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ON TWO REMARKABLE RESULTANTS ARISING OUT OF THE THEORY OF RECTIFIABLE COMPOUND LOGARITHMIC WAVES.

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THE fruitful investigations in which I have been for some time past engaged concerning reducible cyclodes and rectifiable compound logarithmic waves have led me *inter alia* to notice a problem of elimination which from its elegance and peculiarity is, I think, worthy of being offered in a detached form to the *Philosophical Magazine.*

Suppose any number of equations (to fix the ideas say four) of the form which follows:

$$\begin{aligned} U &= ax + by + cz + dt = 0, \\ V &= ax^3 + by^3 + cz^3 + dt^3 = 0, \\ W &= ax^5 + by^5 + cz^5 + dt^5 = 0, \\ \Omega &= ax^7 + by^7 + cz^7 + dt^7 = 0. \end{aligned}$$

If these be regarded as surfaces, they can only be made to intersect in one or another of a definite number of points.

For in the case of intersection we must evidently have

$$dt . \begin{vmatrix} 1 & 1 & 1 & 1 \\ x^2 & y^2 & z^2 & t^2 \\ x^4 & y^4 & z^4 & t^4 \\ x^6 & y^6 & z^6 & t^6 \end{vmatrix}, \text{ that is } dt\,\zeta(x^2, y^2, z^2, t^2) = 0,$$

ζ being the symbol which ex presses the product of the differences of the quantities which it affects. Hence

$$x \pm y = 0 \text{ or } x \pm z = 0 \text{ or } y \pm z = 0 \text{ or } x \pm t = 0 \text{ or } y \pm t = 0$$
$$\text{or } z \pm t = 0 \text{ or } t = 0.$$

Hence it will easily be seen by substitution and successive reduction that the points of intersection are confined to those hereinunder stated and their analogues, namely

$$x = \pm y = \pm z = \pm t,$$

$$x = \pm y = \pm z, \quad t = 0,$$

$$x = \pm y, \quad z = 0, \quad t = 0,$$

$$x = 0, \quad y = 0, \quad z = 0,$$

the total number of points in the group being

$$2^3 + 4 \,.\, 2^2 + 6 \,.\, 2 + 4, \text{ that is } \frac{3^4 - 1}{2};$$

and so in general for n such equations the number of possible points of intersection will be $\dfrac{3^n - 1}{2}$.

As regards the resultant, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} \Omega + (\,.\,) W + (\,.\,) V + (\,.\,) U = dt\,\zeta(x^2, y^2, z^2, t^2).$$

Hence the resultant of U, V, W, Ω is the same as that of

$$U, V, W, dt \,.\, \zeta(x^2, y^2, z^2, t^2),$$

divided by the resultant of

$$U, V, W, \zeta(x^2, y^2, z^2),$$

that is, is the resultant of

$$U, V, W, dt\,(x^2 - t^2)(y^2 - t^2)(z^2 - t^2).$$

This enables us to see that the required resultant is the product of all the resultants of the systems that can be formed by the interchange of a, b, c after the pattern of the system

$$(a \pm d)\,x + (b \pm d)\,y + (c \pm d)\,z,$$

$$(a \pm d)\,x^3 + (b \pm d)\,y^3 + (c \pm d)\,z^3,$$

$$(a \pm d)\,x^5 + (b \pm d)\,y^5 + (c \pm d)\,z^5,$$

(the signs in the coefficients of the same column being alike, but independent as between column and column), multiplied by the resultant of

$$ax + by + cz,$$

$$ax^3 + by^3 + cz^3,$$

$$ax^5 + by^5 + cz^5,$$

multiplied by $$d^{1\,.\,3\,.\,5};$$

and by continuing this process it is obvious that the required resultant will be made up exclusively of factors of the form

$$d^{\lambda}, \quad (d \pm c)^{\mu}, \quad (d \pm c \pm b)^{\nu}, \quad (d \pm c \pm b \pm a)^{\pi}.$$

So in general for n equations, it may be shown in like manner that the resultant is the product of factors of the form

$$(a_1 \pm a_2 \pm a_3 \pm \dots \pm a_i)^{u_{n,i}},$$

where $u_{n,i}$ is a function of n and i to be determined. But by aid of the method of reduction above indicated, and fixing his attention on those factors of the resultant only in which the single coefficient retained in the substituted equation appears, the intelligent reader will find no difficulty in ascertaining

(1) that $u_{n,1} = 1 \,.\, 3 \,.\, 5 \dots (2n-1)$,

(2) that $u_{n,i} = (i-1)\, u_{n-1,i-1}$.

