24.

ON THE INVERSE ELLIPTIC FUNCTIONS.

[From the *Cambridge Mathematical Journal*, t. IV. (1845), pp. 257—277.]

THE properties of the inverse elliptic functions have been the object of the researches of the two illustrious analysts, Abel and Jacobi. Among their most remarkable ones may be reckoned the formulæ given by Abel (*Œuvres,* t. I. p. 212 [Ed. 2, p. 343]), in which the functions $\phi\alpha$, $f\alpha$, $F\alpha$, (corresponding to Jacobi's $\sin\text{am}\,.\,\alpha$, $\cos\text{am}\,.\,\alpha$, $\Delta am\,.\,\alpha$, though not precisely equivalent to these, Abel's radical being $[(1-c^2x^2)(1+e^2x^2)]^{\frac{1}{2}}$, and Jacobi's, like that of Legendre's $[(1-x^2)(1-k^2x^2)]_{\frac{1}{2}}$), are expressed in the form of fractions, having a common denominator; and this, together with the three numerators, resolved into a doubly infinite series of factors; i.e. the general factor contains two independent integers. These formulæ may conveniently be referred to as "Abel's double factorial expressions" for the functions ϕ, f, F. By dividing each of these products into an infinite number of partial products, and expressing these by means of circular or exponential functions, Abel has obtained (pp. 216—218) two other systems of formulæ for the same quantities, which may be referred to as "Abel's first and second single factorial systems." The theory of the functions forming the above numerators and denominator, is mentioned by Abel in a letter to Legendre (*Œuvres,* t. II. p. 259 [Ed. 2, p. 272]), as a subject to which his attention had been directed, but none of his researches upon them have ever been published. Abel's double factorial expressions have nowhere anything analogous to them in Jacobi's *Fund. Nova;* but the system of formulæ analogous to the first single factorial system is given by Jacobi (p. 86), and the second system is implicitly contained in some of the subsequent formulæ. The functions forming the numerator and denominator of $\sin\text{am}\,.\,u$, Jacobi represents, omitting a constant factor, by $\text{H}(u)$, $\Theta(u)$; and proceeds to investigate the properties of these new functions. This he principally effects by means of a very remarkable equation of the form

$$l\Theta(u) = \tfrac{1}{2}Au^2 + B\textstyle\int_0 du\,.\int_0 du\,\sin^2\text{am}\,u,$$

(*Fund. Nova,* pp. 145, 133), by which $\Theta(u)$ is made to depend on the known function $\sin\operatorname{am}.u$. The other two numerators are easily expressed by means of the two functions H, Θ.

From the omission of Abel's double factorial expressions, which are the only ones which display clearly the real nature of the functions in the numerators and denominators; and besides, from the different form of Jacobi's radical, which complicates the transformation from an impossible to a possible argument, it is difficult to trace the connection between Jacobi's formulæ; and in particular to account for the appearance of an exponential factor which runs through them. It would seem therefore natural to make the whole theory depend upon the definitions of the new transcendental functions to which Abel's double factorial expressions lead one, even if these definitions were not of such a nature, that one only wonders they should never have been assumed *à priori* from the analogy of the circular functions sin, cos, and quite independently of the theory of elliptic integrals. This is accordingly what I have done in the present paper, in which therefore I assume no single property of elliptic functions, but demonstrate them all, from my fundamental equations. For the sake however of comparison, I retain entirely the notation of Abel. Several of the formulæ that will be obtained are new.

The infinite product

$$x\Pi\left(1+\frac{x}{m\omega}\right)\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where m receives the integer values $\pm 1, \pm 2, \ldots \pm r$, converges, as is well known, as r becomes indefinitely great to a determinate function $\sin\frac{\pi x}{\omega}$ of x; the theory of which might, if necessary, be investigated from this property assumed as a definition. We are thus naturally led to investigate the properties of the new transcendant

$$u = x\Pi\Pi\left(1+\frac{x}{m\omega+n\upsilon i}\right)\ldots\ldots\ldots\ldots(2):$$

m and n are integer numbers, positive or negative; and it is supposed that whatever positive value is attributed to either of these, the corresponding negative one is also given to it. $i=\sqrt{(-1)}$, ω and υ are real positive quantities. (At least this is the standard case, and the only one we shall explicitly consider. Many of the formulæ obtained are true, with slight modifications, whatever ω and υ represent, provided only $\omega : \upsilon i$ be not a real quantity; for if it were so, $m\omega+n\upsilon i$ for some values of m, n would vanish, or at least become indefinitely small, and u would cease to be a determinate function of x.)[1]

Now the value of the above expression, or, as for the sake of shortness it may be written, of the function

$$u = x\Pi\Pi\left\{1+\frac{x}{(m,\ n)}\right\}\ldots\ldots\ldots\ldots(3),$$

[1] I have examined the case of impossible values of ω and υ in a paper which I am preparing for *Crelle's Journal.* [The paper here referred to is [25], actually published in *Liouville's Journal*].

depends in a remarkable manner on the mode in which the superior limits of m, n are assigned. Imagine m, n to have any positive or negative integer values satisfying the equation

$$\phi(m^2, n^2) < T \qquad (4).$$

Consider, for greater distinctness, m, n as the coordinates of a point; the equation $\phi(m^2, n^2) = T$ belongs to a certain curve symmetrical with respect to the two axes. I suppose besides that this is a continuous curve without multiple points, and such that the minimum value of a radius vector through the origin continually increases as T increases, and becomes infinite with T. The curve may be *analytically* discontinuous, this is of no importance. The condition with respect to the limits is then that m and n must be integer values denoting the coordinates of a point *within* the above curve, the whole system of such integer values being successively taken for these quantities.

