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ON AN ELEMENTARY PROOF OF SIR ISAAC NEWTON'S HITHERTO UNDEMONSTRATED RULE, GIVEN IN THE *ARITHMETICA UNIVERSALIS*, FOR THE DISCOVERY OF IMAGINARY ROOTS IN ALGEBRAICAL EQUATIONS.

[*Transactions of the Royal Irish Academy*, XXIV. (1871), pp. 247—252.]

I HAVE the honour to lay before the Royal Irish Academy a brief statement of the principal results I have obtained in an investigation which leads to a theorem, from a particular case of which Newton's celebrated and hitherto undemonstrated rule, published in the *Arithmetica Universalis*, flows as a corollary in precisely the same way as Des Cartes' rule can be made to flow from the theorem of Fourier.

Suppose $fx = 0$ an ordinary algebraical equation of the nth degree divested of multiple roots, and let us consider the coupled series, or pair of progressions,

$$\left.\begin{matrix} f^n x, & f^{n-1}x, & f^{n-2}x, & \ldots & f^1 x, & fx \\ G_n x, & G_{n-1}x, & G_{n-2}x, & \ldots & G_1 x, & Gx \end{matrix}\right\}, \qquad \text{(A)}$$

where $$f^r x \text{ means } \left(\frac{d}{dx}\right)^r fx,$$

and $$G_r x = (f_r x)^2 - \lambda_r f_{r-1} x f_{r+1} x,$$

it being understood that

$$G_n x = (f_n x)^2, \quad Gx = (fx)^2.$$

Let us further suppose that λ_r satisfies the equation in differences

$$2 - \lambda_r = \frac{1}{\lambda_{r+1}};$$

and furthermore that λ_r is positive for all values of r from 1 to n, both inclusive.

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When $f^r x = 0$ we have, as is well known,

$$f^r(x+\epsilon) = f^{r+1}x \,.\, \epsilon;$$

and in general, when

$$f^r x = 0, \quad f^{r+1}x = 0 \ldots f^{r+i-1}x = 0,$$

$$f^r(x+\epsilon) = f^{r+i}x \frac{\epsilon^i}{1\,.\,2\ldots i}, \tag{B}$$

ϵ being supposed to be an infinitesimal. So similarly it will be found that when $G_r x = 0$, then

$$G_r(x+\epsilon) = \frac{\epsilon f^r x}{\lambda_{r+1} f^{r+1} x} G_{r+1} x;$$

and more generally, when

$$G_r x = 0, \quad G_{r+1}x = 0 \ldots G_{r+i-1}x = 0,$$

$$G_r(x+\epsilon) = \frac{f^r x}{\lambda_{r+1}\,.\,\lambda_{r+2}\ldots\lambda_{r+i} f^{r+i}x} G_{r+i}x \,.\, \frac{\epsilon^i}{1\,.\,2\ldots i}. \tag{C}$$

Now let us understand by a *double permanence* in the coupled series (A) a double succession of associated terms, as $\begin{Bmatrix} f^r x \; f^{r+1}x \\ G_r x \; G_{r+1} x \end{Bmatrix}$ such that $\dfrac{f^{r+1}x}{f^r x}$ and $\dfrac{G_{r+1}x}{G_r x}$ are both positive; and let us study the effect produced upon the total number of such double permanences by increasing x continuously.

It is clear that no change in such number can take place, except by one or more of the terms in the upper or in the lower, or in both, simultaneously becoming zero.

$G_n x$ and Gx, and $f^n x$ being, the two former necessarily positive, and the third a constant, can never pass through zero. Let us suppose that, for a certain value of x, certain of the intermediate terms become zero, but that fx does not become zero.

The following cases arise and may be considered separately:—

(1) $f^r x$ may become zero alone without either of the adjacent terms doing so, in which case $G_r x$ by definition cannot become zero.

(2) $f^r x, f^{r+1}x, \ldots f^{r+i-1}x$ may all become zero simultaneously, in which case, i being supposed greater than unity, $G_r x, G_{r+1}x, \ldots G_{r+i-1}x$ will all become zero likewise.

(3) $G_r x$ may become zero without either of the adjacent terms doing so.

(4) $G_r x, G_{r+1}x, \ldots G_{r+i-1}x$, may all become zero simultaneously without the superior associated terms any of them vanishing.

On considering all these cases in conjunction with the equations (B), (C), it will be found that by the passage through zero double permanences may be gained, but can never be lost, as x goes on increasing, and moreover that the number so gained is always even.

Now let us suppose that the last superior term fx becomes zero, that is that x becomes a root of the given equation, then it will be easily found that one double permanence is gained*.

Hence, if i real roots are included between λ and μ, and if the number of double permanences in the coupled series (A) when λ is written for x be called $P(\lambda)$, and the like quantity when μ is written for x be called $P(\mu)$; and if μ is greater than λ, we must have $P(\mu) - P(\lambda) = i + 2k$, where k is zero or a positive integer. So that the number of real roots between λ, μ cannot exceed the difference between $P(\mu)$ and $P(\lambda)$.

It remains only to integrate the equation $2 - \lambda_r = \dfrac{1}{\lambda_{r+1}}$; the general integral of this is easily found to be

$$\lambda_r = \frac{A + B(r-1)}{A + Br};$$

so that

$$G_r x = (f^r x)^2 - \frac{A + B(r-1)}{A + Br} f^{r-1}x \,.\, f^{r+1}x;$$

the ratio $A : B$ being subject to the sole condition that $\dfrac{A + B(r-1)}{A + Br}$ shall be positive for all integer values of r not exterior to the limits 1, n, which condition is equivalent to the condition that A and $A + (n-1)B$ shall each have the same sign.

