107.

NOTE ON THE THEORY OF A POINT IN PARTITIONS.

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IN writing down all the solutions in positive integers of the indefinite *Equation of Weight*, $x + 2y + 3z + \ldots = n$, or, in other words, in exhibiting all the partitions of n any integer greater than zero, it may sometimes be useful to be provided with an easy test to secure ourselves against the omission of any of them. Such a test is furnished by the following theorem:—

$$\Sigma (1 - x + xy - xyz \ldots) = 0;$$

thus, for example, if $x + 2y + 3z + 4t + \ldots = 4$, the solutions are five in number, namely

(1) $y = 2$,

(2) $t = 1$,

(3) $x = 1,\ z = 1$,

(4) $x = 2,\ y = 1$,

(5) $x = 4$,

the values of the omitted variables in each solution being zero. The five corresponding values of $1 - x + xy \ldots$ are

$$1,\ 1,\ 0,\ 1,\ -3,$$

whose sum is zero.

The theorem may be proved immediately by expressing the denumerant (which is zero) of the simultaneous equations

$$\begin{Bmatrix} x + 2y + 3z + \ldots = n, \\ x + y + z + \ldots \quad = 0, \end{Bmatrix}$$

in terms of simple denumerants according to the author's general method, or by virtue of the known theorem,

$$(1-t)(1-t^2)(1-t^3)\ldots$$
$$=1-\frac{t}{(1-t)}+\frac{t^3}{(1-t)(1-t^2)}-\frac{t^6}{(1-t)(1-t^2)(1-t^3)}$$
$$+\frac{t^{10}}{(1-t)(1-t^2)(1-t^3)(1-t^4)}+\ldots.$$

This gives at once the equation

$$\frac{1}{(1-t)(1-t^2)(1-t^3)\ldots}-\frac{t}{(1-t)^2(1-t^2)(1-t^3)\ldots}$$
$$+\frac{t^3}{(1-t)^2(1-t^2)^2(1-t^3)\ldots}+\ldots=1.$$

Hence the coefficient of t^n in the above written series for all values of n other than zero is zero. But it will easily be seen that the coefficient of t^n in the first term is $\Sigma 1$, in the second term Σx, in the third Σxy, &c.; so that

$$\Sigma(1-x+xy\ldots)=0,$$

as was to be shown. Thus we have obtained for the problem of indefinite partition a new algebraical unsymmetrical test supplementing the well-known pair of transcendental symmetrical tests expressible by the equations

$$\Sigma\frac{\Pi(x+y+z\ldots)}{\Pi x\Pi y\Pi z\ldots}=2^{n-1},$$

$$\Sigma(-)^{x+y+z\ldots}\frac{\Pi(x+y+z\ldots)}{\Pi x\Pi y\Pi z\ldots}=0^*.$$

The identity employed in the text is only a particular case of Euler's identity,

$$(1+tz)(1+t^2z)(1+t^3z)\ldots=1+\frac{tz}{(1-t)}+\frac{t^3z^2}{(1-t)(1-t^3)}+\ldots,$$

which is tantamount to affirming that the number of partitions of n into r distinct integers is the same as the number of partitions of n into any

* Subject of course to the condition that n is greater than 1. If $x, y, z, \ldots, \omega$ represents any solution in positive integers of the equation

$$x+2y+3z+\ldots+r\omega=r,$$

it is easy to see that

$$\Sigma(-)^{x+y+\ldots+}\frac{\Pi(x+y+\ldots+\omega)}{\Pi x\,\Pi y\ldots\Pi\omega}=1,\ -1,\text{ or }0,$$

according as n, in regard to the modulus $r+1$, is congruent to 0, 1, or neither to 0 nor 1, for the left-hand side of the equation is obviously the coefficient of x^n in the development of

$$\frac{1}{1+x+x^2\ldots+x^r},\text{ that is }\frac{1-x}{1-x^{r+1}}.$$

On making $r=\infty$, this theorem becomes the one in the text. It obviously affords a remarkable pair of independent arithmetical quantitative criteria for determining whether or not one number is divisible by another.

integers none greater than r, in which all the integers from 1 to r appear once at least. It has not, I believe, been noticed that these two systems of partitions are conjugate to each other, each partition of the one system having a correspondent to it in the other. The mode of passing from any partition to its correspondent is by converting each of its integers into a horizontal line of units, laying these horizontal lines vertically under each other, and then summing the columns. Thus, for example, 3, 4, 5 will be first expanded horizontally into

$$\begin{array}{lllll} 1 & 1 & 1, & & \\ 1 & 1 & 1 & 1, & \\ 1 & 1 & 1 & 1 & 1, \end{array}$$

and then summed vertically into

$$3 \quad 3 \quad 3 \quad 2 \quad 1.$$

This is the method employed by Mr Ferrers to show that the number of partitions of n into r, or a less number of parts, is the same as the number of partitions of n into parts none greater than r, and is, in fact, only a generalization of the method of intuitive proof of the fact that

$$m \times n = n \times m,$$

the difference merely being that we here deal with a parallelogram separated into two conterminous parts by an irregularly stepped boundary—one filled with units, the other left blank, instead of dealing with one entirely filled up with units.