor, we fall upon what I call the case of inverse orthogonalism.

This conception will be found to present itself naturally in the course of certain investigations connected with the calculus of sign-progressions suggested by the form of Newton's rule; and that calculus in its turn leads to a theory of tessellation highly curious in itself, and fruitful of consequences to the calculus of operations and the theory of numbers, furnishing interesting food for thought, or a substitute for the want of it, alike to the analyst at his desk and the fine lady in her boudoir.

* For a matrix of the order 2 the ratio of each element to its reciprocal in an orthogonal matrix is necessarily $\pm 1$. This is a case of exception, and may be disregarded. In all other cases the ratio can be varied *ad libitum*.

2. In a strictly orthogonal matrix the $n^2-1$ equations resulting from the equal ratios above referred to, on account of the implications existing between them, really amount to no more than $\frac{n^2+n}{2}$ independent conditions, leaving $\frac{n^2-n}{2}$ of the $n^2$ terms arbitrary. This law, which it would perhaps not be easy to obtain from a direct inspection of the equations, is an instantaneous consequence of the fact that a sum of the squares of $n$ variables may be transformed into a sum of squares of $n$ linear functions of the same by means of an orthogonal substitution,—and that, *vice versâ*, such faculty of transformation is sufficient to establish the character of orthogonalism in the matrix of substitution employed. Consequently the number of conditions to be satisfied is the number of terms in a homogeneous quadratic function of $n$ variables, which is $\frac{n.(n+1)}{2}$. In an *orthogonal* matrix (not *strictly* so) the number of implications is consequently $\frac{(n+2)(n-1)}{2}$.

3. The problem of constructing an inverse orthogonal matrix of any order admits of a general and complete solution. It is to be understood in what follows, that the constant product of any term by its first minor is not to be zero; or, in other terms, the complete determinant of the matrix which is a sum of such products is not to vanish.

First, let us investigate the number of arbitrary elements which enter into any such matrix.

To fix the ideas, consider one of the third order, say

$$\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ A, & B, & C \end{vmatrix}$$

and call the reciprocal matrix formed by its first minors

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ \alpha_1, & \beta_1, & \gamma_1 \\ A_1, & B_1, & C_1 \end{vmatrix}.$$

Then
$$\begin{aligned} aa_1 &= bb_1 = cc_1 \\ &= \alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1 \\ &= AA_1 = BB_1 = CC_1. \end{aligned}$$

These 8 equations are not independent; for we have

$$\begin{aligned} aa_1 + bb_1 + cc_1 &= aa_1 + \alpha\alpha_1 + AA_1 \\ = \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 &= bb_1 + \beta\beta_1 + BB_1 \\ = AA_1 + BB_1 + CC_1 &= cc_1 + \gamma\gamma_1 + CC_1; \end{aligned}$$

which 5 equations in their turn again are not independent, because the sum of the three groups written under one another on the left is equal to the corresponding sum on the right.

Hence we have implication upon implication, so that the number of independent equations is

$$(3^2-1)-(2.3-1)+1=(3-1)^2;$$

and so in general for a matrix of the order $n$, the number of independent equations is $(n-1)^2$, leaving $2n-1$ of the elements arbitrary.

4. This result is easily verified. For, reverting to the example of the third order, if any inverse orthogonal matrix of that order is multiplied, term to term, by the following one,

$$\begin{vmatrix} l\lambda, & l\mu, & l\nu \\ m\lambda, & m\mu, & m\nu \\ n\lambda, & n\mu, & n\nu \end{vmatrix},$$

the product so formed will evidently retain its character unaltered, since each of the equal products will receive a constant multiplier, $lmn.\lambda\mu\nu$.

The number of independent quantities thus introduced is 5, namely,

$$l\lambda;\ \frac{m}{l},\ \frac{n}{l};\ \frac{\mu}{\lambda},\ \frac{\nu}{\lambda};$$

and so in the general case we can introduce $(2n-1)$ arbitrary elements. Thus, then, we may without any loss of generality regard only those matrices of the kind in question which are bordered horizontally and vertically by a line of positive units. From these *reduced* forms it is easy to pass to the general forms by term-to-term multiplication with a matrix of the kind above denoted. The question now becomes narrowed to that of determining the number and form of the reduced inverse orthogonal matrices of any given order $n$,—a problem (if attacked by a direct method) involving the solution of $(n-1)^2$ equations between $(n-1)^2$ unknown quantities.

