APPENDIX.

Additional Notes to Prof. Sylvester's Exeter British Association Address

given in the author's Reprint of this address published in the volume *The Laws of Verse* (London, Longmans, 1870), together with the author's appendix to that volume

On the Incorrect Description of Kant's Doctrine of Space and Time common in English Writers,

a correspondence reprinted from *Nature* (Vol. I., 1869—70).

[*The references are to the pages of the present Volume.*]

p. 654, l. 38. The annexed instance of Mathematical euristic is, I think, from its intrinsic interest, worthy of being put on record. The so-called canonical representation of a binary quartic of the eighth degree I found to be a quartic multiplied by itself, together with a sum of powers of its linear factors, just as for the fourth degree it was known to be a quadric into itself, together with a sum of powers of its factors; but for a sextic a cubic multiplied into itself, with a tail of powers as before, was not found to answer. To find the true representation was like looking out into universal space for a planet desiderated according to Bode's or any other empirical law. I found my *desideratum* as follows: I invented a catena of morphological processes which, applied to a quadric or to a quartic, causes each to reproduce itself: I then considered the two quadrics and two quartics to be noumenally distinguishable (one as an automorphic derivative of the other) although phenomenally identical. The same catena of processes applied to the cubic gave no longer an identical but a distinct derivative, and the product of the two I regarded as the analogue of the before-mentioned square of the quadric or of the quartic. This product of a cubic by its derivative so obtained together with a sum of powers of linear factors of the original cubic, I found by actual trial to my great satisfaction satisfied the conditions of canonicity, and it was thus I was led up to the desired representation, which will be found reproduced in one of Prof. Cayley's memoirs on Quantics, and in Dr Salmon's lectures on Modern Algebra. Here certainly induction, observation, invention, and experimental verification all played their part in contributing to the solution of the problem. I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought—a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. *That night we slept no more.* The canonisant of the quartic (its cubic covariant) was the first thing to offer itself in the inquiry. I had but to think the words "Resultant of Quintic and its Canonisant," and the octodecadic skew invariant would have fallen spontaneously into my lap. By quite another mode of consideration, M. Hermite subsequently was led to the discovery of this, the key to the innermost sanctuary of invariants—so hard is it in Euristic to see what lies immediately before one's eyes. The disappointment weighed deeply, far too deeply, on my mind, and caused me to relinquish for long years a cherished field of meditation; but the whirligig of time brings about its revenges. Ten years later this same Canonisant gave me the upper hand of my honoured predecessor and guide, M. Hermite, in the inquiry (referred to at the end of this address) concerning the invariantive criteria of the constitution of a Quintic with regard to the real and imaginary. By its aid I discovered the essential character of the famous amphigenous surface of the ninth order, and its bicuspidal unicursal section of the fourth order (otherwise termed the bicorn), as may be seen in the third part of my Trilogy, printed in the *Philosophical Transactions* [p. 376, above].

p. 654, l. 41. I was under the conviction that a passage to that effect from Lagrange had been cited to me some years ago by M. Hermite of the Institute of France; on applying to him on the subject, I received the following reply:

"Relativement à l'opinion que suivant vous j'aurais attribuée à Lagrange, je m'empresse de vous informer qu'il ne faut aucunement, à ma connaissance, l'en rendre responsable. Nous nous sommes entretenus du rôle de la *faculté d'observation dans les études que nous avons poursuivies de concert pendant bien des années*, et c'est alors, sans doute, que je vous aurai conté une anecdote que je tiens de M. Chevreul. M. Chevreul, allant à l'Institut dans la voiture de Lagrange, a été vivement frappé du sentiment de plaisir avec lequel ce grand géomètre lui faisait voir, dans un travail manuscrit, la beauté extérieure et artistique, si je peux dire, des nombreuses formules qui y figuraient. Ce sentiment nous l'avons tous éprouvé en faisant, avec sincérité, abstraction de l'idée analytique dont les formules sont l'expression écrite. Il y a là, n'est-il point vrai, un imperceptible lien qui rattache au monde de l'art le monde abstrait de l'algèbre et de l'analyse, et j'ôserai même vous dire que je crois a des sympathies réelles, qui vous font trouver un charme dans les notations d'un auteur et des répulsions qui éloignent d'un autre, par l'apparence seule des formules."