Suppose, next, u' denotes the same function as u, except that the limiting condition is

$$\phi'(m^2, n^2) < T' \qquad (5).$$

The curve $\phi'(m^2, n^2) = T'$ is supposed to possess the same properties with the other limiting curve, and, for greater distinctness, to lie entirely outside of it; but this last condition is nonessential.

These conditions being satisfied, the ratio $u' : u$ is very easily determined in the limiting case of T and T' infinite. In fact

$$\frac{u'}{u} = \Pi\Pi\left\{1 + \frac{x}{(m,\ n)}\right\} \qquad (6),$$

or

$$l\,\frac{u'}{u} = \Sigma\Sigma l\left\{1 + \frac{x}{(m,\ n)}\right\} \qquad (7),$$

the limiting conditions being

$$\phi(m^2, n^2) > T \qquad (8),$$
$$\phi'(m^2, n^2) < T'.$$

Now

$$l\left\{1 + \frac{x}{(m,\ n)}\right\} = \frac{x}{(m,\ n)} - \tfrac{1}{2}\cdot\frac{x^2}{(m,\ n)^2} + \dots \qquad (9),$$

$$l\,\frac{u'}{u} = x\,.\,\Sigma\Sigma\,\frac{1}{(m,\ n)} - \tfrac{1}{2}x^2\,.\,\Sigma\Sigma\,\frac{1}{(m,\ n)^2} + \dots \qquad (10),$$

or, the alternate terms vanishing on account of the positive and negative values destroying each other,

$$l\,\frac{u'}{u} = -\tfrac{1}{2}x^2\,.\,\Sigma\Sigma\,\frac{1}{(m,\ n)^2} - \tfrac{1}{4}x^4\,.\,\Sigma\Sigma\,\frac{1}{(m,\ n)^4} - \dots \qquad (11).$$

In general

$$\Sigma\Sigma\psi(m,\ n) = \iint\psi(m,\ n)\,dm\,dn + P \qquad (12),$$

P denoting a series the first term of which is of the form $C\psi(m, n)$, and the remaining ones depending on the differential coefficients of this quantity with respect to m and n. The limits between which the two sides are to be taken, are identical.

In the present case, supposing T and T' indefinitely great, it is easy to see that the first term of the expression for $l\frac{u'}{u}$ is the only one which is not indefinitely small: and we have

$$l\frac{u'}{u} = -\tfrac{1}{2}Ax^2, \text{ or } u' = u\epsilon^{-\frac{1}{2}Ax^2} \quad \text{.......... (13)},$$

where

$$A = \iint \frac{dmdn}{(m, n)^2} = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad \text{.......... (14)};$$

the limits of the integration being given by

$$\phi\,(m^2, n^2) > T \quad \text{.......... (15)},$$
$$\phi'(m^2, n^2) < T'.$$

Some particular cases are important. Suppose the limits of u' are given by

$$m^2\omega^2 < T^2, \quad n^2v^2 < T^2 \quad \text{.......... (16)},$$

and those of u, by

$$m^2\omega^2 + n^2v^2 < T^2 \quad \text{.......... (17)};$$

we have

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad \text{.......... (18)},$$

$$= -\frac{1}{\omega}\int dn \left\{\frac{1}{T + nvi} - \frac{1}{\sqrt{(T^2 - n^2v^2)} + nvi} - \frac{1}{-\sqrt{(T^2 - n^2v^2)} + nvi} - \frac{1}{-T + nvi}\right\}$$

$$= -\frac{2}{\omega}\int dn \left\{\frac{T}{T^2 + n^2v^2} - \frac{\sqrt{(T^2 - n^2v^2)}}{T^2}\right\} \quad (nv = -T, \quad nv = T)$$

$$= -\frac{2}{\omega v}\int_{-1}^{1} d\theta \left\{\frac{1}{1 + \theta^2} - \sqrt{(1 - \theta^2)}\right\} = -\frac{2}{\omega v}(\pi - \pi) = 0;$$

or, in this case,

$$u' = u \quad \text{.......... (19)}.$$

Again, let the limits of u' be

$$m^2\omega^2 < R'^2, \quad n^2v^2 < S'^2 \quad \text{.......... (20)},$$

and those of u,

$$m^2\omega^2 < R^2, \quad n^2v^2 < S^2 \quad \text{.......... (21)}.$$

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad \text{.......... (22)},$$

$$= -\frac{1}{\omega}\int dn \left\{\frac{1}{R' + nvi} - \frac{1}{R + nvi} + \frac{1}{-R + nvi} - \frac{1}{-R' + nvi}\right\},$$

where the limits are $n^2v^2 < S'^2$, for the terms containing R', $n^2v^2 < S^2$, for the terms containing R,

$$= -\frac{2}{\omega v i}\, l\, \frac{R' + S'i}{R' - S'i}\, \frac{R - Si}{R + Si} \quad\text{.............(23)},$$

$$= -\frac{4}{\omega v}(\lambda' - \lambda), \quad \text{if } \lambda' = \tan^{-1}\frac{S'}{R'}, \quad \lambda = \tan^{-1}\frac{S}{R},$$

the arcs λ, λ' being included between the limits 0, $\frac{1}{2}\pi$. Hence

$$u' = u\epsilon^{2(\lambda' - \lambda)\frac{x^2}{\omega v}} \quad\text{.............(24)}.$$

