# 12.

# ON THE THEORY OF DETERMINANTS.

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THE following Memoir is composed of two separate investigations, each of them having a general reference to the Theory of Determinants, but otherwise perfectly unconnected. The name of "Determinants" or "Resultants" has been given, as is well known, to the functions which equated to zero express the result of the elimination of any number of variables from as many linear equations, without constant terms. But the same functions occur in the resolution of a system of linear equations, in the general problem of elimination between algebraic equations, and particular cases of them in algebraic geometry, in the theory of numbers, and, in short, in almost every part of mathematics. They have accordingly been a subject of very considerable attention with analysts. Occurring, apparently for the first time, in Cramer's *Introduction à l'Analyse des Lignes Courbes*, 1750: they are afterwards met with in a Memoir *On Elimination*, by Bezout, *Mémoires de l'Académie*, 1764; in two Memoirs by Laplace and Vandermonde in the same collection, 1774; in Bezout's *Théorie générale des Equations algebriques* [1779]; in Memoirs by Binet, *Journal de l'École Polytechnique*, vol. IX. [1813]; by Cauchy, *ditto*, vol. X. [1815]; by Jacobi, *Crelle's Journal*, vol. XXII. [1841]; Lebesgue, *Liouville*, [vol. II. 1837], &c. The Memoirs of Cauchy and Jacobi contain the greatest part of their known properties, and may be considered as constituting the general theory of the subject. In the first part of the present paper, I consider the properties of certain derivational functions of a quantity $U$, linear in two separate sets of variables (by the term "Derivational Function," I would propose to denote those functions, the nature of which depends upon the form of the quantity to which they refer, with respect to the variables entering into it, e.g. the differential coefficient of any quantity is a derivational function. The theory of derivational functions is apparently one that would admit of interesting developments). The particular functions of this class which are here considered, are closely connected with the theory of the reciprocal polars of surfaces of the second order, which latter is indeed a particular case of the theory of these functions.

In the second part, I consider the notation and properties of certain functions resolvable into a series of determinants, but the nature of which can hardly be explained independently of the notation.

In the first section I have denoted a determinant, by simply writing down in the form of a square the different quantities of which it is made up. This is not concise, but it is clearer than any abridged notation. The ordinary properties of determinants, I have throughout taken for granted; these may easily be learnt by referring to the Memoirs of Cauchy and Jacobi, quoted above. It may however be convenient to write down the following fundamental property, demonstrated by these authors, and by Binet.

$$\left|\begin{array}{lll} \alpha, & \beta, & \ldots \\ \alpha', & \beta', & \\ \vdots & & \end{array}\right| \left|\begin{array}{lll} \rho, & \sigma, & \ldots \\ \rho', & \sigma', & \\ \vdots & & \end{array}\right| = \left|\begin{array}{lll} \rho\alpha+\sigma\beta\ldots, & \rho\alpha'+\sigma\beta'\ldots, & \ldots \\ \rho'\alpha+\sigma'\beta\ldots, & \rho'\alpha'+\rho'\sigma'\ldots, & \\ \vdots & & \end{array}\right| \quad \ldots\ldots (\odot),$$

an equation, particular cases of which are of very frequent occurrence, e.g. in the investigations on the forms of numbers in Gauss' *Disquisitiones Arithmetica* [1801], in Lagrange's *Determination of the Elements of a Comet's Orbit* [1780], &c. I have applied it in the *Cambridge Mathematical Journal* [1] to Carnot's problem, of finding the relation between the distances of five points in space, and to another geometrical problem. With respect to the notation of the second section, this is so fully explained there, as to render it unnecessary to say anything further about it at present.

§ 1. On the properties of certain determinants, considered as Derivational Functions.

Consider the function

$$\begin{array}{l} U = x(\alpha\xi+\beta\eta+\ldots)+ \\ \quad x'(\alpha'\xi+\beta'\eta+\ldots)+ \\ \quad \vdots \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

($n$ lines, and $n$ terms in each line);

and suppose

$$KU = \left|\begin{array}{lll} \alpha, & \beta, & \ldots \\ \alpha', & \beta', & \ldots \\ \vdots & \vdots & \end{array}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

(The single letter $\kappa$ being employed instead of $KU$, in cases where the quantity $KU$, rather than the functional symbol $K$, is being considered.) And write

$$FU = -\left|\begin{array}{llll} & \mathrm{A}x+\mathrm{A}'x'+\ldots, & \mathrm{B}x+\mathrm{B}'x'+\ldots, & \ldots \\ \mathrm{R}\xi+\mathrm{S}\eta+\ldots, & \alpha\quad, & \beta & ,\ \ldots \\ \mathrm{R}'\xi+\mathrm{S}'\eta+\ldots, & \alpha'\quad, & \beta' & ,\ \ldots \\ \vdots & & & \end{array}\right| \quad \ldots\ldots\ldots (3).$$

$$\Finv U = -\left|\begin{array}{llll} & \mathrm{R}x+\mathrm{R}'x'+\ldots, & \mathrm{S}x+\mathrm{S}'x'+\ldots, & \ldots \\ \mathrm{A}\xi+\mathrm{B}\eta+\ldots, & \alpha\quad, & \beta & ,\ \ldots \\ \mathrm{A}'\xi+\mathrm{B}'\eta+\ldots, & \alpha'\quad, & \beta' & ,\ \ldots \\ \vdots & & & \end{array}\right| \quad \ldots\ldots\ldots (4).$$

The symbols $K$, $F$, $\Finv$ possess properties which it is the object of this section to investigate.

Let $A, B, \ldots, A', B', \ldots, \ldots$ be given by the equations:

$$A = \begin{vmatrix} \beta', & \gamma', \ldots \\ \beta'', & \gamma'', \\ \vdots & \end{vmatrix}, \quad B = \pm \begin{vmatrix} \gamma', & \delta', \ldots \\ \gamma'', & \delta'', \\ \vdots & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

$$A' = \pm \begin{vmatrix} \beta'', & \gamma'', \ldots \\ \beta''', & \gamma''', \\ \vdots & \end{vmatrix}, \quad B' = \begin{vmatrix} \gamma'', & \delta'', \ldots \\ \gamma''', & \delta''', \\ \vdots & \end{vmatrix}$$

(The upper or lower signs according as $n$ is odd or even.)

