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ON CERTAIN RESULTS RELATING TO QUATERNIONS.

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In his last paper on Quaternions [*Phil. Mag.* vol. XXV. (1844), p. 491] Sir William R. Hamilton has alluded to a paper of mine on the Analytical Geometry of (n) Dimensions, in the *Cambridge Mathematical Journal* [11], as one that might refer to the same subject. It may perhaps be as well to notice that the investigations there contained have no reference whatever to Sir William Hamilton's very beautiful theory; a more correct title for them would have been, a Generalization of the Analysis which occurs in ordinary Analytical Geometry.

I take this opportunity of communicating one or two results relating to quaternions; the first of them does appear to me rather a curious one.

Observing that

$$(A+Bi+Cj+Dk)^{-1} = (A-Bi-Cj-Dk) \div (A^2+B^2+C^2+D^2) \quad \text{.........(1)}$$

it is easy to form the equation

$$\left.\begin{aligned}
&Pi+Cj+Dk)^{-1}\,(\alpha+\beta i+\gamma j+\delta k)\,(A+Bi+Cj+Dk)\\
&=\frac{1}{A^2+B^2+C^2+D^2}\\
&\left\{\begin{array}{llll}
 & \alpha\,(A^2+B^2+C^2+D^2) & & \\
+i\,[& \beta\,(A^2+B^2-C^2-D^2) & +2\gamma\,(BC+AD) & +2\delta\,(BD-AC)\;]\\
+j\,[& 2\beta\,(BC-AD) & +\;\gamma\,(A^2-B^2+C^2-D^2) & +2\delta\,(CD+AB)\;]\\
+k\,[& 2\beta\,(BD+AC) & +2\gamma\,(CD-AB) & +\delta\,(A^2-B^2-C^2+D^2)]
\end{array}\right.
\end{aligned}\right\} \ldots (2)$$

which I have given with these letters for the sake of reference; it will be convenient to change the notation and write

$$\left.\begin{array}{l}(1+\lambda i+\mu j+\nu k)^{-1}(ix+jy+kz)(1+\lambda i+\mu j+\nu k)\\ =\dfrac{1}{1+\lambda^2+\mu^2+\nu^2}\\ \left\{\begin{array}{llll} i\,[\,x(1+\lambda^2-\mu^2-\nu^2) & +2y(\lambda\mu+\nu) & +2z(\lambda\nu-\mu) &]\\ +j\,[2x(\lambda\mu-\nu) & +\;y(1-\lambda^2+\mu^2-\nu^2) & +2z(\mu\nu+\lambda) &]\\ +k\,[2x(\lambda\nu+\mu) & +2y(\mu\nu-\lambda) & +\;z(1-\lambda^2-\mu^2+\nu^2) &]\end{array}\right.\end{array}\right\}\ldots (3)$$

$$=\left.\begin{array}{l} i(\alpha x+\alpha' y+\alpha'' z)\\ +j(\beta x+\beta' y+\beta'' z)\\ +k(\gamma x+\gamma' y+\gamma'' z)\end{array}\right\}\ldots\ldots (4)$$

suppose. The peculiarity of this formula is, that the coefficients α, β,... are precisely such that a system of formulæ

$$\left.\begin{array}{l} x_{,}=\alpha x+\alpha' y+\alpha'' z\\ y_{,}=\beta' x+\beta' y+\beta'' z\\ z_{,}=\gamma x+\gamma' y+\gamma'' z\end{array}\right\}\ldots\ldots (5)$$

denotes the transformation from one set of rectangular axes to another set, also rectangular. Nor is this all, the quantities λ, μ, ν may be geometrically interpreted. Suppose the axes Ax, Ay, Az could be made to coincide with the axes $Ax_{,}$, $Ay_{,}$, $Az_{,}$ by means of a revolution through an angle θ round an axis AP inclined to Ax, Ay, Az, at angles f, g, h then

$$\lambda=\tan\tfrac{1}{2}\theta\cos f,\qquad \mu=\tan\tfrac{1}{2}\theta\cos g,\qquad \nu=\tan\tfrac{1}{2}\theta\cos h.$$

In fact the formulæ are precisely those given for such a transformation by M. Olinde Rodrigues Liouville, t. v., "Des lois géometriques qui régissent les déplacemens d'un système solide" (or *Camb. Math. Journal*, t. iii. p. 224 [6]). It would be an interesting question to account, *à priori*, for the appearance of these coefficients here.

The ordinary definition of a determinant naturally leads to that of a quaternion determinant. We have, for instance,

$$\begin{vmatrix}\varpi, & \phi\\ \varpi', & \phi'\end{vmatrix}=\varpi\phi'-\varpi'\phi,\ldots\ldots (6),$$

$$\begin{vmatrix}\varpi, & \phi, & \chi\\ \varpi', & \phi', & \chi'\\ \varpi'', & \phi'', & \chi''\end{vmatrix}=\varpi(\phi'\chi''-\phi''\chi')+\varpi'(\phi''\chi-\phi\chi'')+\varpi''(\phi\chi'-\phi'\chi),$$