5. (1) Let $n$ be a prime number. Write down the line of terms

$$1,\ a,\ a^2,\ \ldots\ a^{n-1},$$

and make $a$ equal in succession to each of the $(n-1)$ roots of $\dfrac{x^n-1}{x-1}=0$. The matrix so formed will be a reduced inverse orthogonal matrix of the $n$th order.

In the case of $n=3$, it is easy and will be instructive to verify this statement. Calling the required matrix

$$\begin{vmatrix} 1, & 1, & 1 \\ 1, & a, & b \\ 1, & c, & d \end{vmatrix},$$

we obtain the four equations

$$ad - bc = d(a-1) = c(1-b) = a(d-1) = b(1-c),$$

which are equivalent to the following,

$$ad = c = b, \quad bc = d = a.$$

Hence $$a^2d^2 = bc, \text{ or } a^4 = a.$$

Hence rejecting the values $a = 0$ and $a = 1$, either of which would cause the constant product to become zero, we have the two solutions,

$$(1)\quad a = \rho, \quad d = \rho, \quad b = \rho^2, \quad c = \rho^2,$$

$$(2)\quad a = \rho^2, \quad d = \rho^2, \quad b = \rho, \quad c = \rho.$$

There is thus but *one* single type of matrix of this order, namely

$$\begin{vmatrix} 1, & 1, & 1 \\ 1, & \rho, & \rho^2 \\ 1, & \rho^2, & \rho \end{vmatrix}.$$

(2) In like manner, for any prime number $n$ there will be but a single type of matrix, the interior *nucleus* of which is a square matrix of the order $(n-1)$ made up of lines or columns of terms in which each line or column contains the $(n-1)$ powers taken in definite order of the $(n-1)$ prime roots of unity. That such a matrix is inversely orthogonal is not difficult of proof; but it is less easy to establish, what I have scarcely a doubt is true (but which I have not yet attempted to demonstrate), that such matrix, when its lines and columns are permuted in every possible manner, contains the complete solution of the corresponding system of $(n-1)^2$ equations. The number of distinct systems or roots satisfying these equations will be the number of distinct forms which can be obtained by permuting the lines and columns—in a word, the number of distinct *derivatives* (a word it will be found hereafter useful to employ) of any given phase of the nucleus. This number will be easily seen to be

$$(n-1).(n-2)^2.(n-3)^2 \dots 1^2;$$

for each derivative, when all the permutations are taken of the lines and of the columns, will appear $n$ times repeated. For instance, if $\rho$ be a prime fifth root of unity so that

$$\begin{vmatrix} \rho, & \rho^2, & \rho^3, & \rho^4 \\ \rho^2, & \rho^4, & \rho, & \rho^3 \\ \rho^3, & \rho, & \rho^4, & \rho^2 \\ \rho^4, & \rho^3, & \rho^2, & \rho \end{vmatrix}$$

is the nucleus, if we take

| | | | | | | |
|---|---|---|---|---|---|---|
| the columns in the order | | | 1, 2, 3, 4, | rows in the order | | 1, 2, 3, 4, |
| or | „ | „ | 3, 1, 4, 2 | „ | „ | 2, 4, 1, 3, |
| or | „ | „ | 2, 1, 4, 3 | „ | „ | 3, 1, 4, 2, |
| or | „ | „ | 4, 3, 2, 1 | „ | „ | 4, 3, 2, 1, |

the resulting derivative is in each case the same. Thus, then, when $n$ is a prime number, the system of $(n-1)^2$ equations which give the terms of the nucleus admits of $\Pi(n-1) \,.\, \Pi(n-2)$ systems of roots.

It will be seen that this law does not hold when $n$ is a composite number, the rule for which I now proceed to state.

6. (1) I observe that there *will be as many distinct types of solutions as there are distinct modes of breaking up n into factors**.

(2) Let $n = p \,.\, q \,.\, r \ldots$ be one of the decompositions in question. Write down the disjunctive product

$$(1, a, a^2, \ldots a^{p-1} \between 1, b, b^2, \ldots b^{q-1} \between 1, c, c^2, \ldots c^{r-1} \between \ldots$$

in which the terms are to follow any fixed law of succession. This will produce a line containing $p \,.\, q \,.\, r \ldots$, that is, $n$ terms.