These quantities satisfy the double series of equations,

$$\begin{aligned} &A\alpha + B\beta + \ldots = \kappa, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).\\ &A\alpha' + B\beta' + \ldots = 0,\\ &\vdots\\ &A'\alpha + B'\beta + \ldots = 0,\\ &A'\alpha' + B'\beta' + \ldots = \kappa,\\ &\vdots\\ &\&c. \end{aligned}$$

$$\begin{aligned} &A\alpha + A'\alpha' + \ldots = \kappa, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),\\ &A\beta + A'\beta' + \ldots = 0,\\ &\vdots\\ &B\alpha + B'\alpha' + \ldots = 0,\\ &B\beta + B'\beta' + \ldots = \kappa,\\ &\vdots\\ &\&c. \end{aligned}$$

the second side of each equation being 0, except for the $r^{\text{th}}$ equation of the $r^{\text{th}}$ set of equations in the systems.

Let $\lambda, \mu, \ldots$ represent the $r^{\text{th}}$, $\overline{r+1}^{\text{th}}$, ... terms of the series $\alpha, \beta, \ldots$; $L, M, \ldots$ the corresponding terms of the series $A, B \ldots$, where $r$ is any number less than $n$, and consider the determinant

$$\begin{vmatrix} A & , \ldots L \\ \vdots & \\ A^{(r-1)}, & \ldots L^{(r-1)} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

which may be expressed as a determinant of the $n^{\text{th}}$ order, in the form

$$\begin{vmatrix} A & , \ldots L & , & 0, & 0, \ldots \\ \vdots & & & & \\ A^{(r-1)}, & \ldots L^{(r-1)}, & & 0, & 0, \\ 0 & , \quad 0 & , & 1, & 0, \\ 0 & , \quad 0 & , & 0, & 1, \\ \vdots & & & & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Multiplying this by the two sides of the equation

$$\kappa = \begin{vmatrix} \alpha, & \beta, & \dots \\ \alpha', & \beta', & \\ \vdots & & \end{vmatrix} \qquad \dots\dots\dots (10),$$

and reducing the result by the equation ($\odot$), and the equations (6), the second side becomes

$$\begin{vmatrix} \kappa, & 0, & \dots & & & \\ 0, & \kappa, & & & & \\ \vdots & & \ddots & & & \\ & & & \kappa, & 0\ , & 0\ , \dots \\ & & & 0, & \mu^{(r)}, & \nu^{(r)}, \\ & & & 0, & \mu^{(r+1)}, & \nu^{(r+1)}, \\ & & & \vdots & & \end{vmatrix} \qquad \dots\dots\dots (11),$$

which is equivalent to

$$\kappa^r \begin{vmatrix} \mu^{(r)}, & \nu^{(r)}, & \dots \\ \mu^{(r+1)}, & \nu^{(r+1)}, & \\ \vdots & & \end{vmatrix} \qquad \dots\dots\dots (12),$$

or we have the equation

$$\begin{vmatrix} A & , \dots\ L \\ \vdots & \\ A^{(r-1)}, & \dots\ L^{(r-1)} \end{vmatrix} = \kappa^{r-1} \begin{vmatrix} \mu^{(r)}, & \nu^{(r)}, & \dots \\ \mu^{(r+1)}, & \nu^{(r+1)}, & \\ \vdots & & \end{vmatrix} \qquad \dots\dots\dots (13),$$

which in the particular case of $r = n$, becomes

$$\begin{vmatrix} A, & B, & \dots \\ A', & B', & \\ \vdots & & \end{vmatrix} = \kappa^{r-1} \qquad \dots\dots\dots (14),$$

which latter equation is given by M. Cauchy in the memoirs already quoted; the proof in the "*Exercises,*" being nearly the same with the above one of the more general equation (13). The equation (13) itself has been demonstrated by Jacobi somewhat less directly. Consider now the function $FU$, given by the equation (3). This may be expanded in the form

$$\begin{aligned} FU = & (\mathrm{R}\,\xi + \mathrm{S}\,\eta + \dots)\,[A\ \ (\mathrm{A}x + \mathrm{A}'x' + \dots) + B\ (\mathrm{B}x + \mathrm{B}'x' + \dots) + \dots] + \\ & (\mathrm{R}'\xi + \mathrm{S}'\eta + \dots)\,[A'\ (\mathrm{A}x + \mathrm{A}'x' + \dots) + B'\ (\mathrm{B}x + \mathrm{B}'x' + \dots) + \dots] + \\ & \vdots \end{aligned} \qquad \dots\dots (15),$$

which may be written

$$\begin{aligned} FU = & x\ (\mathrm{A}\,\xi + \mathrm{B}\,\eta + \dots) + \\ & x'\ (\mathrm{A}'\xi + \mathrm{B}'\eta + \dots) + \\ & \vdots \end{aligned} \qquad \dots\dots\dots (16).$$

by putting
$$\begin{aligned}
\mathrm{A} &= \text{a}\,(\text{r}A + \text{r}'A' + \ldots) + \text{b}\,(\text{r}B + \text{r}'B' + \ldots) + \ldots, \quad \ldots\ldots\ldots (17).\\
\mathrm{B} &= \text{a}\,(\text{s}A + \text{s}'A' + \ldots) + \text{b}\,(\text{s}B + \text{s}'B' + \ldots) + \ldots,\\
&\vdots\\
\mathrm{A}' &= \text{a}'(\text{r}A + \text{r}'A' + \ldots) + \text{b}'(\text{r}B + \text{r}'B' + \ldots) + \ldots,\\
\mathrm{B}' &= \text{a}'(\text{s}A + \text{s}'A' + \ldots) + \text{b}'(\text{s}B + \text{s}'B' + \ldots) + \ldots,\\
&\vdots
\end{aligned}$$

Hence,
$$KFU = \begin{vmatrix} \mathrm{A}, & \mathrm{B}, \ldots \\ \mathrm{A}', & \mathrm{B}', \\ \vdots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

$$= \begin{vmatrix} \text{a}, & \text{b}, \ldots \\ \text{a}', & \text{b}', \\ \vdots & \end{vmatrix} \begin{vmatrix} \text{r}, & \text{s}, \ldots \\ \text{r}', & \text{s}', \\ \vdots & \end{vmatrix} \begin{vmatrix} A, & B, \ldots \\ A', & B', \\ \vdots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (19),$$

or observing the equation (14), and writing
$$\begin{vmatrix} \text{a}, & \text{b}, \ldots \\ \text{a}', & \text{b}', \\ \vdots & \end{vmatrix} = J \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

$$\begin{vmatrix} \text{r}, & \text{s}, \ldots \\ \text{r}', & \text{s}', \\ \vdots & \end{vmatrix} = \text{ſ} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

this becomes
$$KFU = J\text{ſ}\,.\,(KU)^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Hence likewise
$$K\text{ꟻ}U = J\text{ſ}\,.\,(KU)^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Consider next the equation
$$\text{ꟻ}FU = -\begin{vmatrix} & \text{r}x + \text{r}'x' + \ldots, & \text{s}x + \text{s}'x' + \ldots, \ldots \\ \text{a}\xi + \text{b}\eta + \ldots, & \mathrm{A} & , \quad \mathrm{B} \\ \text{a}'\xi + \text{b}'\eta + \ldots, & \mathrm{A}' & , \quad \mathrm{B}' \quad , \\ \vdots & & \end{vmatrix} \ldots\ldots\ldots (24).$$