These two conditions furnish us with the following Table of double entry:—

	$i=$ 1,	2,	3,	4,	5,	6
$n=1$	1					
$=2$	1	1				
$=3$	3	1	2			
$=4$	15	3	2	6		
$=5$	105	15	6	6	24	
$=6$	945	105	30	18	24	120

which, of course, may be indefinitely extended. Thus, for example, when $n=3$, the resultant is

$$(abc)^3 (a^2-b^2)(a^2-c^2)(b^2-c^2)(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2)^2.$$

The above investigation leads as a corollary to the following arithmetical theorem.

Call $1 \,.\, 3 \,.\, 5 \dots (2x-1) = Q_x$ and $1 = Q_0$. Then

$$2x\frac{Q_{x-1}}{2} + 2x(2x-2)\frac{Q_{x-2}}{4} + 2x(2x-2)(2x-4)\frac{Q_{x-4}}{6} + \dots$$

$$+ 2x(2x-2)\dots 2 \,.\, \frac{Q_0}{2x} = \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2x-1}\right) Q_x.$$

For example. If $x=4$,

$$8\frac{1.3.5}{2} + 8.6.\frac{1.3}{4} + 8.6.4.\frac{1}{6} + 8.6.4.2.\frac{1}{8}$$

$$= 60 + 36 + 32 + 48 = 176.$$

So, too,

$$3.5.7 + 1.5.7 + 1.3.7 + 1.3.5 = 176.$$

The value of $u_{n,i}$ is, of course, $\Pi(i-1)\, Q_{n-i}$.

There is a more elaborate system of $2n$ equations, the resultant of which can be made to depend on that of the system of n equations just ascertained. Thus, take $2n = 6$, and consider the system

$$ax + by + cz + dt + eu + fv\ ; \quad x + y + z + t + u + v\ ;$$
$$ax^3 + by^3 + cz^3 + dt^3 + eu^3 + fv^3;\quad x^3 + y^3 + z^3 + t^3 + u^3 + v^3;$$
$$ax^5 + by^5 + cz^5 + dt^5 + eu^5 + fv^5;\quad x^5 + y^5 + z^5 + t^5 + u^5 + v^5;$$

the order of the resultant of this system in the letters a, b, c, d, e, f is obviously $1.3.5(1.3 + 1.5 + 3.5)$.

Now *pair* the six variables in every possible manner; the number of such pairs is $1.3.5$.

Let x, y, z, t, u, v be any one such set of pairs. Make

$$x + y = 0,\quad z + t = 0,\quad u + v = 0;$$

then the latter set of three functions become zero, and the former three may be made zero with right assignments of x, z, t, provided the resultant of

$$(a-b)x + (c-d)z + (e-f)u,$$
$$(a-b)x^3 + (c-d)z^3 + (e-f)u^3,$$
$$(a-b)x^5 + (c-d)z^5 + (e-f)u^5$$

is zero. Hence the required resultant will *contain* the product of the resultants of the $1.3.5$ systems formed after the above pattern; and as this product will be of $1.3.5(1.3 + 1.5 + 3.5)$ dimensions in the constants, it must be not merely contained in, but identical with, the required resultant. Thus the new set of functions regarded as hyper-loci (like the former set) can only be made to intersect in one or another of a fixed group of points. Moreover, passing to the case of $2n$ equations, it is obvious that the resultant of such system will be made up exclusively of factors of the form