Let $a, b, c, \ldots$ respectively represent the $p$th, $q$th, $r$th, ... roots of unity; by giving to each of these quantities successively its $p, q, r, \ldots$ values we shall obtain $p \,.\, q \,.\, r \ldots$, that is, $n$ lines, constituting a matrix of the $n$th order; the totality of the matrices so formed contain between them the complete solution of the $(n-1)^2$ system of equations.

As an example let $n = 4$.

Here there are two modes of decomposition, namely,

$$n = 4, \quad n = 2 \,.\, 2.$$

Let $i, i'$ denote the two primitive fourth roots of unity, and denote negative unity by $\bar{1}$. The two types will be

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & i, & \bar{1}, & i' \\ 1, & \bar{1}, & 1, & \bar{1} \\ 1, & i', & \bar{1}, & i \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & \bar{1}, & 1, & \bar{1} \\ 1, & 1, & \bar{1}, & \bar{1} \\ 1, & \bar{1}, & \bar{1}, & 1 \end{vmatrix}.$$

The number of distinct derivatives of the nucleus of the first of these types is $\frac{(1 \,.\, 2 \,.\, 3)^2}{2}$, that is, 18, the divisor 2 originating in the symmetry of the square in respect to its diagonals.

The number of distinct derivatives of the second type, which contains a higher capacity of symmetry than the former (that is, a symmetry persistent under certain permutations of its constituent lines or columns), is 6.

The following Table, in which $+\ -$ are substituted for $1, \bar{1}$, will make this evident.

* When $n$ is the $\nu$th power of a prime, the number of decompositions becomes the number of *indefinite* partitions of $\nu$.

Phases of nucleus to type 2 . 2 :—

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
- & + & - & + & - & - & - & - & + \\
+ & - & - & - & - & + & - & + & - \\
- & - & + & - & + & - & + & - & - \\
\hline
+ & - & - & - & + & - & - & - & + \\
- & + & - & - & - & + & + & - & - \\
- & - & + & + & - & - & - & + & - \\
\hline
\end{array}
$$

Phases of nucleus to type 4 :—

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
i & - & i' & - & i' & i & i' & i & - \\
- & + & - & + & - & - & - & - & + \\
i' & - & i & - & i & i' & i & i' & - \\
\hline
- & i & i' & i' & - & i & i & i' & - \\
+ & - & - & - & + & i & - & - & + \\
- & i' & i & i & - & i' & i' & i & - \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|ccc|ccc|}
\hline
- & + & - & + & - & - & - & - & + \\
i & - & i' & - & i' & i & i' & i & - \\
i' & - & i & - & i & i' & i & i' & - \\
\hline
+ & - & - & - & + & i & - & - & + \\
- & i & i' & i' & - & i & i & i' & - \\
- & i' & i & i & - & i' & i' & i & - \\
\hline
\end{array}
$$

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
i & - & i' & - & i' & i & i' & i & - \\
i' & - & i & - & i & i' & i & i' & - \\
- & + & - & + & - & - & - & - & + \\
\hline
- & i & i' & i' & - & i & i & i' & - \\
- & i' & i & i & - & i' & i' & i & - \\
+ & - & - & - & + & - & - & - & + \\
\hline
\end{array}
$$

Thus, then, the total number of distinct solutions of our $(4-1)^2$, that is, 9, algebraical equations applicable to this case is $18+6$, or 24. The formula $\Pi(n-1) . \Pi(n-2)$ would give only 12. How it should happen that the order of the system of equations for different values of $n$ is not an algebraical, but a transcendental function of $n$ depending on the factors of which $n$ is made up, will become less surprising when it is considered that the quantities equated to zero in any such system, although algebraical in themselves, are not analytical but *tactical* functions of $n$ their degree.

7. It remains to assign the value of the constant product in the reduced form of matrix of the order $n$, or, which comes to the same thing, the value of the complete determinant of such matrix, which is obviously $n$ times the former quantity.

(1) When $n$ is undecomposed, the value of this determinant, by virtue of a well-known theorem given years ago by Professor Cayley, for expressing the discriminant of an algebraical function as a determinant composed of powers of its roots, is easily recognized to be $i^{(n-1)(n-2)} n^{\frac{n}{2}}$, which we may call $\Delta_n$.