In particular if $\frac{S'}{R'} = \frac{S}{R}$, $u' = u$. If $\frac{S'}{R'} = 0$, $\frac{S}{R} = 1$, $u' = u\epsilon^{-\frac{1}{2}\beta x^2}$; if $\frac{S'}{R'} = \infty$, $\frac{S}{R} = 1$, $u' = u\epsilon^{\frac{1}{2}\beta x^2}$: where $\beta = \frac{\pi}{\omega v}$, for which quantity it will continue to be used.

We may now completely define the functions whose properties are to be investigated. Writing, for shortness,

$$(m,\ n) = m\omega + nvi \quad\text{.............}(A),$$

$$(\bar{m},\ n) = (m + \tfrac{1}{2})\,\omega + nvi,$$

$$(m,\ \bar{n}) = m\omega + (n + \tfrac{1}{2})\, vi,$$

$$(\bar{m},\ \bar{n}) = (m + \tfrac{1}{2})\,\omega + (n + \tfrac{1}{2})\, vi\,;$$

we may put

$$\gamma x = x\Pi\Pi\left\{1 + \frac{x}{(m,\ n)}\right\} \quad\text{.............}(B),$$

$$gx = \Pi\Pi\left\{1 + \frac{x}{(\bar{m},\ n)}\right\},$$

$$Gx = \Pi\Pi\left\{1 + \frac{x}{(m,\ \bar{n})}\right\},$$

$$\mathfrak{G}x = \Pi\Pi\left\{1 + \frac{x}{(\bar{m},\ \bar{n})}\right\};$$

the limits being given respectively by the equations

$$\text{mod.}\,(m,\ n) < T, \quad \text{mod.}\,(\bar{m},\ n) < T, \quad \text{mod.}\,(m,\ \bar{n}) < T, \quad \text{mod.}\,(\bar{m},\ \bar{n}) < T,$$

T being finally infinite. The system of values $m = 0$, $n = 0$, is of course omitted in γx.

The functions γx, gx, Gx, $\mathfrak{G}x$, are all of them real finite functions of x, possessing properties analogous to that of u. Thus, representing any one of them by Jx, we have

$$Jx = \epsilon^{\pm\frac{1}{2}\beta x^2} J_{\pm\beta}\, x \quad\text{.............}(C),$$

where $J_{\pm\beta}x$ is the same as Jx, only for $J_\beta x$ the limits are given by $m^2\omega^2$ or $(m+\frac{1}{2})^2\omega^2 < R^2$, $n^2\upsilon^2$ or $(n+\frac{1}{2})^2\upsilon^2 < S$, ($R$, S, and $\frac{R}{S}$ infinite), and for $J_{-\beta}x$, by the same formulæ, (R, S, and $\frac{S}{R}$ infinite). It is to this equation that the most characteristic properties of the functions Jx are due.

The following equations are deduced immediately from the above definitions:

$$\gamma(-x) = -\gamma x, \quad g(-x) = gx, \quad G(-x) = Gx, \quad \mathfrak{G}(-x) = \mathfrak{G}x \;\ldots\ldots\ldots\; (D),$$

$$\gamma(0) = 0, \quad g(0) = 1, \quad G(0) = 1, \quad \mathfrak{G}(0) = 1,$$

$$\gamma'(0) = 1.$$

Suppose $\gamma_1 x$, $g_1 x$, $G_1 x$, $\mathfrak{G}_1 x$, are the values that would have been obtained for γx, gx, Gx, $\mathfrak{G}x$ by interchanging ω and υ,—then changing x into xi, and interchanging m and n, by which means the limiting equations are the same in the two cases, we obtain the following system of equations:

$$\gamma_1(xi) = i\gamma x \;\ldots\ldots\ldots\; (E),$$

$$g_1(xi) = Gx,$$

$$G_1(xi) = gx,$$

$$\mathfrak{G}_1(xi) = \mathfrak{G}x\,;$$

or otherwise,

$$\gamma(xi) = i\gamma_1 x \;\ldots\ldots\ldots\; (F),$$

$$g(xi) = G_1 x,$$

$$G(xi) = g_1 x,$$

$$\mathfrak{G}(xi) = \mathfrak{G}_1 x,$$

equations which are useful in transforming almost any other property of the functions J.