I am, however, none the less persuaded that on one or more than one occasion, M. Hermite, speaking of Lagrange, expressed to me, if not as I supposed on Lagrange's, then certainly on his own high authority, "that the faculty of observation was no less necessary for the successful cultivation of the pure mathematical than of the natural sciences." I am glad also to notice that Lagrange was able to accommodate a friend *dans sa voiture.* England has much to learn from France and Russia as to the proper mode of treating its greatest men.

p. 655, l. 9. It is very common, not to say universal, with English writers, even such authorised ones as Whewell, Lewes, or Herbert Spencer, to refer to Kant's doctrine as affirming space "to be a form of thought," or "of the understanding." This is putting into Kant's mouth (as pointed out to me by Dr C. M. Ingleby), words which he would have been the first to disclaim, and is as inaccurate a form of expression as to speak of "the plane of a sphere," meaning its surface or a superficial layer, as not long ago I heard a famous naturalist do at a meeting of the Royal Society. Whoever wishes to gain a notion of Kant's leading doctrines in a succinct form, weighty with thought, and free from all impertinent comment, should study Schwegler's *Handbook of Philosophy*, translated by Stirling. He will find in the same book a most lucid account of Aristotle's doctrine of matter and form, showing how matter passes unceasingly upwards into form, and form downwards into matter; which will remind many of the readers of *Nature* of the chain of depolarisations and repolarisations which are supposed to explain the decomposition of water under galvanic action, eventuating in oxygen being thrown off at one pole and hydrogen at the other (it recalls also the high algebraical theories in which the same symbols play the part of operands to their antecedents and operators to their consequents): at one end of the Aristotelian chain comes out πρώτη ὕλη, at the other πρῶτον εἶδος. We have, then, only to accept and apply the familiar mathematical principle of the two ends of infinity being one and the same point (like the extremities of a divided and extended ring), and the otherwise immovable stumbling-block of duality is done away with, and the universe reintegrated in the wished-for unity. For this corollary, which to many will appear fanciful, neither Aristotle nor Schwegler is responsible. We perfectly understand how in perspective the latent polarities of any point in a closed curve (taken as the object) may be developed into and displayed in the form of a duad of *quasi* points or half-points at an infinite distance from each other in the picture. In like manner we conceive how *actuality* and *potentiality* which exist indistinguishably as one in the *absolute* may be projected into seemingly separate elements or moments on the plane of the human understanding. Whatever may be the merits of the theory in itself, this view seems to me to give it a completeness which its author could not have anticipated, and to accomplish what Aristotle attempted but avowedly failed to effect, viz. the complete subversion of the "Platonic Duality," and the reintegration of matter and mind into one.

p. 655, l. 19. I know there are many, who, like my honoured and deeply lamented friend the late eminent Prof. Donkin, regard the alleged notion of generalised space as only a disguised form of algebraical formulisation ; but the same might be said with equal truth of our notion of infinity in algebra, or of impossible lines, or lines making a zero angle in geometry, the utility of dealing with which as positive substantiated notions no one will be found to dispute. Dr Salmon, in his extension of Chasles' theory of characteristics to surfaces, Mr Clifford, in a question of probability (published in the *Educational Times*), and myself in my theory of partitions, and also in my paper on Barycentric Projection in the *Philosophical Magazine* [p. 358, above], have all felt and given evidence of the practical utility of handling space of four dimensions, as if it were conceivable space. Moreover, it should be borne in mind that every perspective representation of figured space of four dimensions is a figure in real space, and that the properties of figures admit of being studied to a great extent, if not completely, in their perspective representations. In philosophy, as in aesthetic, the highest knowledge comes by faith. I know (from personal experience of the fact) that Mr Linnell or Madame Bodichon can distinguish purple tints in clouds where my untutored eye and unpurged vision can perceive only confused grey. If an Aristotle, or Descartes, or Kant, assures me that he recognises God in the conscience, I accuse my own blindness if I fail to see with him. If Gauss, Cayley, Riemann, Schläfli, Salmon, Clifford, Kronecker, have an inner assurance of the reality of transcendental space, I strive to bring my faculties of mental vision into accordance with theirs. The positive evidence in such cases is more worthy than the negative, and actuality is not cancelled or balanced by privation, as matter plus space is none the less matter. I acknowledge two separate sources of authority—the collective sense of mankind, and the illumination of privileged intellects. As a parallel case, I would ask whether it is by demonstrative processes that the doctrine of limits and of infinitely greats and smalls, has found its way to the ready acceptance of the multitude; or whether, after deducting whatever may be due to modified hereditary cerebral organisation, it is not a consequence rather of the insensible moulding of the ideas under the influence of language which has become permeated with the notions originating in the minds of a few great thinkers? I am assured that Germans, even of the non-literary class, such as ladies of fashion and novel readers, are often appalled by the hebetude of their English friends in muddling up together, as if they were nearly or quite the same thing, the reason and the understanding, in doing into English the words Vernunft and Verstand, thereby confounding distinctions now become familiar (such is the force of language) to the very milkmaids of Fatherland.