(2) When $n$ is decomposed under the form $pqr \ldots$, the corresponding determinant may easily be proved equal to

$$\Delta_p{}^{qr\ldots} . \Delta_q{}^{pr\ldots} . \Delta_r{}^{pq\ldots} \ldots .$$

Hence the determinant in the general case is

$$(-)^{\phi} p^{\frac{pqr\ldots}{2}} . q^{\frac{qpr\ldots}{2}} . r^{\frac{rpq\ldots}{2}} \ldots = (-)^{\phi} n^{\frac{n}{2}},$$

where
$$\phi = n\Sigma \frac{(p-1)(p-2)}{p}.$$

Thus, if each term in any reduced inverse orthogonal matrix of the order $n$ be divided by the square root of $n$, the fourth power of the resulting determinant is unity for all the types without distinction. If $n$ is decomposed into $\mu$ equal factors $p$, $\phi = \mu(p-1)(p-2)p^{\mu-1}$; so that when $\mu > 1$, the determinant is $\pm i$ if $\mu \equiv 1$ [mod 2], and $p \equiv -1$ [mod 4], and is $\pm 1$ in all other cases. When $\mu = 1$, its value is $(\pm i)$ if $p \equiv -1$ or $0$ [mod 4], and $\pm 1$ in the other two cases. When $n$ is undecomposed, the value of the constant product, which is $\frac{1}{n}$ of the determinant, takes the simple form $(i^{n-1} n)^{\frac{n-2}{2}}$.

8. When $n$ is a power of 2, the type corresponding to its decomposition into the equal factors 2 deserves especial consideration. In this type the only roots of unity which appear are 1 and $\bar{1}$; and as each of those numbers is its own arithmetical inverse, the matrix may be said with equal propriety to be inversely orthogonal or directly orthogonal, that is, orthogonal in the sense conveyed in Art. 1. Moreover, on dividing each term by $\sqrt{n}$, it becomes strictly orthogonal, since the sum of the squares of the terms in each row or column then becomes unity.

A very little reflection will make it clear, *à priori*, that using simply + and − in place of $+1$ and $-1$, the known theorems relating to the form of the products of two sums of 2, or of 4, or of 8 squares must exhibit instances of orthogonal matrices of this nature. Thus, to begin with the case of the equation

$$(\alpha^2 + \beta^2)(a^2 + b^2) = A^2 + B^2,$$

we may represent the values of $A$ and $B$ by writing the three matrices

$$\begin{vmatrix} \alpha, & \beta \\ \alpha, & \beta \end{vmatrix} \quad \begin{vmatrix} a, & b \\ b, & a \end{vmatrix} \quad \begin{vmatrix} +, & + \\ +, & - \end{vmatrix};$$

on multiplying these three together, term by term, we obtain

$$\begin{vmatrix} +\alpha a + \beta b \\ +\alpha b - \beta a \end{vmatrix},$$

where

$$+\alpha a + \beta b = A,$$
$$+\alpha b - \beta a = B.$$

Moreover the term-to-term product of the second and third matrices, namely, $\begin{vmatrix} a, & b \\ b, & -a \end{vmatrix}$, is an orthogonal matrix.

So again in the equation

$$(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(a^2 + b^2 + c^2 + d^2) = A^2 + B^2 + C^2 + D^2$$

the three matrices become

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \end{vmatrix} \quad \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} \quad \begin{vmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{vmatrix}.$$

The resulting product,

$$\begin{aligned} &\alpha a + \beta b + \gamma c + \delta d \\ &\alpha b - \beta a + \gamma d - \delta c \\ &\alpha c - \beta d - \gamma a + \delta b \\ &\alpha d + \beta c - \gamma b - \delta a, \end{aligned}$$

represents in its four lines the respective values of $A$, $B$, $C$, $D$. Moreover the matrix produced by the product of the second and third, that is,

$$\begin{vmatrix} a, & b, & c, & d \\ b, & -a, & d, & -c \\ c, & -d, & -a, & b \\ d, & c, & -b, & -a \end{vmatrix},$$

is an orthogonal matrix. The same remarks apply to the representation of the product of two sums of eight squares under the form of a sum of eight. Omitting the first matrix, consisting of repetitions of one given set of eight letters, $\alpha$, $\beta$, $\gamma$, $\delta$, $\lambda$, $\mu$, $\nu$, $\pi$, the remaining two matrices may be written as below:

