109.

ON THE PARTITION OF AN EVEN NUMBER INTO TWO PRIMES.

[*Proceedings of the London Mathematical Society*, IV. (1871—73), pp. 4—6.]

In one of his minor papers, Euler has enunciated, as a theorem resting entirely on intuition from a comparatively small number of instances, that every even number may be decomposed into a sum of two primes. The object of Mr Sylvester's communication was to obtain some measure of the probable number of ways in which such decomposition can be effected for any given number; if it can be shown to be probably greater than the square root of the number itself, it will follow, from generally admitted principles of the theory of chances, that the probability of the theorem being universally true above any assigned limit, if *proved* to be true up to that limit, may be represented by an infinite product of terms which will approach as near as we please to unity the higher the limit is taken.

The mere fact of the theorem, as Euler gave it, being proved up to 100,000,000 or any other number, however great, would leave the probability of its being universally true absolutely zero, just as the fact of the sun having risen 100,000,000 times would not contribute an atom of probability to the supposition that it would continue to rise for all time to come. In the case before us, on the contrary, the probability of the theorem being universally true by a sufficiently copious induction may be made to approach as near as we please to absolute certitude. The Author considers that he has established beyond the reach of reasonable doubt that the order of the magnitude which represents the mean probable value of the number of modes of effecting the resolution of a very large even number into two prime numbers is that of the square of the number of primes inferior to the given number divided by the number itself, or, which (thanks to the discoveries of Legendre and Tchebicheff) we know to be the same thing, the number of the decompositions in question bears a finite ratio (assignable within limits) to the number to be decomposed divided by the square of its Napierian logarithm. If we agree provisionally to call preter-primes in respect to n those numbers which are prime themselves and also when subtracted from n leave prime remainders, the Author shows that the probable number of such preter-primes (that is, the most probable value attainable under our present conditions of knowledge) may be found approximately by multiplying the number of ordinary primes inferior to n by the product of a set of fractions depending in part on the order of the magnitude and in part on the constitution of the number n. If n is the double of a prime, the product in question is got by multiplying together all the quantities $\frac{p-2}{p-1}$, where p is every odd prime between unity and the square root of n; but if n itself contains any such primes among its factors, then the corresponding factors are to be omitted out of the product. We thus see that if two even numbers of considerable magnitude lie adjacent and tolerably near to each other, one of which is the double of a prime, but the other six times a prime, the number of preter-primes relative to the latter will be about twice as many as those relative to the former. For the purpose of greater simplicity of explanation, the formula of approximation has been stated above with less accuracy than it admits of being stated with; instead of the total number of odd primes being multiplied by the product of factors last described, those only should have been taken which are not intermediate between 2 and $\sqrt{n}$, and the result so modified should have been stated to be the probable value, not of the total number of preter-primes, but only of such of them (by far the larger number) as are not of the excluded class above described, nor, subtracted from n, give rise to remainders belonging to such class. The Author has found, by actual trial on an extensive scale, that the estimated values of the number of decompositions never differ by more than a moderate, and in some cases exceedingly slight, percentage from their actual values, determined by the use of Borchardt's tables.

The same methods enable him also to assign a probable value to the number of modes of resolving an odd number into the sum of one prime and the double of another, and in general leads to an approximate representation of the number of solutions in prime numbers of any system of linear equations of which the total number of solutions is limited, and even to resolve approximately such questions as that of determining how many prime numbers there are inferior to a given limit which are followed by prime numbers differing from them by any assigned interval.

Since the communication was made to the Mathematical Society, the Secretaries have been informed by Mr Sylvester that he has verified his results by quite a different method. The exact number of the solutions of the equation $x+y=n$ in prime numbers may be expressed algebraically by means of the method of Generating Functions in terms of the inferior primes to n. The expression will be found to consist of two parts—one a constant multiple of n, the other a function of the roots of unity corresponding to the several inferior primes and their combinations. The former non-periodic part may obviously be regarded as the mean value of the expression, and Mr Sylvester has found that it is identical with the value obtained by the method of averages previously employed.