As a public teacher of mere striplings, I am often amazed by the facility and absence of resistance with which the principles of the infinitesimal calculus are accepted and assimilated by the present race of learners. When I was young, a boy of sixteen or seventeen who knew his infinitesimal calculus would have been almost pointed at in the streets as a prodigy, like Dante, as a man who had seen hell. Now-a-days, our Woolwich cadets at the same age, talk with glee of tangents and asymptotes and points of contrary flexure and discuss questions of double maxima and minima, or ballistic pendulums, or motion in a resisting medium, under the familiar and ignoble name of *sums*.

p. 657, l. 15. Induction and analogy are the special characteristics of modern mathematics, in which theorems have given place to theories, and no truth is regarded otherwise than as a link in an infinite chain. "Omne exit in infinitum" is their favourite motto and accepted axiom. No mathematician now-a-days sets any store on the discovery of isolated theorems, except as affording hints of an unsuspected new sphere of thought, like meteorites detached from some undiscovered planetary orb of speculation. The form, as well as matter, of mathematical science, as must be the case in any true living organic science, is in a constant state of flux, and the position of its centre of gravity is liable to continual change. At different periods in its history defined, with more or less accuracy, as the science of number or quantity, or extension or operation or arrangement, it appears at present to be passing through a phase in which the development of the notion of Continuity plays the leading part. In exemplification of the generalising tendency of modern mathematics, take so simple a fact as that of two straight lines or two planes being incapable of including "a space." When analysed this statement will be found to resolve itself into the assertion that if two out of the four triads that can be formed with four points lie respectively *in directo*, the same must be true of the remaining two triads; and that if two of the five tetrads that can be formed with five points lie respectively *in plano*, the remaining three tetrads (subject to a certain obvious exception) must each do the same. This, at least, is one way of arriving at the notion of an unlimited rectilinear and planar schema of points. The two statements above made, translated into the language of determinants, immediately suggest as their generalised expression my great "Homaloidal Law," which affirms that the vanishing of a certain specifiable number of minor determinants of a given order of any matrix (i.e. rectangular array of quantities) implies the simultaneous evanescence of all the rest of that order. I made (*inter alia*) a beautiful application of this law (which is, I believe, recorded in Mr Spottiswoode's valuable treatise on Determinants, but where besides I know not) to the establishment of the well-known relations, wrung out with so much difficulty by Euler, between the cosines of the nine angles, which two sets of rectangular axes in space make with one another. This is done by contriving and constructing a matrix such that the six known equations connecting the nine cosines taken both ways in sets of threes shall be expressed by the evanescence of six of its minors; the simultaneous evanescence of the remaining minors given by the Homaloidal Law will then be found to express the relations in question (which, Euler has put on record, it drove him almost to despair to obtain), but which are thus obtained by a simple process of inspection and reading off, without any labour whatever. The fact that such a law, containing in a latent form so much refined algebra, and capable of such interesting immediate applications, should present itself to the *observation* merely as the extended expression of the ground of the possibility of our most elementary and seemingly intuitive conceptions concerning the right line and plane, has often filled me with amazement to reflect upon.

p. 657, l. 17. As the prerogative of Natural Science is to cultivate a taste for observation, so that of Mathematics is, almost from the starting-point, to stimulate the faculty of invention.

p. 658, l. 41. Is it not the same disregard of principles, the same *indifference to truth for its own sake* which prompts the question, "Where's the good of it?" in reference to speculative science, and "Where's the harm of it?" in reference to white lies and pious frauds? In my own experience I have found that the very same class of people who delight to put the first question are in the habit of acting upon the denial implied in the second. *Abit in mores incuria.*

p. 658, l. 43. This theorem still awaits proof; it is stated, I believe, in Euler's correspondence with Goldbach: I re-discovered it in ignorance of Euler's having mentioned it, in connection with a theory of my own concerning cubic forms. The evidence in its favour is *induction* of the undemonstrative or purely accumulative kind, and it may or may not turn out eventually to be true. As a most learned scholar who heard this address given at Exeter remarked to me, not many days ago, it is certainly by no process of deduction that we make out that five times six is thirty. I mention this, because I know some, who agree, or did agree, with Professor Huxley's published opinions about mathematics, are under the impression that the higher processes of mind in mathematics only concern "the aristocracy of mathematicians": on the contrary, they lie at the very foundations of the subject. There are besides, and in abundance, mathematical processes which only by a forced interpretation can be brought under the head of demonstration, whether deductive or inductive, and really belong to a sort of artistic and constructive faculty, such for example as evaluating definite integrals, or making out the best way one can the number of distinct branches and the general character of each branch of a curve from its algebraical equation.

p. 660, l. 8. I am happy to be able to add that this gentleman who, when the above lines were printed, was at the bottom of the Staff of the Mathematical Instructors at Woolwich, has been since appointed, with the full concurrence of his colleagues, on the nomination of the Governor, Sir John Lintorn Simmons, K.C.B., to succeed me as Professor of Mathematics at the Royal Military Academy.