$$\begin{vmatrix} a & b & c & d & l & m & n & p \\ b & a & d & c & m & l & p & n \\ c & d & a & b & n & p & l & m \\ d & c & b & a & p & n & m & l \\ l & m & n & p & a & b & c & d \\ m & l & p & n & b & a & d & c \\ n & p & l & m & c & d & a & b \\ p & n & m & l & d & c & b & a \end{vmatrix} \quad \begin{vmatrix} + & + & + & + & + & + & + & + \\ + & - & - & + & + & - & + & - \\ + & + & - & - & + & - & - & + \\ + & - & + & - & + & + & - & - \\ + & - & - & - & - & + & + & + \\ + & + & + & - & - & - & + & + \\ + & - & + & + & - & - & - & + \\ + & + & - & + & - & + & - & - \end{vmatrix}.$$

The lettered matrix forms (as in the preceding cases) a "*conjugate* system [in Cauchy's sense] of regular substitutions." The right-hand matrix, interpreting + and − to mean plus and minus units, is a direct and inverse orthogonal matrix corresponding to 8 represented as 2 . 2 . 2; the lines produced by the term-to-term multiplication of the three matrices give the quantities $A$, $B$, $C$, $D$, $L$, $M$, $N$, $P$, which satisfy the equation

$$\Sigma A^2 = (\Sigma\alpha^2)\,\Sigma\,(a^2),$$

and the term-to-term product of the two matrices actually above written is an orthogonal matrix of the 8th order.

9. I now pass to another and more important illustration of such matrices, which presents itself in the application of Newton's rule (or my extension of it) for finding a superior limit to the number of real roots in an algebraical equation. That rule deals with permanences and variations of sign in two series of quantities. It will be more simple to consider the two simultaneous successions of signs obtained by multiplying together the signs of the consecutive terms in the series

$$\begin{array}{llll} f, & f_1, & f_2, & \ldots f_n, \\ G, & G_1, & G_2, & \ldots G_n. \end{array}$$

We obtain in this way two series of $n$ signs each, written respectively over one another; and the quantities with which the theory is concerned are the numbers, say $\pi$ and $\phi$, of compound signs $\begin{smallmatrix}+\\+\end{smallmatrix}$ and $\begin{smallmatrix}-\\+\end{smallmatrix}$ which occur in these simultaneous progressions: the $f$ series and $G$ series both consist of functions of $x$; the increase of $\pi$ and the decrease of $\phi$, when $x$ ascends from one given value $a$ to another $b$, each of them gives a superior limit to the number of real roots in $fx$ contained between $a$ and $b$.

It is of course obvious that $\pi$ corresponds to the number of double permanences, and $\phi$ to that of variation permanences in the original series of $f$'s and $G$'s. Now it appeared to me desirable, in the same way as double and higher orders of denumerants have been shown in my lectures on Partitions of Numbers to be expressible as linear functions of simple denumerants, so in like manner to get rid of compound variations and permanences, and to express them, or at least their number, by means of simple variations or permanences. This comes to the same thing as finding a means of making the enumeration of the four species of compound signs $\begin{smallmatrix}+\\+\end{smallmatrix}\Big|\begin{smallmatrix}-\\+\end{smallmatrix}\Big|\begin{smallmatrix}+\\-\end{smallmatrix}\Big|\begin{smallmatrix}-\\-\end{smallmatrix}\Big|$, in two simultaneous series of signs, depend on the enumeration of the simple signs + or − in those series themselves, or in series derived from them, or in the two sorts combined.

10. As a first step in the generalization of this question, let us suppose $i$ series of simultaneous progressions of + and − signs, giving rise to $2^i$ varieties of vertical combinations of sign. Now let the $i$ given series be combined, $r$ and $r$ together, in every possible manner, where $r$ takes all values from 0 to $i$, both inclusive.

When $r = 1$, it is of course understood that the so-called combinations are the original $i$ series themselves.

When $r = 0$, it is to be understood that a series exclusively of the signs + is intended.

When $r$ is not 0, nor 1, let the $r$ series corresponding to any $r$-ary combination be multiplied term-to-term together.

When $r = 0$, the + succession, and when $r = 1$ the given $n$ series are to be reckoned as the corresponding products. The number of series of signs so obtained will of course be

$$1 + i + \frac{i(i-1)}{2} + \ldots = 2^i.$$

By the sum of any series let us understand the number of signs +, less the number of signs −. When the $i$ given series are written over one another, each of the $2^i$ varieties of columns that can be formed of the signs + and − will occur a certain number of times. I shall show that these $2^i$ numbers are linear functions of the $2^i$ sums last mentioned. Of this theorem, on account of its importance, I shall give a rigorous proof.