$$= -\begin{vmatrix} 1, & & \\ & \text{a}, & \text{b}, \ldots \\ & \text{a}', & \text{b}', \\ & & \end{vmatrix} \begin{vmatrix} 1, & & \\ & \text{r}, & \text{s}, \ldots \\ & \text{r}', & \text{s}', \\ & & \end{vmatrix} \begin{vmatrix} & \xi, & \eta, \ldots \\ x, & A, & B, \\ x', & A', & B', \\ \vdots & & \end{vmatrix} \ldots\ldots\ldots\ldots (25).$$

$$= -J\text{ſ} \begin{vmatrix} & \xi, & \eta, \ldots \\ x, & A, & B, \\ x', & A', & B', \\ \vdots & & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

If the two sides of this equation are multiplied by the two sides of the equation (2), written under the form

$$\kappa = \begin{vmatrix} 1, & & \\ & \alpha, & \beta, \ldots \\ & \alpha', & \beta', \\ & \vdots & \end{vmatrix} \text{ ..................................... (27),}$$

the second side is reduced to

$$-J\Gamma \begin{vmatrix} & \alpha\xi + \beta\eta \ldots, & \alpha'\xi + \beta'\eta \ldots, \ldots \\ x, & \kappa \quad , & . \quad , \\ x', & . \quad , & \kappa \quad , \\ \vdots & & \end{vmatrix} \text{ ..................... (28),}$$

$$= - J\Gamma \,.\, \kappa^{n-1} \,.\, U \text{ ..................................... (29),}$$

and hence

$$\daleth F U = J\Gamma \,.\, (KU)^{n-2} \,.\, U \text{ .............................. (30).}$$

Similarly

$$F \daleth U = J\Gamma \,.\, (KU)^{n-2} \,.\, U \text{ ..............................(31);}$$

also combining these with the equations (22), (23),

$$\frac{\daleth F U}{K F U} = \frac{F \daleth U}{K \daleth U} = \frac{U}{KU} \text{ ................................ (32).}$$

It may be remarked here that if $U$, $V$ are functions connected by the equation

$$FU = cFV, \text{ or } \daleth U = c \daleth V, \text{ ..................... (33),}$$

then in general

$$U = c^{\frac{1}{n-1}} V \text{ ........................................ (34).}$$

To prove this, observing that the first of the equations (33) may be written

$$FU = F(c^{\frac{1}{n-1}} V) \text{ ................................ (35),}$$

we have

$$\daleth \,.\, FU = \daleth \,.\, F(c^{\frac{1}{n-1}} V) \text{ ............................ (36),}$$

or

$$J\Gamma \,.\, (KU)^{n-2}\, U = J\Gamma\, [K(c^{\frac{1}{n-1}} V)]^{n-2}\, c^{\frac{1}{n-1}}\, V \text{.................... (37).}$$

If neither $J$, $\Gamma$ nor $(KU)$ vanish, this equation is of the form

$$U = kV \text{ ............................................ (38),}$$

whence substituting in (33),

$$k^{n-1} = c \text{ ............................................ (39),}$$

which demonstrates the equation (34); and this equation might be proved in like manner from the second of the equations (33). If however, $J=0$, or $\Gamma=0$, the above proof fails, and if $KU=0$, the proof also fails, unless at the same line $n=2$. In all these cases probably, certainly in the case of $KU=0$, $n \neq 2$, the equation (34) is not a necessary consequence of (33). In fact $FU$, or $\daleth U$ may be given, and yet $U$ remain indeterminate.

Let $U_,$, $\alpha_,$, $\beta_, \ldots A_,$, $B_,$, &c.… be analogous to $U$, $\alpha$, $\beta \ldots$, $A_,$, $B$, &c.… and consider the equation

$$K(KU_, . FU + gKU . FU_,) \qquad (40),$$

$$= \begin{vmatrix} \kappa_, A + g\kappa A_, , & \kappa_, B + g\kappa B_, , \ldots \\ \kappa_, A' + g\kappa A_,' , & \kappa_, B + g\kappa B_,' , \\ \vdots & \end{vmatrix}$$

Multiply the two sides by the two sides of the equation (2), the second side becomes, after reduction,

$$\begin{vmatrix} \kappa_,\kappa + g\kappa\,(A_,\alpha + B_,\beta + \ldots), & g\kappa\,(A_,'\alpha + B_,'\beta + \ldots), \ldots \\ g\kappa\,(A_,\alpha' + B_,\beta' + \ldots), & \kappa_,\kappa + g\kappa\,(A_,'\alpha' + B_,'\beta' + \ldots), \\ \vdots & \end{vmatrix} \qquad (41).$$

Multiplying by the two sides of the analogous equation

$$\kappa_, = \begin{vmatrix} \alpha_, , & \alpha_,' , \ldots \\ \beta_, , & \beta_, , \\ \vdots & \end{vmatrix} \qquad (42),$$

and reducing, the second side becomes

$$\begin{vmatrix} \kappa\kappa_,\,(\alpha_, + g\alpha), & \kappa\kappa_,\,(\beta_, + g\beta), \ldots \\ \kappa\kappa_,\,(\alpha_,' + g\alpha'), & \kappa\kappa_,\,(\beta_,' + g\beta'), \\ \vdots & \end{vmatrix} \qquad (43),$$

$$= \kappa^n . \kappa_,^n . K(U_, + gU) \qquad (44),$$

whence

$$K(KU_, . FU + gKU . FU_,) = (KU)^{n-1}(KU_,)^{n-1} K(U_, + gU) \qquad (45),$$

and similarly

$$K(KU_, . \daleth U + gKU . \daleth U_,) = (KU)^{n-1}(KU_,)^{n-1} K(U_, + gU) \qquad (46).$$

In a similar manner is the following equation to be demonstrated,

$$\daleth(KU_, . FU + gKU . FU_,) = F(KU_, . \daleth U + gKU . \daleth U_,) = \qquad (47),$$

$$-J\Gamma . (KU)^{n-2}(KU_,)^{n-2} \times \begin{vmatrix} & \alpha_, x + \alpha_,' x' , & \beta_, x + \beta_,' x' \ldots, \ldots \\ \alpha\xi + \beta\eta \ldots, & \alpha_, + g\alpha \ldots, & \beta_, + g\beta \ldots, \\ \alpha'\xi + \beta'\eta \ldots, & \alpha_,' + g\alpha' \ldots, & \beta_,' + g\beta_,' \ldots, \\ \vdots & & \end{vmatrix}$$

