104.

THE STORY OF AN EQUATION IN DIFFERENCES OF THE SECOND ORDER.

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MY recent researches into the order of the various systems of equations which serve to determine the forms of reducible cyclodes have led me to notice an equation in the second order of differences which I imagine is new, and possesses a peculiarly interesting complete integral.

If we call

$$fx = (x^2 - a^2)^i (x^2 - b^2)^j (x^2 - c^2)^k \dots,$$

and ($i, j, k, \dots$ &c. being given) determine $a, b, c, \dots$ &c. so as to make

$$(fx)^2 + (f'x)^2$$

a complete square, and if we suppose the indices $i, j, k, \dots$ to consist of λ integers of one value, μ integers of a second value, ν of a third, and so on, the number of solutions of the problem will *in general* depend not on $i, j, k, \dots$ but on the derived integers $\lambda, \mu, \nu, \dots$; and we may denote the maximum value of this number by the type $[\lambda, \mu, \nu, \dots]$*.

Now I have been able to establish the following theorem of derivation as a particular case of a more general one of which the clue is in my hands:—

$$[1, \lambda, \mu, \nu, \dots] = [\lambda, \mu, \nu, \dots] + \Sigma (\lambda^2 - \lambda) [1, \lambda - 2, \mu, \nu, \dots]$$
$$+ 2\Sigma\lambda\mu [1, \lambda - 1, \mu - 1, \nu, \dots].$$

Suppose now that $\lambda, \mu, \nu, \dots$ all become unity, and that we call

$$[1, 1, 1, \dots \text{ to } n \text{ terms}] = \Omega_n,$$

then the theorem above stated gives the relation

$$\Omega_n = \Omega_{n-1} + (n-1)(n-2)\,\Omega_{n-2}.$$

* For example, if $fx = (x^2 - a^2)^i (x^2 - b^2)^j (x^2 - c^2)^k (x^2 - d^2)^l$,
the type is [1, 1, 1, 1], of which the maximum value is 9; but if the sum of any two of the quantities i, j, k, l happens to become equal to the sum of the other two, the order sinks and is either 8 or 7; I am not quite certain which at present, although it is more probably the former.

But by virtue of the form of the equations for finding fx, I know independently that Ω_n is the product of n terms of the progression

$$1, 1, 2, 2, 3, 3, 4, \ldots$$

Hence we have one particular solution of the above equation in differences. To find the second, if we make Ω_1 and Ω_2, 1 and 2 respectively instead of 1, 1, it will be found that the nth term becomes the product of n terms of the analogous progression 1, 2, 2, 4, 4, 6, 6 .... Thus, then, we are in possession of the complete integral of the equation

$$u_{x+1} = u_x + (x^2 - x)\, u_{x-1},$$

namely $$u_{2x} = C \,.\, 1^2 \,.\, 3^2 \,.\, 5^2 \ldots (2x-1)^2 + K \,.\, 2^2 \,.\, 4^2 \ldots (2x-2)^2\, 2x,$$

$$u_{2x+1} = C \,.\, 1^2 \,.\, 3^2 \,.\, 5^2 \ldots (2x-1)^2 (2x+1) + K \,.\, 2^2 \,.\, 4^2 \ldots (2x)^2.$$

Writing $u_x = 1 \,.\, 2 \,.\, 3 \ldots (x-1)\, v_x$, the above equation takes the remarkably simple form

$$v_{x+1} - v_{x-1} = \frac{v_x}{x}\text{*}.$$

The romance of algebra presents few episodes more wonderful than this specimen of the way in which the determination of the degree of an equation resulting from elimination can be made to contribute a new and by no means obvious fact to the Calculus of Differences.

* Whether taken under this or the original form, the equation will be found to lie *outside* the cases of integrable linear difference equations of the second order with linear or quadratic coefficients given by the late Mr Boole in his valuable treatise on finite differences. In the second form the solution ought by Laplace's method to be representable by a definite integral. Expressed under the more ordinary form the integral is as follows :

$$v_{2x} = C\,\frac{3 \,.\, 5 \,.\, 7 \ldots (2x-1)}{2 \,.\, 4 \,.\, 6 \ldots (2x-2)} + K\,\frac{2 \,.\, 4 \,.\, 6 \ldots (2x)}{1 \,.\, 3 \,.\, 5 \ldots (2x-1)},$$

$$v_{2x-1} = C\,\frac{3 \,.\, 5 \,.\, 7 \ldots (2x-1)}{2 \,.\, 4 \,.\, 6 \ldots (2x-2)} + K\,\frac{2 \,.\, 4 \,.\, 6 \ldots (2x-2)}{1 \,.\, 3 \,.\, 5 \ldots (2x-3)}.$$