p. 660, l. 20. *Complete* in the sense of *universal*, more *than perfect or complete* in the ordinary sense. Two criteria are absolutely fixed; but in addition to these two an additional criterion or set of criteria must be introduced to make the system of conditions sufficient. The number of such set may be either one or whatever number we please, and into such one or into each of the set (if more than one) an indefinite number of arbitrary parameters (limited) may be introduced. Now the geometrical construction I arrive at contains implicitly the totality of all these infinitely varied forms of criteria, or sets of criteria, and without it the existence and possibility of such variety in the shape of the solution could never have been anticipated or understood. My truly eminent friend M. Charles Hermite (Membre de l'Institut), with all the efforts of his extraordinary analytical power, and with the knowledge of my results to guide him, has only been able by the non-geometrical method to arrive at one form of solution consisting of a third criterion absolutely definite and destitute of a single variable parameter. As is well known, I have made a very important use of a criterion of the same form as M. Hermite's, but containing one arbitrary parameter (limited). The subject will be found resumed from the point where I left it, and pursued in considerable detail by Prof. Cayley, in one of his more recent memoirs on Quantics in the *Philosophical Transactions.* M. Hermite it was who first surprised Invariantists (l'Eglise Invariantive, as we are sometimes styled) by an *à priori* demonstration that the nature of the roots or factors of quartics could in general be found by means of invariantive criteria. This was known to be possible up to the *fourth* order of binary quantics, and impossible for the *fourth.* M. Hermite showed that this negation which seemed to stop the way to further progress was an exceptional case; that whereas for the second, third, fifth, sixth, and all higher degrees the thing could be done, for the fourth alone it was impossible: as regards linear Quantics, the question does not arise. I look upon this failure of a law for one term in the middle of an infinite progression as an unparalleled *miracle of arithmetic*, far more real and deeper seated than the one alluded to by Mr Babbage in connection with the discontinuous action of a supposed machine in his ninth Bridgwater Treatise.

p. 660, l. 22. So I found, as a pure matter of observation, that allineation (*alignement*) in ornamental gardening—i.e. the method of putting trees in positions to form a very great number or the greatest number possible of straight rows, of which a few special cases only had been previously considered as detached porismatic problems, forms part of a great connected theory of the pluperfect points on a cubic curve, those points, of which the nine points of inflection and Plücker's twenty-seven points may serve as the lowest instances.

On the Incorrect Description of Kant's Doctrine of Space and Time common in English Writers *.

In the very remarkable contribution by Professor Sylvester (*Nature*, No. 9) this sentence occurs: "It is very common, not to say universal, with English writers, even such authorised ones as Whewell, Lewes, or Herbert Spencer, to refer to Kant's doctrine as affirming space to be a 'form of thought' 'or of the understanding.' This is putting into Kant's mouth (as pointed out to me by Dr C. M. Ingleby) words which he would have been the first to disclaim."

It is not on personal grounds that I wish to rectify the misconception into which Dr Ingleby has betrayed Professor Sylvester. When objections are made to what I have written, it is my habit either silently to correct my error, or silently to disregard the criticism. In the present case I might be perfectly contented to disregard a criticism which any one who even glanced at my exposition of Kant would see to be altogether inexact; but as misapprehensions of Kant are painfully abundant, readers of Kant being few, and those who take his name in vain being many, it may be worth while to stop *this* error from getting into circulation through the channel of *Nature*. Kant assuredly did teach, as Professor Sylvester says, and as I have repeatedly stated, that space is a form of intuition. But there is no discrepancy at all in also saying that he taught space to be a "form of thought," since every student of Kant knows that intuition without thought is mere sensuous *impression*. Kant considered the mind under three aspects, Sensibility, Understanding, and Reason. The *à priori* forms of Sensibility, which rendered Experience possible, were Space and Time: these were forms of thought, conditions of cognition. It was by such forms of thought that he reoccupied the position taken by Leibnitz in defending and amending the doctrine of innate ideas, namely, that knowledge has another source besides sensible experience—the *intellectus ipse*.

While, therefore, any one who spoke of space as a "form of the understanding" would certainly use language which Kant would have disclaimed, Kant himself would have been surprised to hear that space was not held by him as a "form of thought."

George Henry Lewes.

January 3.

The following paragraphs, I believe, faithfully render sundry passages of Kant's writings:

"Objects are given to us by means of sense (Sinnlichkeit), which is the sole source of intuitions (Anschauungen); but they are thought by the understanding, from which arise conceptions (Begriffe)." (*Kritik*, p. 55, Hartenstein's edition.)

* From *Nature*, Vol. I. (1869—70). See [p. 655, l. 9].