As a matter of typographical convenience, I write the columnar combinations of sign in horizontal in lieu of their proper vertical order, as, for example, $+ + -$ in lieu of $\begin{matrix}+\\+\\-\end{matrix}$, and, moreover, use such horizontal line enclosed within brackets to signify the number of the recurrences of the corresponding combination; thus $(+ - - +)$ means the number of times the combination $\begin{pmatrix}+\\-\\-\\+\end{pmatrix}$ occurs in four given simultaneous progressions. Again, as regards the sums, $s$ will denote the sum of the line of *plus* signs, which is of course the same as the number of terms in each progression, $i$ the number of columns, and $s_{p,q,r}\ldots$ will denote the sum of the line formed by the multiplication together of the $p$th, $q$th, $r$th, ... lines of the given $i$ set of lines. This being premised, and using each of the symbols $\lambda_1, \lambda_2, \lambda_3, \ldots$ to denote + or −, as the case may be, the number of recurrences of each species of combination in terms of the *sums* is expressed by the following formula,

$$[\lambda_1, \lambda_2, \ldots \lambda_i] = \frac{1}{2^i}(s + \Sigma\lambda_p s_p + \Sigma\lambda_p\lambda_q s_{p,q} + \Sigma\lambda_p\lambda_q\lambda_r s_{p,q,r} + \&c.),$$

as I shall proceed to prove. But first, to make the meaning of this formula more clear, let us suppose $i=2$, the formula then gives the following equations:—

$(+\,+)$, that is the number of combinations $\begin{bmatrix}+\\+\end{bmatrix}$, $=\frac{1}{4}\{s+s_1+s_2+s_{1,2}\}$,

$(+\,-)$, „ „ „ „ $\begin{bmatrix}+\\-\end{bmatrix}$, $=\frac{1}{4}\{s+s_1-s_2-s_{1,2}\}$,

$(-\,+)$, „ „ „ „ $\begin{bmatrix}-\\+\end{bmatrix}$, $=\frac{1}{4}\{s-s_1+s_2-s_{1,2}\}$,

$(-\,-)$, „ „ „ „ $\begin{bmatrix}-\\-\end{bmatrix}$, $=\frac{1}{4}\{s-s_1-s_2+s_{1,2}\}$.

11. Now for the proof of the general formula.

For shortness call the quantity $s+\Sigma\lambda_p s_p+\Sigma\lambda_p\lambda_q s_{p,q}\ldots$ (where the signs $\lambda_1, \lambda_2, \ldots \lambda_i$ are all supposed to be given) $E$.

Let us consider the effect of the existence of any single column of signs $\mu_1, \mu_2, \ldots \mu_i$ in the given $i$ progressions upon the value of $E$; besides contributing the signs $\mu_1, \mu_2, \ldots \mu_i$ respectively to the series $s_1, s_2, \ldots s_i$ this column will contribute to the series

$$s_{\theta_1,\theta_2,\ldots\theta_j} \text{ the sign } \mu_{\theta_1}\mu_{\theta_2}\ldots\mu_{\theta_j}.$$

Hence altogether it will contribute to $E$

$$(1+\lambda_1\mu_1)(1+\lambda_2\mu_2)\ldots(1+\lambda_i\mu_i) \text{ units};$$

and thus the total value of $E$, depending on the entire number of columns of all kinds, will be

$$\Sigma\left[\{(1+\lambda_1\mu_1)(1+\lambda_2\mu_2)\ldots(1+\lambda_i\mu_i)\}\,.\,(\mu_1\mu_2\ldots\mu_i)\right],$$

where the $\lambda$ system is given, but the $\mu$ system is variable.

But any factor $(1+\lambda_q\mu_q)$ is zero unless $\lambda_q=\mu_q$. Hence for any system of values of $\mu$ not coincident with the $\lambda$ system the corresponding multiplier of $(\mu_1\mu_2\ldots\mu_i)$ vanishes, and for that system it becomes $2^i$. Hence

$$E=2^i(\lambda_1, \lambda_2, \ldots \lambda_i),$$

as was to be proved.