&c., the same as for common determinants, only here the order of the factors on each term of the second side of the equation is essential, and not, as in the other case, arbitrary. Thus, for instance,

$$\begin{vmatrix}\varpi, & \varpi'\\ \varpi, & \varpi'\end{vmatrix}=\varpi\varpi'-\varpi\varpi',\ =0\ \ldots\ldots (7),$$

but
$$\begin{vmatrix} \varpi, & \varpi \\ \varpi', & \varpi' \end{vmatrix} = \varpi\varpi' - \varpi'\varpi, \neq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

that is, a quaternion determinant does not vanish when two vertical rows become identical. One is immediately led to inquire what the value of such determinants is. Suppose

$$\varpi = x + iy + jz + kw, \quad \varpi' = x' + iy' + jz' + kw', \text{ \&c.},$$

it is easy to prove

$$\begin{vmatrix} \varpi & \varpi \\ \varpi & \varpi' \end{vmatrix} = -2 \begin{vmatrix} i, & j, & k \\ x, & y, & z \\ x', & y', & z' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$\begin{vmatrix} \varpi, & \varpi, & \varpi \\ \varpi', & \varpi', & \varpi' \\ \varpi'', & \varpi'', & \varpi'' \end{vmatrix} = -2 \begin{vmatrix} 3, & i, & j, & k \\ x, & y, & z, & w \\ x', & y', & z', & w' \\ x'', & y'', & z'', & w'' \end{vmatrix} \quad \ldots\ldots\ldots\ldots (10),$$

$$\begin{vmatrix} \varpi, & \varpi, & \varpi, & \varpi \\ \varpi', & \varpi', & \varpi', & \varpi' \\ \varpi'', & \varpi'', & \varpi'', & \varpi'' \\ \varpi''', & \varpi''', & \varpi''', & \varpi''' \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

or a quaternion determinant vanishes when four or more of its vertical rows become identical.

Again, it is immediately seen that

$$\begin{vmatrix} \varpi, & \phi \\ \varpi', & \phi' \end{vmatrix} + \begin{vmatrix} \phi, & \varpi \\ \phi', & \varpi' \end{vmatrix} = \begin{vmatrix} \varpi, & \varpi \\ \phi', & \phi' \end{vmatrix} - \begin{vmatrix} \varpi', & \varpi' \\ \phi, & \phi \end{vmatrix} \quad \ldots\ldots\ldots (12)$$

&c. for determinants of any order, whence the theorem, if any four (or more) adjacent vertical columns of a quaternion determinant be transposed in every possible manner, the sum of all these determinants vanishes, which is a much less simple property than the one which exists for the horizontal rows, viz. the same that in ordinary determinants exists for the horizontal or vertical rows indifferently. It is important to remark that the equations

$$\begin{vmatrix} \varpi, & \phi \\ \varpi', & \phi' \end{vmatrix} = 0 \text{ or } \begin{vmatrix} \varpi, & \varpi' \\ \phi, & \phi' \end{vmatrix} = 0, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

i.e.
$$\varpi\phi' - \varpi'\phi = 0, \text{ or } \varpi\phi' - \phi\varpi' = 0, \text{ \&c.}$$

are none of them the result of the elimination of Π, Φ, from the two equations

$$\begin{aligned} \varpi\Pi + \phi\Phi &= 0, \\ \varpi'\Pi + \phi'\Phi &= 0, \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

On the contrary, the result of this elimination is the very different equation

$$\varpi^{-1}\,.\,\phi - \varpi'^{-1}\,.\,\phi' = 0 \quad\text{..............................}\quad (15),$$

equivalent of course to four independent equations, one of which may evidently be replaced by

$$M\varpi\,.\,M\phi' - M\varpi'\,.\,M\phi = 0 \quad\text{..............................}\quad (16),$$

if $M\varpi$, &c. denotes the modulus of ϖ, &c. An equation analogous to this last will undoubtedly hold for any number of equations, but it is difficult to say what is the equation analogous to the one immediately preceding this, in the case of a greater number of equations, or rather, it is difficult to give the result in a symmetrical form independent of extraneous factors.

I may just, in conclusion, mention what appears to me a possible application of Sir William Hamilton's interesting discovery. In the same way that the circular functions depend on infinite products, such as

$$x\Pi\left(1 + \frac{x}{m\pi}\right), \text{ \&c. } \quad\text{..............................}\quad (17),$$

$$\{m \text{ any integer from } \infty \text{ to } -\infty, \text{ omitting } m = 0\}$$

and the inverse elliptic functions on the doubly infinite products

$$x\Pi\left(1 + \frac{x}{mw + n\varpi i}\right), \text{ \&c. } \quad\text{..............................}\quad (18)$$

$$\{m \text{ and } n \text{ integers from } \infty \text{ to } -\infty, \text{ omitting } m = 0,\ n = 0\},$$

may not the inverse ultra-elliptic functions of the next order of complexity depend on the quadruply infinite products

$$x\Pi\left(1 + \frac{x}{mw + n\varpi i + o\phi j + p\psi k}\right)? \quad\text{........................}\quad (19)$$

$$\{m,\ n,\ o,\ p \text{ integers from } \infty \text{ to } -\infty, \text{ omitting } m = 0,\ n = 0,\ o = 0,\ p = 0\}.$$

It seems as if some supposition of this kind would remove a difficulty started by Jacobi (Crelle, t. ix.) with respect to the multiple periodicity of these functions. Of course this must remain a mere suggestion until the theory of quaternions is very much more developed than it is at present; in particular the theory of quaternion exponentials would have to be developed, for even in a product, such as (18), there is a certain singular exponential factor running through the theory, as appears from some formulæ in Jacobi's *Fund. Nova* (relative to his functions Θ, H), the occurrence of which may be accounted for, *à priori*, as I have succeeded in doing in a paper to be published shortly in the *Cambridge Mathematical Journal* [24].