The functions $J_\beta x$ are changed one into another, except as regards a constant multiplier, by the change of x into $x + \frac{\omega}{2}$. This will be shown in a Note, or it may be seen from some formulæ deduced immediately from the definitions of the functions $J_\beta x$, which will be given in the sequel[1]. Observing the relation between Jx and $J_\beta x$, we have in particular

$$\gamma\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\beta\omega x} A gx \;\ldots\ldots\ldots\; (G),$$

$$g\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\omega\beta x} B\gamma x,$$

$$G\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\beta\omega x} C\mathfrak{G}x,$$

$$\mathfrak{G}\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\beta\omega x} DGx,$$

[1] Not given in the present paper. [The Note *was* given, see p. 154, and the formulæ referred to must have been the formulæ (M) p. 144.]

where A, B, C, D, are most simply determined by writing $x=0$, $x=-\frac{\omega}{2}$. Putting at the same time $\epsilon^{\beta\omega^2}=\epsilon^{\frac{\pi\omega}{\upsilon}}=q_1^{-1}$,

$$A=\gamma\left(\frac{\omega}{2}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(H),$$

$$B=-q_1^{-\frac{1}{4}}\div\gamma\left(\frac{\omega}{2}\right),$$

$$C=G\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div\mathfrak{G}\left(\frac{\omega}{2}\right),$$

$$D=\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div G\left(\frac{\omega}{2}\right);$$

whence also

$$G\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\ \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Similarly, the functions $J_{-\beta}x$ are changed one into the other by the change of x into $x+\frac{1}{2}\upsilon i$. We have in the same way

$$\gamma\left(x+\frac{\upsilon i}{2}\right)=\epsilon^{-\frac{1}{2}\beta\upsilon xi}\,A'Gx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(I),$$

$$g\left(x+\frac{\upsilon i}{2}\right)=\epsilon^{-\frac{1}{2}\beta\upsilon xi}\,B'\mathfrak{G}x,$$

$$G\left(x+\frac{\upsilon i}{2}\right)=\epsilon^{-\frac{1}{2}\beta\upsilon xi}\,C'\gamma x,$$

$$\mathfrak{G}\left(x+\frac{\upsilon i}{2}\right)=\epsilon^{-\frac{1}{2}\beta\upsilon xi}\,D'gx.$$

Whence

$$A'=\gamma\left(\frac{\upsilon i}{2}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(J).$$

$$B'=g\left(\frac{\upsilon i}{2}\right)=q^{-\frac{1}{4}}\div\mathfrak{G}\left(\frac{\upsilon i}{2}\right),$$

$$C'=-q^{-\frac{1}{4}}\div\gamma\left(\frac{\upsilon i}{2}\right),$$

$$D'=\mathfrak{G}\left(\frac{\upsilon i}{2}\right)=q^{-\frac{1}{4}}\div g\left(\frac{\upsilon i}{2}\right);$$

where $\epsilon^{\beta\upsilon^2}=\epsilon^{\frac{\pi\upsilon}{\omega}}=q^{-1}$. It is obvious that the relation between q and q_1 is $lq\,.\,lq_1=-\pi^2$.

We obtain from the above

$$g\left(\frac{\upsilon i}{2}\right)\mathfrak{G}\left(\frac{\upsilon i}{2}\right)=q^{-\frac{1}{4}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

Also, by making $x=\frac{\upsilon i}{2}$ in the expression for $\gamma\left(x+\frac{\omega}{2}\right)$ and $x=\frac{\omega}{2}$ in that for $\gamma\left(x+\frac{\upsilon i}{2}\right)$, we have

$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{\upsilon i}{2}\right)=-i\gamma\left(\frac{\upsilon i}{2}\right) G\left(\frac{\omega}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

and the same or an equivalent one would have been obtained from the functions g, G, $\mathfrak{G}$.

By combining the above systems, we deduce one of the form

$$\gamma\left(x+\frac{\omega}{2}+\frac{\upsilon i}{2}\right)=\epsilon^{\frac{1}{2}\beta x(\omega-\upsilon i)} A''\mathfrak{G}x \ldots\ldots\ldots\ldots\ldots\ldots (K),$$

$$g\left(x+\frac{\omega}{2}+\frac{\upsilon i}{2}\right)=\epsilon^{\frac{1}{2}\beta x(\omega-\upsilon i)} B''Gx,$$

$$G\left(x+\frac{\omega}{2}+\frac{\upsilon i}{2}\right)=\epsilon^{\frac{1}{2}\beta x(\omega-\upsilon i)} C''gx,$$

$$\mathfrak{G}\left(x+\frac{\omega}{2}+\frac{\upsilon i}{2}\right)=\epsilon^{\frac{1}{2}\beta x(\omega-\upsilon i)} D''\gamma x;$$

and, observing the equation $\epsilon^{\beta\omega\upsilon i}=\epsilon^{\pi i}=(-1)$, with the following values for the coefficients,

$$A''=(-1)^{\frac{1}{4}}\,.\,\gamma\left(\frac{\omega}{2}\right) g\left(\frac{\upsilon i}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (L),$$

$$B''=-(-1)^{\frac{1}{4}}\,.\,q_1^{-\frac{1}{4}}\,.\,\gamma\left(\frac{\upsilon i}{2}\right)\div\gamma\left(\frac{\omega}{2}\right),$$

$$C''=(-1)^{\frac{1}{4}}\,.\,\mathfrak{G}\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\upsilon i}{2}\right),$$

$$D''=-(-1)^{\frac{1}{4}}\,.\,q^{-\frac{1}{4}}\,.\,\mathfrak{G}\left(\frac{\omega}{2}\right)\div\gamma\left(\frac{\upsilon i}{2}\right).$$