12. These formulæ admit of a useful application to Newton's rule.

The two superior limits to the number of roots included between $(a)$ and $(b)$ which it (or my extension of it) furnished are $\Delta(+\,+)$ and $-\Delta(+\,-)$, where $\Delta$ refers to the ascent from $a$ to $b$, and the series are those mentioned in Art. 9. Hence, calling the two limits $\lambda, \lambda'$, remembering that $s$ is constant,

$$\lambda=\frac{\Delta s_1+\Delta s_2+\Delta s_{1,2}}{4},$$

$$\lambda'=\frac{\Delta s_1-\Delta s_2+\Delta s_{1,2}}{4};$$

so that the limits are

$$\tfrac{1}{4}\{\Delta s_1 + \Delta s_{1,2}\} \pm \frac{\Delta s_2}{4}.$$

The mean of these is $\frac{1}{4}(\Delta s_1 + \Delta s_{1,2})$, which *à fortiori* is also a superior limit. Here $s_1$ refers to the series

$$f,\ f_1,\ f_2,\ \dots f_n,$$

and $s_{1,2}$ refers to the series

$$fG,\ f_1G_1,\ f_2G_2,\ \dots f_nG_n,$$

which I have called, in an article in this *Magazine**, the $H$ series. If $p$ is the number of permanences in the $f$, and $\phi$ in the $H$ series, it is readily seen that

$$\frac{\Delta s_1 + \Delta s_{1,2}}{4} = \frac{\Delta p + \Delta\phi}{2}.$$

Hence, since $\lambda$ and $\lambda'$ are each of them superior limits, it follows as an immediate consequence that $\dfrac{\Delta p + \Delta\phi}{2}$ is so likewise; but this assertion conveys no new information, and ought not to be treated as a new theorem, as I inadvertently stated it to be; the fact, however, of its being implied in what was previously known is so far from being immediately evident, that M. Angelo Genocchi has followed me in regarding the theorem as an independent one, and devoted an article to the demonstration of it as such in the *Nouvelles Annales* for January of this year †.

13. The complete system of relations between the two sets of $2^i$ quantities given by the theorem in Art. 10 may it is evident be expressed by means of the inverse orthogonal matrix (also orthogonal) whose type corresponds to $2 \cdot 2 \cdot 2 \dots$ ($i$ terms). Thus, for example, for the case of $i = 3$, we may write—

[* p. 542 above.]

† If we call $\nu$ the number of real roots in $f$ comprised between $a$ and $b$, we know from Fourier's theorem that $\nu = \Delta p - 2\theta$, where $\theta$ is the number of times that an *even* change occurs in the value of $p$ as we pass from $a$ to $b$, this change being always in the positive direction. And, again, as I have shown in the article in the *Philosophical Magazine* above referred to,

$$\nu = \frac{\Delta p + \Delta\phi}{2} - \vartheta,$$

where $\vartheta$ is the total number of *times* that $\phi$ undergoes a change within the same interval,—such change being always even, on account of the two terminals of the $G$ series being both positive—the one extremity being a positive constant, and the other the square of $f$. This change, however, is sometimes additive and sometimes ablative, $\phi$ not necessarily *increasing* always (as $p$ does) on ascending from $a$ to $b$: thus the two unknown transcendants $\theta$ and $\vartheta$ are connected by the simple relation

$$2\theta - \vartheta = \frac{\Delta p - \Delta\phi}{2}.$$

Of course each evanescence of a term in the $f$ or $G$ series between two terms of like sign is to be reckoned as a distinct *time* of change. I also make abstraction of the singular cases where several consecutive terms vanish together in either series.

| | $s$ | $s_1$ | $s_2$ | $s_3$ | $s_{1,2}$ | $s_{1,3}$ | $s_{2,3}$ | $s_{1,2,3}$ |
|---|---|---|---|---|---|---|---|---|
| $2^3 . (+++)$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $2^3 . (++-)$ | $+$ | $+$ | $+$ | $-$ | $+$ | $-$ | $-$ | $-$ |
| $2^3 . (+-+)$ | $+$ | $+$ | $-$ | $+$ | $-$ | $+$ | $-$ | $-$ |
| $2^3 . (+--)$ | $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | $+$ | $+$ |
| $2^3 . (-++)$ | $+$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ | $-$ |
| $2^3 . (-+-)$ | $+$ | $-$ | $+$ | $-$ | $-$ | $+$ | $-$ | $+$ |
| $2^3 . (--+)$ | $+$ | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $2^3 . (---)$ | $+$ | $-$ | $-$ | $-$ | $+$ | $+$ | $+$ | $-$ |