Suppose $$\overline{U} = \Sigma\,(\rho\xi + \sigma\eta + \ldots)\,(ax + a'x' + \ldots) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48),$$
this expression being the abbreviation of
$$\begin{aligned}\overline{U} = &(\rho\,\xi + \sigma\,\eta + \ldots)\,(a\,x + a'x' + \ldots) + \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49),\\ &(\rho_{,}\xi + \sigma_{,}\eta + \ldots)\,(a_{,}x + a_{,}'x' + \ldots) + \\ &+\\ &\vdots\end{aligned}$$

[$(n-1)$ lines, or a smaller number].

then $$K\overline{U} = \left|\begin{array}{ll} \Sigma a\,\rho, & \Sigma a\,\sigma,\ldots \\ \Sigma a'\rho, & \Sigma a'\sigma, \\ \vdots & \end{array}\right| \text{ is } = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

which follows from the equation (⊙).

Conversely, whenever $K\overline{U} = 0$, $\overline{U}$ is of the above form.

Also $$F\overline{U} = -\left|\begin{array}{lll} & \mathrm{A}x + \mathrm{A}'x' + \ldots, & \mathrm{B}x + \mathrm{B}'x' + \ldots, \ldots \\ \mathrm{R}\,\xi + \mathrm{S}\,\eta + \ldots, & \Sigma a\,\rho \quad, & \Sigma a\,\sigma \quad, \\ \mathrm{R}'\xi + \mathrm{S}'\eta + \ldots, & \Sigma a'\rho \quad, & \Sigma a'\sigma \quad, \\ \vdots & & \end{array}\right| \ldots\ldots\ldots\ldots (51),$$

which may be transformed into

$$F\overline{U} = \left|\begin{array}{ll} \mathrm{A}x + \mathrm{A}'x' \ldots, & \mathrm{B}x + \mathrm{B}'x' \ldots, \ldots \\ \rho \quad, & \sigma \quad, \\ \vdots & \end{array}\right| \left|\begin{array}{ll} \mathrm{R}\xi + \mathrm{S}\eta \ldots, & \mathrm{R}'\xi + \mathrm{S}'\eta \ldots, \ldots \\ a \quad, & a' \quad, \\ \vdots & \end{array}\right| \ldots (52),$$

(for shortness, I omit the demonstration of this equation).

And similarly,

$$\daleth\overline{U} = \left|\begin{array}{ll} \mathrm{R}x + \mathrm{R}'x' + \ldots, & \mathrm{S}x + \mathrm{S}'x' + \ldots, \ldots \\ \rho \quad, & \sigma \quad, \\ \vdots & \end{array}\right| \left|\begin{array}{ll} \mathrm{A}\xi + \mathrm{B}\eta + \ldots, & \mathrm{A}'\xi + \mathrm{B}'\eta + \ldots, \ldots \\ a \quad, & a' \quad, \\ \vdots & \end{array}\right| \ldots (53),$$

where it is obvious that if the sum $\Sigma$ contain fewer than $(n-1)$ terms, $FU = 0$, $\daleth U = 0$.

The equations (52), (53) express the theorem, that whenever $K\overline{U} = 0$, the functions $F\overline{U}$, $\daleth\overline{U}$ are each of them the product of two determinants.

If next $$U_{,} = U + \overline{U},$$

then in (45) taking $g=-1$ [the Numbers (56) &c. which follow are as in the original memoir]

$$K\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}=K\{K(U+\overline{U}).\mathcal{F}U-KU.\mathcal{F}(U+\overline{U})\}\ldots\ldots\ldots(56),$$

$$=(KU)^{n-1}.(K(U+\overline{U}))^{n-1}.K\overline{U},$$

or observing the equation (50),

$$K\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}=K\{K(U+\overline{U}).\mathcal{F}U-KU.\mathcal{F}(U+\overline{U})\}=0\ \ldots\ldots(57).$$

Hence $F\{(K(U+\overline{U}).\mathcal{F}U-KU.\mathcal{F}(U+\overline{U})\}=\mathcal{F}\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}$ are each of them the product of two determinants. But this result admits of a further reduction: we have

$$F\{K(U+\overline{U}).\mathcal{F}U-KU.\mathcal{F}(U+\overline{U})\}=\mathcal{F}\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}\ldots\ldots\ldots(58)$$

$$=-J\Gamma(KU)^{n-2}.(K(U+\overline{U}))^{n-2}\left|\begin{array}{llll} & \alpha_{,}x+\alpha_{,}'x'+\ldots, & \beta_{,}x+\beta_{,}'x'+\ldots, & \ldots \\ \alpha\xi+\beta\eta+\ldots, & \alpha_{,}\ -\alpha, & \beta_{,}\ -\beta, & \\ \alpha'\xi+\beta'\eta+\ldots, & \alpha_{,}'\ -\alpha', & \beta_{,}'\ +\beta', & \\ \vdots & & & \end{array}\right|,$$

substituting $\alpha_{,}=\alpha+\Sigma\rho a$, &c...., also observing that if the second line be multiplied by $x$, the third by $x'$,... and the sum subtracted from the first line, the value of the determinant is not altered, and that the effect of this is simply to change $\alpha_{,}, \alpha_{,}'\ldots$ into $\alpha, \alpha'\ldots$ in the first line, and introduce into the corner place a quantity $-U$, which in the expansion of the determinant is multiplied by zero: this may be written in the form

$$-J\Gamma(KU)^{n-2}(K(U+\overline{U}))^{n-2}\left|\begin{array}{llll} & \alpha x+\alpha'x'+\ldots, & \beta x+\beta'x'+\ldots, & \ldots \\ \alpha\xi+\beta\eta+\ldots, & \Sigma\rho a, & \Sigma\sigma a, & \\ \alpha'\xi+\beta'\eta+\ldots, & \Sigma\rho a', & \Sigma\sigma a', & \\ \vdots & & & \end{array}\right|\ldots(59),$$

which may be reduced to

$$J\Gamma.(KU)^{n-2}.(K(U+\overline{U}))^{n-2}\times\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60),$$

$$\left|\begin{array}{lll} \alpha x+\alpha'x'+\ldots, & \beta x+\beta'x'+\ldots, & \ldots \\ \rho, & \sigma, & \\ \vdots & & \end{array}\right|\ \left|\begin{array}{lll} \alpha\xi+\beta\eta+\ldots, & \alpha'\xi+\beta'\eta\ldots, & \ldots \\ a, & a', & \\ \vdots & & \end{array}\right|$$