Let us suppose $A = n$, $B = -1$, then

$$\left.\begin{aligned}
G_n x &= (f^n x)^2, \\
G_{n-1} x &= (f^{n-1}x)^2 - \frac{2}{1} f^n x f^{n-2} x, \\
G_{n-2} x &= (f^{n-2}x)^2 - \frac{3}{2} f^{n-1} x f^{n-3} x, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
G_1 x &= (f'x)^2 - \frac{n}{n-1} f''x fx, \\
G x &= (fx)^2.
\end{aligned}\right\} \qquad \text{(D)}$$

Let fx be written under the form

$$c_0 x^n + nc_1 x^{n-1} + \frac{1}{2} n(n-1) c_2 x^{n-2} + \ldots + nc_{n-1} x + c_n;$$

* This is on the supposition of the root in question being simple or unifold; if it is a multifold root, equivalent to i simple roots, i double permanences will be gained.

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write down the coupled series

$$\begin{matrix} c_0, & c_1, & c_2, & \dots c_n \\ c_0^2, & c_1^2 - c_0c_2, & c_2^2 - c_1c_3, & \dots c_n^2 \end{matrix} \text{ say } \begin{matrix} c_0, & c_1, & c_2, & \dots c_n. \\ T_0, & T_1, & T_2, & \dots T_n. \end{matrix}$$

On consulting the *Arithmetica Universalis,* it will be found that Newton's complete theorem amounts to asserting that the number of negative roots in $fx = 0$ cannot exceed the number of double permanences in these associated series, and (which is a necessary inference from the former, and needs no separate proof, since to obtain it we need only to change x into $-x$ in the equation $fx = 0$) that the number of positive roots in the same cannot exceed the number of variation-permanences in such association; by a variation-permanence understanding a double succession in which the superior terms c_r, c_{r+1} form a variation, and the associated inferior terms T_r, T_{r+1} a permanence.

But it is easily seen that

$$c_0 = \frac{1}{\Pi n} f^n 0,$$

$$c_1 = \frac{1}{\Pi n} f^{n-1} 0,$$

$$c_2 = \frac{1 \cdot 2}{\Pi n} f^{n-2} 0,$$

$$c_3 = \frac{1 \cdot 2 \cdot 3}{\Pi n} f^{n-3} 0,$$

....

$$c_n = \frac{1 \cdot 2 \cdot 3 \dots n}{\Pi n} f 0;$$

and consequently,

$$T_0 = \frac{1}{(\Pi n)^2} (f^n 0)^2,$$

$$T_1 = \frac{1}{(\Pi n)^2} \left\{ (f^{n-1}0)^2 - \frac{2}{1} f^n 0 f^{n-2} 0 \right\},$$

$$T_2 = \frac{(1 \cdot 2)^2}{(\Pi n)^2} \left\{ (f^{n-2}0)^2 - \frac{3}{2} f^{n-1} 0 f^{n-3} 0 \right\},$$

$$T_3 = \frac{(1 \cdot 2 \cdot 3)^2}{(\Pi n)^2} \left\{ (f^{n-3}0)^2 - \frac{4}{3} f^{n-2} 0 f^{n-4} 0 \right\},$$

....

$$T_n = \frac{(1 \cdot 2 \cdot 3 \dots n)^2}{(\Pi n)^2} f_0^2.$$

So that, using the symbol G in the special sense in which it is employed in equations (D), if we compare the two coupled systems

$$\left.\begin{matrix} c_0, & c_1, & c_2, \dots & c_n \\ T_0, & T_1, & T_2, \dots & T_n \end{matrix}\right\}, \quad \left.\begin{matrix} f^n0, & f^{n-1}0, & f^{n-2}0, \dots f0 \\ G_n0, & G_{n-1}0, & G_{n-2}0, \dots G0 \end{matrix}\right\},$$

it is evident that each term in the one association is a positive multiplier of the corresponding term in the other. But by the theorem previously established we know that the number of negative roots in fx, that is the number of roots between 0 and $-\infty$, cannot exceed the difference between the number of double permanences in the (f, G) couple, less the number of double permanences in this couple, when $-\infty$ is substituted in place of 0; but in the couple so modified, the superior terms present only variation successions, so that there can be no double permanences, and consequently the above assertion is tantamount to the more simple one, that the number of negative roots in fx cannot exceed the number of double permanences in the (f, G) couple, which are evidently identical with those in the (c, T) couple. This demonstrates Newton's complete rule.

More generally, if we form a double series precisely analogous to the (c, T) couple, except that the c's are derived from the equation $f(x+p)=0$ in lieu of the equation $fx=0$, and if we call the number of double permanences so resulting the number of such due to p, we have the more general theorem that the number of real roots passed over in ascending from p to q cannot exceed the number of double permanences due to q less the number of the same due to p, and can only fall short of such difference by some *even* number. This, however, it will be seen, is only a particular case of the general theorem where λ_r is of the form

$$\frac{A+B(r-1)}{A+Br}.$$

But furthermore it is proper to state, that although such is the most general form that according to what has been above demonstrated can be ascribed to λ_r, it follows from other considerations that it is *not* the most general.

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