Collecting the formulæ which connect $\gamma\left(\frac{\omega}{2}\right)$, $\gamma\left(\frac{\upsilon i}{2}\right)$, ... these are

$$g\left(\frac{\omega}{2}\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (L\ bis),$$

$$G\left(\frac{\upsilon i}{2}\right)=0,$$

$$G\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}},$$

$$g\left(\frac{\upsilon i}{2}\right)\mathfrak{G}\left(\frac{\upsilon i}{2}\right)=q^{-\frac{1}{4}},$$

$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{\upsilon i}{2}\right)=-i\gamma\left(\frac{\upsilon i}{2}\right) G\left(\frac{\omega}{2}\right).$$

And by the assistance of these

$$B''C''\div A''D''=\ B'D'\div A'C'=CD\div AB=-1 \qquad (28),$$

$$A'B'\div C'D'\ =-A''B''\div C''D''=-\gamma^2\left(\frac{\upsilon i}{2}\right)\div \mathfrak{G}^2\left(\frac{\upsilon i}{2}\right),$$

$$A''C''\div B''D''=-A'C'\ \div B'D'=\ \gamma^2\left(\frac{\omega}{2}\right)\div \mathfrak{G}^2\left(\frac{\omega}{2}\right),$$

$$A\ D\ \div B\ C\ =-A'D'\div B'C'=\ \gamma^2\left(\frac{\omega}{2}\right)\div G^2\left(\frac{\omega}{2}\right)=-\gamma^2\left(\frac{\upsilon i}{2}\right)\div g^2\left(\frac{\upsilon i}{2}\right),$$

which will be required presently.

It is now easy to proceed to the general systems of formulæ,

$$\Theta=(-1)^{mn}\,\epsilon^{\beta x(m\omega-n\upsilon i)}\,q_1^{-\frac{1}{2}m^2}\,q^{-\frac{1}{2}n^2} \qquad (M),$$

$$\gamma\{x+(m,\ n)\}=(-1)^{m+n}\,.\,\Theta\gamma x\,,$$

$$g\{x+(m,\ n)\}=(-1)^{m}\ .\,\Theta g x\,,$$

$$G\{x+(m,\ n)\}=(-1)^{n}\ .\,\Theta G x\,,$$

$$\mathfrak{G}\{x+(m,\ n)\}=\qquad \Theta\mathfrak{G}x.$$

$$\Phi=(-1)^{n\,(m+\frac{1}{2})}\,\epsilon^{\beta x[(m+\frac{1}{2})\,\omega-n\upsilon i]}\,q_1^{-\frac{1}{2}m^2-\frac{1}{2}m}\,q^{-\frac{1}{2}n^2}.$$

$$\gamma\{x+(\overline{m},\ n)\}=(-1)^{m+n}.\,\Phi A g x\,,$$

$$g\{x+(\overline{m},\ n)\}=(-1)^{m}\ .\,\Phi B\gamma x\,,$$

$$G\{x+(\overline{m},\ n)\}=(-1)^{n}\ .\,\Phi C\mathfrak{G}x,$$

$$\mathfrak{G}\{x+(\overline{m},\ n)\}=\qquad \Phi D G x.$$

$$\Psi=(-1)^{m\,(n+\frac{1}{2})}\,\epsilon^{\beta x\,[m\omega-(n+\frac{1}{2})\,\upsilon i]}\,q_1^{-\frac{1}{2}m^2}\,q^{-\frac{1}{2}n^2-\frac{1}{2}n}$$

$$\gamma\{x+(m,\ \overline{n})\}=(-1)^{m+n}.\,\Psi A'Gx,$$

$$g\{x+(m,\ \overline{n})\}=(-1)^{m}\ .\,\Psi B'\mathfrak{G}x,$$

$$G\{x+(m,\ \overline{n})\}=(-1)^{n}\ .\,\Psi C'\gamma x\,,$$

$$\mathfrak{G}\{x+(m,\ \overline{n})\}=\qquad \Psi D'g x\,.$$

$$\Omega=(-1)^{mn+\frac{1}{2}m+\frac{1}{2}n}\,\epsilon^{\beta x[(m+\frac{1}{2})\,\omega-(n+\frac{1}{2})\,\upsilon i]}\,q_1^{-\frac{1}{2}m^2-\frac{1}{2}m}\,q^{-\frac{1}{2}n^2-\frac{1}{2}n}.$$

$$\gamma\{x+(\overline{m},\ \overline{n})\}=(-1)^{m+n}.\,\Omega A''\mathfrak{G}x,$$

$$g\{x+(\overline{m},\ \overline{n})\}=(-1)^{m}\ .\,\Omega B''Gx\,,$$

$$G\{x+(\overline{m},\ \overline{n})\}=(-1)^{n}\ .\,\Omega C''g x\,,$$

$$\mathfrak{G}\{x+(\overline{m},\ \overline{n})\}=\qquad .\,\Omega D''\gamma x\,.$$