If each of these determinants are multiplied by the quantity $(KU)^{n-1}$, expressed under the two forms

$$\left|\begin{array}{ll} A, & B,\ldots \\ A', & B', \\ \vdots & \end{array}\right|,\ \left|\begin{array}{ll} A, & A',\ldots \\ B, & B', \\ \vdots & \end{array}\right|\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

they would become respectively

$$KU\,.\begin{vmatrix} x & , & x' & ,\ldots \\ A\rho+B\sigma+\ldots, & & A'\rho+B'\sigma+\ldots, & \\ \vdots & & & \end{vmatrix},\ KU\,.\begin{vmatrix} \xi & , & \eta & ,\ldots \\ Aa+A'a'+\ldots, & & Ba+B'a'+\ldots, & \\ \vdots & & & \end{vmatrix}\ \ldots (62),$$

so that finally

$$F\{K(U+\bar{U})\,.\,\digamma U-KU\,.\,\digamma(U+\bar{U})\}=\digamma\{K(U+\bar{U})\,FU-KU\,.\,F(U+\bar{U})\}\ \ldots\ldots\ldots (63).$$

$$=J\Gamma\cdot\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2}\times\begin{vmatrix} x & , & x' & ,\ldots \\ A\rho+B\sigma+\ldots, & & A'\rho+B'\sigma+\ldots, & \\ \vdots & & & \end{vmatrix}\begin{vmatrix} \xi & , & \eta & ,\ldots \\ Aa+A'a'+\ldots, & & Ba+B'a'+\ldots, & \\ \vdots & & & \end{vmatrix}$$

The second side of this may be written under the forms

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2}\begin{vmatrix} \mathrm{A}x+\mathrm{A}'x'+\ldots & , & \mathrm{B}x+\mathrm{B}'x'+\ldots & ,\ldots \\ \mathrm{A}(A\rho+B\sigma\,..)+\mathrm{A}'(A'\rho+B'\sigma\,..)+\ldots, & & \mathrm{B}(A\rho+B\sigma\,..)+\mathrm{B}'(A'\rho+B'\sigma\,..)+\ldots, & \\ \vdots & & & \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} R\xi+S\eta+\ldots & , & R'\xi+S'\eta+\ldots & ,\ldots \\ R(Aa+A'a'\,..)+S(Ba+B'a'\,..)+\ldots, & & R'(Aa+A'a'\,..)+S'(Ba+B'a'\,..)+\ldots, & \\ \vdots & & & \end{vmatrix}\ \ldots (64).$$

And

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2}\begin{vmatrix} Rx+R'x'+\ldots & , & Sx+S'x'+\ldots & ,\ldots \\ R(A\rho+B\sigma\,..)+R'(A'\rho+R'\sigma\,..)+\ldots, & & S(A\rho+B\sigma\,..)+S'(A'\rho+B'\sigma\,..)+\ldots, & \\ \vdots & & & \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} A\xi+B\eta+\ldots & , & A'\xi+B'\eta+\ldots & ,\ldots \\ A(Aa+A'a'\,..)+B(Ba+B'a'\,..)+\ldots, & & A'(Aa+A'a'\,..)+B'(Ba+B'a'\,..)+\ldots, & \\ \vdots & & & \end{vmatrix}\ \ldots (65).$$

And again, by the equations (52), (53), in the new forms

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2} F\,.\,\Sigma\{[(A\rho+B\sigma\ \ldots)(\mathrm{A}\xi+\mathrm{B}\eta\ \ldots)+(A'\rho+B'\sigma\ldots)(\mathrm{A}'\xi+\mathrm{B}'\eta\ldots)\ldots]$$
$$\times[(Aa+A'a'\ldots)(\mathrm{R}x+\mathrm{R}'x'\ldots)+(Ba+B'a'\ \ldots)(\mathrm{S}x+\mathrm{S}'x'\ldots)\ldots]\}\ldots\ldots (66),$$

$$\left(\frac{K(U+\bar{U})}{(KU)}\right)^{n-2}\digamma\,.\,\Sigma\,\{[(A\rho+B\sigma\ \ldots)(\mathrm{R}\xi+\mathrm{S}\eta\ \ldots)+(A'\rho'+B'\sigma\ldots)(\mathrm{R}'\xi+\mathrm{S}'\eta\ldots)\ldots]$$
$$\times[(Aa+A'a'\ldots)(\mathrm{A}x+\mathrm{A}'x'\ldots)+(Ba+B'a'\ \ldots)(\mathrm{B}x+\mathrm{B}'x'\ldots)\ldots]\}\ \ldots (67).$$

Comparing these latter forms with the two equivalent quantities forming the first side of (53), and observing (33), (34), it would appear at first sight that

$$K(U+\bar{U}) \,.\, \text{ꟻ}U - KU \,.\, \text{ꟻ}(U+\bar{U})$$

$$=\left(\frac{K(U+\bar{U})}{KU}\right)^{\frac{n-2}{n-1}} \{\Sigma\, [(A\rho + B\sigma\,\ldots)(\mathrm{A}\xi + \mathrm{B}\eta\,\ldots) + (A'\rho + B'\sigma\ldots)(\mathrm{A}'\xi + \mathrm{B}'\eta\,\ldots)\ldots]$$

$$\times\, [(Aa + A'a'\,\ldots)(\mathrm{R}x + \mathrm{R}'x'\,\ldots) + (Ba + B'a'\,\ldots)(\mathrm{S}x + \mathrm{S}'x'\,\ldots)\ldots]\},$$

$$K(U+\bar{U}) \,.\, FU - KU \,.\, F(U+\bar{U})$$

$$=\left(\frac{K(U+\bar{U})}{KU}\right)^{\frac{n-2}{n-1}} \Sigma\, \{[(A\rho + B\sigma\ldots)(\mathrm{R}\xi + \mathrm{S}\eta\,\ldots) + (A'\rho + B'\sigma\ldots)(\mathrm{R}'\xi + \mathrm{S}'\eta\ldots)\ldots]$$

$$\times\, [(Aa + A'a'\,\ldots)(\mathrm{A}x + \mathrm{A}'x'\,\ldots) + (Ba + B'a'\,\ldots)(\mathrm{B}x + \mathrm{B}'x'\,\ldots)\ldots]\},$$

which however are not true, except for $n=2$, on account of the equation (57). In the case of $n=2$, these equations become

$$K(U+\bar{U}) \,.\, \text{ꟻ}U - KU \,.\, \text{ꟻ}(U+\bar{U})$$

$$= [(A\rho + B\sigma\,\ldots)(\mathrm{A}\xi + \mathrm{B}\eta\,\ldots) + (A'\rho + B'\sigma\,\ldots)(\mathrm{A}'\xi + \mathrm{B}'\eta\,\ldots) + \ldots]$$