$$(a_1 + a_2 + \ldots + a_i - a_{i+1} - a_{i+2} - \ldots - a_{2i})^{J_{n,i}}*,$$

where $J_{n,i}$ is a function of n and i to be determined. The value of $u_{n,i}$, which has been found above, leads to this without difficulty. By an obvious method of calculation it may be shown that

$$J_{n,i} = u_{n,i}\{1.3.5\ldots(2n-1)\}\frac{n.(n-1)\ldots(n-i+1)}{1.2\quad\ldots\quad i}2^{i-1}$$
$$\div\left\{\frac{2n.(2n-1)\ldots(2n-2i+1)}{1.2\quad\ldots\quad 2i}\cdot\frac{1}{2}\cdot\frac{2i.(2i-1)\ldots(i+1)}{1.2\quad\ldots\quad i}\right\}$$
$$= 2i.2^{i-1}.\frac{\Pi n}{\Pi 2n}.\frac{\Pi(2n-2i)}{\Pi(n-i)}\{\Pi(i-1)\}^2 Q_{n-i}.Q_n$$
$$= \Pi(i-1)\,\Pi i\,(Q_{n-i})^2 = i\,(u_{n,i})^2.$$

* It will, of course, be understood that a_1, a_2, a_3, &c. are written above in place of a, b, c, &c.

We thus obtain the following Table for finding the *frequency* $J_{n,i}$ of any given form of factor:—

	$i=1,$	$2,$	$3,$	$4,$	5
$n=1$	1				
$=2$	1	2			
$=3$	9	2	12		
$=4$	225	18	12	144	
$=5$	$(105)^2$	450	108	140	2880

The resultant thus determined is the coefficient of the leading term of an equation of the degree $1^2 . 3^2 . 5^2 \ldots (2n-1)^2$, upon which depends the determination of a set of $2n$ quantities $\xi_1, \xi_2 \ldots, \xi_n$, so chosen as to make the arc of the curve whose equation is

$$y = a_1 \log (x^2 - \xi_1^2) + a_2 \log (x^2 - \xi_2^2) + \ldots + a_{2n} \log (x^2 - \xi_{2n}^2)$$

equal to

$$x + a_1 \log \frac{x - \xi_1}{x + \xi_1} + \ldots + a_{2n} \log \frac{x - \xi_{2n}}{x + \xi_{2n}},$$

$a_1, a_2, \ldots a_{2n}$ being $2n$ given unequal quantities. It follows from the above that the number of distinct solutions is $1^2 . 3^2 \ldots (2n-1)^2$, unless one group of i of the coefficients a and a second group of i other of them can be found such that the sum of the one group is equal to the sum of the other; in that case, and in that case only, the number of solutions undergoes a reduction. A similar conclusion can be extended to the case of an odd number $(2n+1)$ of the parameters (a), in which case the number of solutions is $1^2 . 3^2 \ldots (2n-1)^2 (2n+1)$, except when, as above, two sets of parameters can be found the same in number and equal in amount, in which case the number of solutions undergoes a reduction as before.

I mention these facts with the view of making it understood that the problems of elimination herein proposed and solved are not mere idle dreams and speculations of the fancy, but have a real ontological significance in connexion with a great Algebraico-Diophantine problem of the Integral Calculus.

P.S. Suppose ν to be any positive integer, even or odd, and that the curve or compound symmetrical logarithmic wave

$$y = \Sigma_{\theta=\nu}^{\theta=1} a_\theta \log (x^2 - \xi_\theta^2)$$

is to be made subject to the relation

$$arc\ minus\ abscissa = \Sigma_{\theta=\nu}^{\theta=1} a_\theta \log \left(\frac{x - \xi_\theta}{x + \xi_\theta}\right).$$

Then the a coefficients (or form-parameters) being given, the ξ quantities (or asymptotic distances from the Y axis of the logarithmic wavelets) depend on the solution of an algebraical equation whose degree is the product of ν terms of the series

$$1, 1, 3, 3, 5, 5, 7, \ldots.$$

When $\nu = 2n$, the coefficient of the leading term of this equation is the resultant of the system, or rather double system, of $2n$ functions of $2n$ variables which has been already discussed.