The meaning of this Table is self-apparent. Thus, for example, if we wish to find the value of $(-++)$, that is, the number of recurrences of $\begin{vmatrix} \mp \\ + \\ + \end{vmatrix}$ in the three given series, we read it off from the 5th line above and find it equal to

$$\frac{s - s_1 + s_2 + s_3 - s_{1,2} - s_{1,3} + s_{2,3} - s_{1,2,3}}{8}.$$

The Table of signs itself is obviously the matrix corresponding to the product 2 . 2 . 2.

From the fact of this Table being orthogonal, we infer that the two sets of quantities are (to a numerical multiplier *près*) the same linear functions, the first set of the second, and the second of the first.

14. The theorem of Art. 10 may be extended to simultaneous progressions of signs denoting any root of $+$, as for example, $+\rho$, $\rho^2$, where $\rho$ is a cube root of $+$ instead of $+$ and $-$. Let each series be supposed to consist of $q$th roots of $+$, and let $(\lambda_1, \lambda_2, \ldots \lambda_i)$ denote the number of recurrences of the column $\begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{vmatrix}$ in which each $\lambda$ is some $q$th root of $+$; then there will be $q^i$ quantities of the form $(\lambda_1, \lambda_2, \ldots \lambda_i)$. Again, we may form series by combining together not merely the given $i$ series themselves, but their squares, cubes, &c. up to the $(q-1)$th powers, and form the term-to-term products of all the series entering into any such combination; in this way, including $s$ (the series constituted exclusively of $+$ signs), we shall obtain $q^i$ series, the

general symbol for the sum of the terms in any one of which, when we substitute the roots of 1 for the corresponding roots of +, may be written $[s_1^{q_1}, s_2^{q_2}, s_3^{q_3}, \ldots s_i^{q_i}]$, where each $s$ is a $q$th root of +, and each $q$ with a subscript is some one of the numbers in the series 0, 1, 2, ... $(q-1)$. If now we understand by the above bracket, when $q_1, q_2, \ldots q_i$ are all zero, the value corresponding to $s$ in the particular case previously considered, that is, the number of terms in each series, the relation between the two sets of $q^i$ numbers is given by the equation

$$(\lambda_1, \lambda_2, \ldots \lambda_i) = \frac{1}{q^i} \Sigma \frac{[s_1^{q_1}, s_2^{q_2}, s_3^{q_3}, \ldots s_i^{q_i}]}{(\lambda_1^{q_1}, \lambda_2^{q_2}, \lambda_3^{q_3}, \ldots \lambda_i^{q_i})} *.$$

If we write out a Table expressing these relations in a manner similar to that employed for the particular case of $q = 2$ in a preceding article, we shall obtain a square array of signs ($q^i$ to a side) which will form an inverse orthogonal matrix corresponding to the type $q \cdot q \cdot q \ldots$ (to $i$ terms).

* The reader will please to observe that the terms included under the sign of summation are in general not real but complex numbers formed with the $q$th roots of unity. Their *sum*, however, is necessarily a real number, being the number of recurrences of the column of signs $\lambda_1, \lambda_2, \ldots \lambda_i$ in the given system of sign-progressions. The proof of the theorem is precisely the same as for the case previously considered, where $q=2$; namely, it may be shown that the sum above denoted by $\Sigma$ is equal to

$$\Sigma \left[ \frac{\left\{1-\left(\frac{\lambda_1}{\mu_1}\right)^q\right\}\left\{1-\left(\frac{\lambda_2}{\mu_2}\right)^q\right\}\ldots\left\{1-\left(\frac{\lambda_i}{\mu_i}\right)^q\right\}}{\left(1-\frac{\lambda_1}{\mu_1}\right)\left(1-\frac{\lambda_2}{\mu_2}\right)\ldots\left(1-\frac{\lambda_i}{\mu_i}\right)} \right] [\mu_1\, \mu_2 \ldots \mu_i],$$

each of the $q^i$ terms of which new sum vanishes except that one in which the variable $\mu$ system is identical with the given $\lambda$ system of the $q$th roots of unity, for which term the fraction becomes equal to $q^i$.