$$\times\, [(Aa + A'a'\,\ldots)(\mathrm{R}x + \mathrm{R}'x'\,\ldots) + (Ba + B'a'\,\ldots)(\mathrm{S}x + \mathrm{S}'x'\,\ldots)\ldots] \quad \ldots\ldots\ldots(68),$$

$$K(U+\bar{U})\, FU - KU \,.\, F(U+\bar{U})$$

$$= [(A\rho + B\sigma\,\ldots)(\mathrm{R}\xi + \mathrm{S}\eta\,\ldots) + (A'\rho + B'\sigma\,\ldots)(\mathrm{R}'\xi + \mathrm{S}'\eta\,\ldots)\ldots]$$

$$\times\, [(Aa + A'a'\,\ldots)(\mathrm{A}x + \mathrm{A}'x'\,\ldots) + (Ba + B'a'\,\ldots)(\mathrm{B}x + \mathrm{B}'x' + \ldots)\ldots] \quad \ldots\ldots(69),$$

and it is remarkable that these equations ((68), (69)) are true whatever be the value of $n$, provided $\Sigma$ contains a single term only. The demonstration of this theorem is somewhat tedious, but it may perhaps be as well to give it at full length. It is obvious that the equation (69) alone need be proved, (68) following immediately when this is done.

I premise by noticing the following general property of determinants. The function

$$\left|\begin{matrix} \alpha + \Sigma\rho a, & \beta + \Sigma\sigma a, \ldots \\ \alpha' + \Sigma\rho a', & \beta' + \Sigma\sigma' a, \\ \vdots & \end{matrix}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70),$$

(where $\Sigma\rho a = \rho_1 a_1 + \rho_2 a_2 \ldots + \rho_s a_s$), contains no term whose dimension in the quantities $a, a' \ldots$, or in the other quantities $\rho, \sigma \ldots$, is higher than $s$. (Of course if the order of the determinant be less than $s$ or equal to it, this number becomes the limit of the dimension of any term in $a, a' \ldots$ or $\rho, \sigma \ldots$, and the theorem is useless.) This is easily proved by means of a well-known theorem,

$$\left|\begin{matrix} \Sigma\rho\, a, & \Sigma\sigma\, a, \ldots \\ \Sigma\rho' a, & \Sigma\sigma' a, \\ \vdots & \end{matrix}\right| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(71),$$

whenever $s$ is less than the number expressing the order of the determinant. Hence in the formula (70), if $\Sigma$ contain a single term only, the first side of the equation is linear in $\rho, \sigma, \ldots$ and also in $\alpha, \alpha', \ldots$, i.e. it consists of a term independent of all these quantities, and a second term linear in the products $\rho\alpha, \rho\alpha', \ldots \sigma\alpha, \sigma\alpha', \ldots$ This is therefore the form of $K(U+\overline{U})$.

Consider the several equations

$$\begin{aligned}\kappa = KU &= A\alpha + B\beta + \ldots \qquad (72),\\ &= A'\alpha' + B'\beta' + \ldots\\ &= \&c.\end{aligned}$$

it is easy to deduce

$$\begin{aligned}\kappa_{,} = K(U+\overline{U}) = KU &+ A\rho\alpha + B\sigma\alpha + \qquad (73).\\ &+ A'\rho\alpha' + B'\sigma\alpha' +\\ &\vdots\end{aligned}$$

To find the values of $A$, $B$, &c. corresponding to $U+\overline{U}$, we must write

$$\begin{aligned}A &= \text{M}'\beta + \text{N}'\gamma' + \qquad (74),\\ &= \text{M}''\beta + \text{N}''\gamma'' +\\ &= \&c.\end{aligned}$$

where

$$\text{M}' = \begin{vmatrix}\gamma'', & \delta'', \ldots\\ \gamma''', & \delta''',\\ \vdots & \end{vmatrix}, \quad \text{N}' = \pm\begin{vmatrix}\delta'', & \epsilon'', \ldots\\ \delta''', & \epsilon''',\\ \vdots & \end{vmatrix} \qquad (75),$$

$$\text{M}'' = \pm\begin{vmatrix}\gamma''', & \delta''', \ldots\\ \gamma'''', & \delta'''',\\ \vdots & \end{vmatrix}, \quad \text{N}'' = \begin{vmatrix}\delta''', & \epsilon''', \ldots\\ \delta'''', & \epsilon'''',\\ \vdots & \end{vmatrix}, \ \&c.$$

the order of each of these determinants being $n-2$, and the upper or lower signs being used according as $n-1$ is odd or even, i.e. as $n$ is even or odd. Hence

$$\begin{aligned}A_{,} = A &+ \text{M}'\,\sigma\alpha' + \text{N}'\,\tau\alpha' + \ldots \qquad (76),\\ &+ \text{M}''\sigma\alpha'' + \text{N}''\tau\alpha'' + \ldots\\ &\vdots\end{aligned}$$

and therefore

$$\begin{aligned}\kappa_{,}A - \kappa A_{,} = \quad & A^2\rho\alpha + (AB \qquad)\,\sigma\alpha + (AC \qquad)\,\tau\alpha + \ldots \qquad (77),\\ &+ AA'\rho\alpha' + (AB' - \kappa\text{M}')\,\sigma\alpha' + (AC' - \kappa\text{N}')\,\tau\alpha' + \ldots\\ &+ AA''\rho\alpha'' + (AB'' - \kappa\text{M}'')\,\sigma\alpha'' + (AC'' - \kappa\text{N}'')\,\tau\alpha'' +\\ &\vdots\end{aligned}$$

the additional quantities $C$, $\tau$ having been introduced for greater clearness. Now the equations

$$\begin{aligned}&AB' - \kappa M' = A'B, \quad AC' - \kappa N' = A'C, \ldots \qquad (78),\\ &AB'' - \kappa M'' = A''B, \quad AC'' - \kappa N'' = A''C,\\ &\vdots\end{aligned}$$

written under the form

$$AB' - A'B = \kappa M', \quad AC' - A'C = \kappa N', \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(79),$$
$$AB'' - A''B = \kappa M'', \quad AC'' - A''C = \kappa N'',$$
$$\vdots$$

are particular cases of the equation (13), and are therefore identically true. Hence, substituting in (77),

$$\begin{aligned} \kappa_{,}A - \kappa A_{,} = \quad & A^2\rho a + A\ B\sigma a + A\ C\tau a\ \ldots + \quad \ldots\ldots\ldots\ldots\ldots\ldots (80),\\ & + A\ A'\rho a' + A'\ B\sigma a' + A'\ C\tau a' \ldots + \\ & + A''A\rho a'' + A''B\sigma a'' + A''C\tau a'' \ldots + \\ & \quad \vdots \\ & = (\rho A + \sigma B + \ldots)(Aa + A'a' + \ldots). \end{aligned}$$