Suppose $x=0$, we have the new systems,

$$\Theta_0 = (-1)^{mn} q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2} \quad \ldots\ldots\ldots\ldots \quad (M\ bis).$$

$$\gamma(m, n) = 0, \qquad \gamma'(m, n) = (-1)^{m+n}\Theta_0,$$
$$g(m, n) = (-1)^m \Theta_0,$$
$$G(m, n) = (-1)^n \Theta_0,$$
$$\mathfrak{G}(m, n) = \Theta_0.$$

$$\Phi_0 = (-1)^{n(m+\frac{1}{2})} q_1^{-\frac{1}{2}m^2-\frac{1}{2}m} q^{-\frac{1}{2}n^2}.$$

$$\gamma(\bar{m}, n) = (-1)^{m+n}\Phi_0 A,$$
$$g(\bar{m}, n) = 0, \qquad g'(\bar{m}, n) = (-1)^m \Phi_0 B,$$
$$G(\bar{m}, n) = (-1)^n \Phi_0 C,$$
$$\mathfrak{G}(\bar{m}, n) = \Phi_0 D.$$

$$\Psi_0 = (-1)^{m(n+\frac{1}{2})} q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2-\frac{1}{2}n}.$$

$$\gamma(m, \bar{n}) = (-1)^{m+n}\Psi_0 A',$$
$$g(m, \bar{n}) = (-1)^m \Psi_0 B',$$
$$G(m, \bar{n}) = 0, \qquad G'(m, \bar{n}) = (-1)^n \Psi_0 C',$$
$$\mathfrak{G}(m, \bar{n}) = \Psi_0 D'.$$

$$\Omega_0 = (-1)^{mn+\frac{1}{2}m+\frac{1}{2}n} q_1^{-\frac{1}{2}m^2-\frac{1}{2}m} q^{-\frac{1}{2}n^2-\frac{1}{2}n}.$$

$$\gamma(\bar{m}, \bar{n}) = (-1)^{m+n}\Omega_0 A'',$$
$$g(\bar{m}, \bar{n}) = (-1)^m \Omega_0 B'',$$
$$G(\bar{m}, \bar{n}) = (-1)^n \Omega_0 C'',$$
$$\mathfrak{G}(\bar{m}, \bar{n}) = 0, \qquad \mathfrak{G}'(\bar{m}, \bar{n}) = \Omega_0 D''.$$

We obtain immediately, by taking the logarithmic differentials of the functions γx, gx, Gx, $\mathfrak{G}x$, the equations

$$\gamma' x \div \gamma x = \Sigma\Sigma\, \{x-(m, n)\}^{-1}, \quad m=0, \quad n=0 \text{ admissible}, \ \ldots\ldots (N),$$
$$g' x \div g x = \Sigma\Sigma\, \{x-(\bar{m}, n)\}^{-1},$$
$$G' x \div G x = \Sigma\Sigma\, \{x-(m, \bar{n})\}^{-1},$$
$$\mathfrak{G}' x \div \mathfrak{G} x = \Sigma\Sigma\, \{x-(\bar{m}, \bar{n})\}^{-1},$$

the limits being the same as in the case of the factorial expressions.

Consider an equation

$$gx\, Gx \div \gamma x\, \mathfrak{G}x = \Sigma\Sigma\, [\mathfrak{A}\{x-(m, n)\}^{-1} + \mathfrak{B}\{x-(\bar{m}, \bar{n})\}^{-1}] \ldots\ldots\ldots (29),$$

we have $$\mathfrak{A} = g(m,\ n)\, G(m,\ n) \div \gamma'(m,\ n)\, \mathfrak{G}(m,\ n) = 1 \quad\text{.........(30)},$$

$$\mathfrak{B} = g(\bar{m},\ \bar{n})\, G(\bar{m},\ \bar{n}) \div \gamma(\bar{m},\ \bar{n})\, \mathfrak{G}'(\bar{m},\ \bar{n}) = B''C'' \div A''D'' = -1 \quad\text{... (31)}.$$

(The application of the ordinary method of decomposition into partial fractions, which is in general exceedingly precarious when applied to transcendental functions, is justified here by a theorem of Cauchy's, which will presently be quoted.) We have thus

$$gxGx \div \gamma x\mathfrak{G}x = (\gamma' x \div \gamma x) - (\mathfrak{G}'x \div \mathfrak{G}x),$$

and similarly

$$gx\,\mathfrak{G}x \div \gamma x\ \ Gx = (\gamma' x \div \gamma x) - (G'x \div Gx), \quad\text{.........(O)},$$

$$Gx\mathfrak{G}x \div \gamma x\ \ gx = (\gamma' x \div \gamma x) - (g'x \div gx),$$

$$-b^2\gamma x\mathfrak{G}x \div gx\ \ Gx = (g'x \div gx) - (G'x \div Gx),$$

$$e^2\gamma x\ gx \div Gx\,\mathfrak{G}x = (G'x \div Gx) - (\mathfrak{G}'x \div \mathfrak{G}x),$$

$$c^2\gamma x Gx \div \mathfrak{G}x\ gx = (\mathfrak{G}'x \div \mathfrak{G}x) - (g'x \div gx);$$

in which we have written

$$\gamma\left(\frac{vi}{2}\right) \div \mathfrak{G}\left(\frac{vi}{2}\right) = \frac{i}{e} \quad\text{.........(32)},$$

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c},$$

$$\gamma\left(\frac{\omega}{2}\right) \div G\left(\frac{\omega}{2}\right) = -i\left\{\gamma\left(\frac{vi}{2}\right) \div g\left(\frac{vi}{2}\right)\right\} = \frac{1}{b}.$$

Eliminating the derived coefficients,

$$G^2x - \mathfrak{G}^2x = e^2\gamma^2 x, \quad\text{.........(33)},$$

$$g^2x - G^2x = -b^2\gamma^2 x,$$

$$\mathfrak{G}^2x - g^2x = c^2\gamma^2 x.$$

Adding these equations, $b^2 = e^2 + c^2$, or $b = \sqrt{(e^2 + c^2)}$, in which sense it will continue to be used.