When $\nu = 2n + 1$, the coefficient of the leading term is the resultant of a system of $2n+1$ functions of $2n+1$ variables: $(n+1)$ of them of the form $\Sigma x, \Sigma x^3, \ldots \Sigma x^{2n+1}$; n of them of the form $\Sigma ax, \Sigma ax^3, \ldots \Sigma ax^{2n+1}$ respectively.

To obtain this last-named resultant we may pair the variables (leaving *one* out) in every possible way, then make the sum of each pair and also the solitary or unpaired one zero, and finally, substituting in the n equations last stated (which come down to the form of a system of n equations between n variables discussed at the outset of this paper), calculate its resultant*. The product of all the resultants so found will be the resultant required, as may be proved by counting its order in the given coefficients, which is easily ascertained to be

$$1 . 3 . 5 \ldots (2n+1) \left(\frac{1}{1} + \frac{1}{3} + \ldots + \frac{1}{2n-1}\right) \{1 . 3 . 5 \ldots (2n-1)\},$$

as it ought in order to be the complete resultant. It will be seen then that this complete resultant, like the former one, is still made up of linear factors of the form

$$(a_1 + a_2 + \ldots + a_i - a_{i+1} - a_{i+2} - \ldots - a_{2i}),$$

and it only remains to ascertain the *frequency* of each such factor. By a calculation precisely similar in nature to that indicated for the case of $\nu = 2n$, it will be found that for this case of $\nu = 2n+1$ the *frequency* in question

$$= \Pi (i-1) \Pi i . Q(n-i) Q(n-i+1).$$

For $\nu = 2n$ it has been already proved to be

$$\Pi (i-1) \Pi i \{Q(n-i)\}^2.$$

* Regarded as loci, the ν functions can only intersect in one or another of an invariable system of points independent of the particular values of the coefficients. The equations to any one of these points (from what has been shown in the text) will easily be seen to be of the form

$$\begin{Bmatrix} x_1 = x_2 = \ldots = x_{2i} = -x_{2i+1} = -x_{2i+2} = \ldots = -x_{2j}, \\ x_{2j+1} = 0, \quad x_{2j+2} = 0, \ \ldots \ x_\gamma = 0. \end{Bmatrix}$$

Hence by a simple enough combinatorial calculation it may be deduced that the number of these fixed possible points of intersection, or, so to say, ganglions of the system is $\frac{3^\gamma + (-1)^\gamma}{8} - \frac{1}{4}$, which is, of course, always an integer; or, more briefly, the ganglionic exponent is the integer part of $\frac{3^\gamma}{8}$.

Thus we obtain the complete double-entry Table of *Frequency* underwritten:

	$i=$ 1,	2,	3,	4,	5,	6
$\nu=2$	1					
$=3$	1					
$=4$	1	2				
$=5$	3	2				
$=6$	9	2	12			
$=7$	45	6	12			
$=8$	225	18	12	144		
$=9$	1575	90	36	144		
$=10$	11025	450	108	144	2880	
$=11$	99225	3150	540	432	2880	
$=12$	893025	22050	2700	1296	2880	3628800

This Table, although obtained by two slightly varying processes according as ν is even or odd, forms, and ought to be regarded as, an organic whole.

To prevent misconception, I ought to add that when ν is sufficiently great, the compound symmetrical logarithmic wave $\Sigma a \log(x^2 - \xi^2)$ admits of other rectifiable cases besides those of the form $\Sigma a \log\left(\frac{x-\xi}{x+\xi}\right)$ above adduced.

It remains to study the relation between the frequency of each factor and the nature of the corresponding contact between the functions (regarded as *loci*) into whose resultant it enters. I have reason to hope that Dr Olaus Henrici, who has done such valuable work in the theory of Discriminants and Resultants, may be disposed to take up this interesting and pregnant question.