Forming in a similar manner, the combinations $\kappa_{,}B - \kappa B_{,}, \ldots \kappa_{,}A' - \kappa A_{,}'$, $\kappa_{,}B - \kappa B_{,}', \ldots$, multiplying by the products of the different quantities $Ax + A'x' \ldots$, $Bx + B'x' \ldots$, ... $R\xi + S\eta \ldots$, $R'\xi + S'\eta \ldots$, ... and adding so as to form the function $K(U + \overline{U}) . FU - KU . F(U + \overline{U})$, we obtain the required formula, viz. that the value of this quantity is

$$\begin{aligned} &= [(\rho A + \sigma B\ \ldots)(\mathrm{R}\xi + \mathrm{S}\eta\ \ldots) + (A'\rho + B'\sigma \ldots)(\mathrm{R}'\xi + \mathrm{S}'\eta \ldots) + \ldots] \ldots\ldots (81);\\ &\times [(Aa + A'a' \ldots)(\mathrm{A}x + \mathrm{A}'x' \ldots) + (Ba + B'a' \ldots)(\mathrm{B}x + \mathrm{B}'x' \ldots) + \ldots] \end{aligned}$$

with this theorem, I conclude the present section,—noticing only, as a problem worthy of investigation, the discovery of the forms of the second sides of the equations (68), (69), in the case of $\Sigma$ containing more than a single term.

§ 2. On the notation and properties of certain functions resolvable into a series of determinants.

Let the letters $\quad r_1, \ r_2, \ldots r_k \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$

represent a permutation of the numbers

$$1, \ 2, \ldots k \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then in the series (1), if one of the letters succeeds mediately or immediately a letter representing a higher number than its own, for each time that this happens there is said to be a "derangement" or "inversion." It is to be remarked that if any letter succeed $s$ letters representing higher numbers, this is reckoned for the same number $s$ of inversions.

Suppose next the symbol

$$\pm_r \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

denotes the sign + or −, according as the number of inversions in the series (1) is even or odd.

This being premised, consider the symbol

$$\left\{\begin{matrix} A\rho_1\sigma_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \dots \end{matrix}\right\} \dots\dots\dots\dots \quad (4),$$

denoting the sum of all the different terms of the form

$$\pm_r \pm_s \dots A\rho_{r_1}\sigma_{s_1} \dots \quad \dots A\rho_{r_k}\sigma_{s_k} \dots \quad \dots\dots\dots\dots \quad (5),$$

the letters

$$r_1, \quad r_2 \dots r_k; \quad s_1, \quad s_2 \dots s_k; \text{ \&c.} \quad \dots\dots\dots\dots \quad (6),$$

denoting any permutations whatever, the same or different, of the series of numbers (2) [and the several combinations of $\rho\sigma \dots$ being understood as denoting suffixes of the $A$'s]. The number of terms represented by the symbol (5) is evidently

$$(1 . 2 \dots k)^n \dots\dots\dots\dots \quad (7).$$

In some cases it will be necessary to leave a certain number of the vertical rows $\rho$, $\sigma \dots$ unpermuted. This will be represented by writing the mark (†) immediately above the rows in question. So that for instance

$$\left\{\begin{matrix} A\rho_1\sigma_1 \dots \overset{\dagger}{\theta}_1\overset{\dagger}{\phi}_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \dots \theta_k\phi_k \end{matrix}\right\} \dots\dots\dots\dots \quad (8),$$

the number of rows with the (†) being $x$, denotes the sum of the

$$(1 . 2 \dots k)^{n-x} \dots\dots\dots\dots \quad (9)$$

terms, of the form

$$\pm_r \pm_s \dots A\rho_{r_1}\sigma_{s_1} \dots \theta_1\phi_1 \dots A\rho_{r_k}\sigma_{s_k} \dots \theta_k\phi_k \dots \quad \dots\dots\dots \quad (10).$$

Then it is obvious, that if all the rows have the mark (†) the notation (8) denotes a single product only, and if the mark (†) be placed over all but one of the rows the notation (8) belongs to a determinant. It is obvious also that we may write

$$\left\{\begin{matrix} A\rho_1\sigma_1 \dots \theta_1\phi_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \dots \theta_k\phi_k \end{matrix}\right\} = \Sigma \pm_u \pm_v \dots \left\{\begin{matrix} A\rho_1\sigma_1 \dots \overset{\dagger}{\theta}_{u_1}\overset{\dagger}{\phi}_{v_1} \dots (n) \\ \vdots \\ \rho_k\sigma_k \dots \theta_{u_k}\phi_{v_k} \end{matrix}\right\} \dots\dots \quad (11),$$

where $\Sigma$ refers to the different permutations,

$$u_1, \quad u_2, \dots u_k; \quad v_1, \quad v_2, \dots v_k; \text{ \&c.} \dots\dots\dots\dots \quad (12),$$

which can be formed out of the numbers (2). The equation (11) would still be true if the mark (†) were placed over any number of the columns $\rho$, $\sigma \dots$

Suppose in this equation a single column only is left without the mark (†) on the second side of the equation; the first side is then expressed as the sum of a number

$$(1 . 2 \dots k)^{n-1}, \text{ or generally } (1 . 2 \dots k)^{n-x-1} \quad \dots\dots\dots\dots \quad (13),$$

of determinants, according as we consider the symbol (4) or the more general one (8). And this may be done in $n$ or $(n-x)$ different ways respectively.

It may be remarked, that the symbol (8) is the same in form as if a single column only had the mark (†) over it; the number $n$ being at the same time reduced from $n$ to $(n-x+1)$: for the marked columns of symbols may be replaced by a single marked column of new symbols. Hence, without loss of generality, the theorems which follow may be stated with reference to a single marked column only.