Also, $$g^2x = \mathfrak{G}^2x - c^2\gamma^2 x, \quad\text{.........(P)}.$$

$$G^2x = \mathfrak{G}^2x + e^2\gamma^2 x.$$

Suppose

$$\phi x = \gamma x \div \mathfrak{G}x, \quad\text{.........(Q)},$$

$$fx = gx \div \mathfrak{G}x,$$

$$Fx = Gx \div \mathfrak{G}x,$$

then

$$f^2x = 1 - c^2\phi^2 x, \quad\text{.........(R)},$$

$$F^2x = 1 + e^2\phi^2 x,$$

and also
$$\phi' x = \quad fx\, Fx\ , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(S),$$
$$f'x = -c^2\phi x\, Fx,$$
$$F'x = \quad e^2 \phi x fx\ .$$

Hence, putting for fx, Fx, their values,

$$1 = \frac{\phi' x}{\sqrt{(1-c^2\phi^2 x)(1+e^2\phi^2 x)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (T);$$

or writing $\phi x = y$, and integrating,

$$x = \int_0^{\phi x} \frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (U),$$

or

$$\phi^{-1} y = \int_0^{y} \frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}},$$

which shows that ϕ is an inverse elliptic function.

The equations which are the foundation of the theory of the functions ϕ, f, F, are deduced immediately from the equations (S). (Abel, *Œuvres*, tom. I. p. 143 [Ed. 2, p. 268.]) These are

$$\phi(x+y) = \frac{\phi x\, fy\, Fy + \phi y\, fx\, Fx}{1 + e^2c^2\,\phi^2 x\,\phi^2 y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (V),$$

$$f(x+y) = \frac{fx\, fy - c^2\,\phi x\,\phi y\, Fx\, Fy}{1+e^2c^2\,\phi^2 x\,\phi^2 y},$$

$$F(x+y) = \frac{Fx\, Fy + e^2\,\phi x\,\phi y\, fx\, fy}{1+e^2c^2\,\phi^2 x\,\phi^2 y};$$

so that from this point we may take for granted any properties of these functions. We see, for instance, immediately,

$$\phi\left(\frac{\varpi i}{2}\right) = \frac{i}{e},\ \phi\left(\frac{\omega}{2}\right) = \frac{1}{c};$$

whence

$$\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (W),$$

$$\frac{\varpi i}{2} = \int_0^{\frac{i}{e}} \frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}},\ \text{or}\ \frac{\varpi}{2} = \int_0^{\frac{1}{e}} \frac{dy}{\sqrt{(1+c^2y^2)(1-e^2y^2)}} \ldots\ldots (X),$$

which give the values of ω, ϖ in terms of c, e; values which may be developed in a variety of ways, in infinite series. We may also express $\gamma\left(\frac{\omega}{2}\right)$, &c., and consequently A, B ... &c., by means of the quantities c, e. We have only to combine the equations

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c},\quad \gamma\left(\frac{\varpi i}{2}\right) \div \mathfrak{G}\,\frac{\varpi i}{2} = \frac{i}{e},\quad G\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{b}{c},\quad g\left(\frac{\varpi i}{2}\right) \div \mathfrak{G}\left(\frac{\varpi i}{2}\right) = \frac{b}{e} \ldots (34),$$

with the former relations between these quantities, and we have

$$\gamma\left(\frac{\omega}{2}\right)=b^{-\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}},\quad \gamma\left(\frac{\upsilon i}{2}\right)=ib^{-\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}},\ \ldots\ldots\ldots\ldots\ldots\ldots(Y),$$

$$g\left(\frac{\omega}{2}\right)=0,\qquad g\left(\frac{\upsilon i}{2}\right)=b^{\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}},$$

$$G\left(\frac{\omega}{2}\right)=b^{\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}},\quad \mathfrak{G}\left(\frac{\upsilon i}{2}\right)=0,$$

$$\mathfrak{G}\left(\frac{\omega}{2}\right)=b^{-\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}},\quad \mathfrak{G}\left(\frac{\upsilon i}{2}\right)=b^{-\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}.$$

$$\begin{array}{lll}
A=b^{-\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}, & A'=ib^{-\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}, & A''=(-1)^{\frac{1}{4}}c^{-\frac{1}{2}}e^{-\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}\ , \quad \ldots (Z).\\
B=-b^{\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}\ , & B'=b^{\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}\ , & B''=-(-1)^{\frac{1}{4}}ic^{\frac{1}{2}}e^{-\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}},\\
C=b^{\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}\ , & C'=-ib^{\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}, & C''=(-1)^{\frac{1}{4}}c^{-\frac{1}{2}}e^{\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}\ ,\\
D=b^{-\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}\ , & D'=b^{-\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}\ , & D''=-(-1)^{\frac{1}{4}}ic^{\frac{1}{2}}e^{\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}\ ,
\end{array}$$

which are to be substituted in any formulæ into which these quantities enter.