Suppose the letters

$$\rho_1, \quad \rho_2, \dots \rho_k; \quad \sigma_1, \quad \sigma_2, \dots \sigma_k; \text{ \&c.} \qquad (14)$$

denote certain permutations of

$$\alpha_1, \quad \alpha_2, \dots \alpha_k; \quad \beta_1, \quad \beta_2, \dots \beta_k; \text{ \&c.} \qquad (15),$$

in such a manner that

$$\rho_1 = \alpha_{g_1}, \quad \rho_2 = \alpha_{g_2}, \dots \rho_k = \alpha_{g_k}; \quad \sigma_1 = \beta_{h_1}, \quad \sigma_2 = \beta_{h_2}, \dots \sigma_k = \beta_{h_k} \dots \qquad (16).$$

Then the two following theorems may be proved:

$$\left\{\begin{matrix} \overset{\dagger}{A}\rho_1\sigma_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \end{matrix}\right\} = \pm_g \pm_h \dots \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \qquad (17),$$

if $n$ be even: but in the contrary case

$$\left\{\begin{matrix} \overset{\dagger}{A}\rho_1\sigma_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \end{matrix}\right\} = + \pm_g \dots \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \qquad (18).$$

By means of these, and the equation (11), a fundamental property of the symbol (3) may be demonstrated. We have

$$\left\{\begin{matrix} A\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} = \Sigma \pm_g \left\{\begin{matrix} \overset{\dagger}{A}\rho_1\beta_1 \dots (n) \\ \vdots \\ \rho_k\beta_k \end{matrix}\right\} \qquad (19),$$

which when $n$ is even, reduces itself by (17) to

$$\left\{\begin{matrix} A\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} = \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \Sigma (\pm_g \pm_g . 1) \qquad (20)$$

$$= 1 . 2 \dots k \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\}.$$

But when $n$ is odd, from the equation (18),

$$\left\{\begin{matrix} A\alpha_1, & \beta_1 \dots (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} = \left\{\begin{matrix} \overset{\dagger}{A\alpha_1}\beta_1 \dots (n) \\ \vdots \\ \alpha_k, \beta_k \end{matrix}\right\} \Sigma (\pm_g 1) = 0 \quad \dots\dots\dots(21),$$

since the number of negative and positive values of $\pm_g$ are equal.

From the equation (20), it follows that when $n$ is even, the value of a symbol of the form

$$\left\{\begin{matrix} A\overset{\dagger}{\alpha_1}\beta_1, & (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} \quad \dots\dots\dots (22)$$

is the same, over whichever of the columns $\alpha$, $\beta\dots$ the mark (†) is placed. To denote this indifference, the preceding quantity is better represented by

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1, & \beta_1 \dots (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} \quad \dots\dots\dots (23),$$

this last form being never employed when $n$ is odd, in which case the same property does not hold. Hence also an ordinary determinant is represented by

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\}, \quad \left\{\begin{matrix} \overset{\dagger}{A}\,1\,1 \\ \vdots\ \vdots \\ k\,k \end{matrix}\right\} \quad \dots\dots\dots (24),$$

the latter form being obviously equally general with the former one.

It is obvious from the equations (17), (18), that the expression (22) vanishes, in the case of $n$ even whenever any two of the symbols $\alpha$ are equivalent, or any two of the symbols $\beta$, &c.; but if $n$ be odd, this property holds for the symbols $\beta$, &c., but not for the marked ones $\alpha$. In fact, the interchange of the two equal symbols, in each case, changes the sign of the expression (22), but they evidently leave it unaltered, i.e. the quantity in question must be zero.

Consider now the symbol

$$\left\{\begin{matrix} \overset{\dagger}{A}\,1\,1 \dots (2p) \\ \vdots \\ k\,k \end{matrix}\right\} \quad \dots\dots\dots (25),$$

which, for shortness, may be denoted by

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\} \quad \dots\dots\dots (26).$$

I proceed to prove a theorem, which may be expressed as follows:

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\}\,.\,\{\overset{\dagger}{B}\,.\,k\,.\,2q\} = \{\overline{AB}|\,k\,.\,2p+2q-2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27),$$

where

$$\overline{AB}|_{rs\ldots xy\ldots} = S\,.\,A_{rs\ldots l}\,B_{xy\ldots l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

the number of the symbols $r, s, \ldots$ being obviously $2p-1$, and that of $x, y, \ldots$ being $2q-1$. The summatory sign $S$ refers to $l$, and denotes the sum of the several terms corresponding to values of $l$ from $l=1$ to $l=k$. Also the theorem would be equally true if $l$ had been placed in any position whatever in the series $r, s \ldots l$; and again, in any position whatever in the series $x, y \ldots l$, instead of at the end of each of these. With a very slight modification this may be made to suit the case of an odd number instead of one of the two even numbers $2p$, $2q$; (in fact, it is only necessary to place the mark (†) in $\{\overline{AB}|\ldots\}$ over the column corresponding to the marked column in $\{A\ldots\}$, $\{A\ldots\}$ being the symbol for which the number of columns is odd), but it is inapplicable where the two numbers are odd. Consider the second side of (27); this may be expanded in the form

$$\Sigma + \pm_s \ldots \pm_x \pm_y \ldots \overline{AB}|_{1s_1\ldots x_1y_1\ldots}\,.\,\overline{AB}|_{2s_2\ldots x_2y_2\ldots} \ldots \overline{AB}|_{ks_k\ldots x_ky_k\ldots} \quad \ldots\ldots (29),$$

where $\Sigma$ refers to the different quantities $s, \ldots, x, y, \ldots$ as in (11).

Substituting from (28), this becomes

$$\Sigma\,.\,S_{l_1}\ldots S_{l_k}\,(+\pm_s\ldots\pm_x\pm_y A_{1s_1\ldots l_1}\ldots A_{ks_k\ldots l_k}\ldots B_{x_1y_1\ldots l_1}\ldots B_{x_ky_k\ldots l_k})\ldots (30).$$

Effecting the summation with respect to $x, y \ldots$ this becomes

$$\Sigma\,.\,S_{l_1}\ldots S_{l_k} + \pm_s \ldots A_{1s_1\ldots l_1}\ldots A_{ks_k\ldots l_k}\left\{\begin{matrix}\overset{\dagger}{B}1\,1\ldots l_1\\ \vdots\\ kk\ldots l_k\end{matrix}\right\} \ldots\ldots\ldots (31),$$

$\Sigma$ now referring to $s, \ldots$ only. The quantity under the sign $\Sigma$ vanishes if any two of the quantities $l$ are equal, and in the contrary case, we have

$$\left\{\begin{matrix}\overset{\dagger}{B}1\,1\ldots l_1\\ \vdots\\ kk\ldots l_k\end{matrix}\right\} = \pm_l\{\overset{\dagger}{B}\,.\,k\,.\,2q\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

which reduces the above to

$$\{\overset{\dagger}{B}\,.\,k\,.\,2q\}\,.\,\Sigma + \pm_s \ldots \pm_l A_{1.s_1\ldots l_1}\ldots A_{k.s_k\ldots l_k} \quad \ldots\ldots\ldots\ldots\ldots (33),$$

$\Sigma$ referring to the quantities $s\ldots$, and also to the quantities $l$. And this is evidently equivalent to

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\}\,\{\overset{\dagger}{B}\,.\,k\,.\,2q\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (34),$$

the theorem to be proved. It is obvious that when $p=1$, $q=1$, the equation (27), coincides with the theorem (⊙), quoted in the introduction to this paper.