The following is Cauchy's Theorem, (*Exercises de Math.* t. II. p. 289).

"If in attributing to the modulus r of the variable

$$z=r\{\cos p+\surd(-1)\sin p\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

infinitely great values, these can be chosen so that the two functions

$$\frac{fz+f(-z)}{2},\quad \frac{fz-f(-z)}{2z},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

sensibly vanish, whatever be the value of p, or vanish in general, though ceasing to do so and obtaining *finite* values for certain particular values of p; then

$$fx=\mathcal{E}\,\frac{\{(fz)\}}{x-z}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37),$$

the integral residue being reduced to its principal value."

To understand this, it is only necessary to remark that the integral residue in question is the series of fractions that would be obtained by the ordinary process of decomposition; and by the principal value is meant, that *all* those roots are to be taken, the modulus of which is not greater than a certain limit, this limit being afterwards made infinite.

Suppose now fx is a fraction, the numerator and denominator of which are monomials of the form $(\gamma x)^l\,(gx)^m\ldots$, l, $m\ldots$ being positive integers, and of course no common factor being left in the numerator and denominator.

Let λ be the excess of the degree of the denominator over that of the numerator. Suppose the modulus r of (z) has any value not the same with any of the moduli of

$$(m,\,n),\quad (\overline{m},\,n),\quad (m,\,\overline{n}),\quad (\overline{m},\,\overline{n})\ \ldots\ldots\ldots\ldots\ldots\ldots(38).$$

Then we have

$$r(\cos p + i \sin p) = m\omega + n\upsilon i + \theta \quad \text{.........................(39)},$$

θ being a finite quantity, such that none of the functions $J\theta$ vanish. m and n are the greatest integer values which allow the possible part of θ and the coefficient of its impossible part to remain positive. We have therefore

$$m^2\omega^2 + n^2\upsilon^2 = r^2 - M \quad \text{..............................(40)},$$

M being finite; or when r is infinite, at least one of the values m, n is infinite. The function fz reduces itself to the form

$$q_1^{\frac{1}{2}\lambda m^2}\, q^{\frac{1}{2}\lambda n^2}\, \epsilon^{mA+nB}F \quad \text{..................................(41)},$$

where F is finite. Hence q_1 and q being always less than unity, fz, and consequently both $\frac{1}{2}\{fz + f(-z)\}$ and $\frac{1}{2z}\{fz - f(-z)\}$ vanish for $r = \infty$, as long as λ is positive.

In the case of $\lambda = 0$, the conditions are still satisfied, if we suppose fx to denote an uneven function of x: for when $\lambda = 0$, the index of exponential in the above expression vanishes, or fz is constantly finite. But fz being an odd function of z, $fz + f(-z) = 0$. And $\frac{1}{2z}\{fz - f(-z)\}$ vanishes for z infinite, on account of the z in the denominator: hence the expansion is admissible in this case. But it is certainly so also, in a great many cases at least, where fz is an even function of z; for these may be deduced from the others by a simple change in the value of the variable. For instance, from the expansion of $\gamma x \div gx$, which is an odd function, by writing $x + \frac{\upsilon i}{2}$ for x, we obtain that of $Gx \div \mathfrak{G}x$, which is even.

A case of some importance is when the function is of the above form, multiplied by an exponential $\epsilon^{\frac{1}{2}ax^2+bx}$. Here writing $z = m\omega + n\upsilon i + \theta$, the admissibility of the formula depends on the evanescence of

$$\epsilon^{\frac{1}{2}a(m\omega+n\upsilon i)^2}\, q_1^{\frac{1}{2}\lambda m^2}\, q^{\frac{1}{2}\lambda n^2} \quad \text{..............................(42)};$$

or, if $a = h + ki$, this becomes, omitting a finite factor,

$$\epsilon^{-\frac{1}{2}m^2(\lambda\beta - h)\omega^2 - \frac{1}{2}n^2(\lambda\beta + h)\upsilon^2 - kmn\,\omega\upsilon} \quad \text{..........................(43)},$$

which vanishes if $h^2 + k^2 < \lambda^2\beta^2$, i.e. the modulus of a is less than $\lambda\beta$. The limiting case is admissible when the series is convergent.

We obtain in this way a very great variety of formulæ. For instance,

$$\epsilon^{\frac{1}{2}ax^2+bx} \div \gamma x = \Sigma\Sigma\,[(-1)^{-mn-m-n}\,\epsilon^{\frac{1}{2}a(m,\,n)^2+b(m,\,n)}\, q_1^{\frac{1}{2}m^2}\, q^{\frac{1}{2}n^2}\{x - (m,\,n)\}^{-1}] \quad \text{........} (A'),$$

$$\epsilon^{\frac{1}{2}ax^2+bx} \div gx = -b^{\frac{1}{2}}c^{-\frac{1}{2}}\Sigma\Sigma\,[(-1)^{-mn-m-\frac{1}{2}n}\,\epsilon^{\frac{1}{2}a(\overline{m},\,n)^2+b(\overline{m},\,n)}\, q_1^{\frac{1}{2}(m+\frac{1}{2})^2}\, q^{\frac{1}{2}n^2}\{x - (\overline{m},\,n)\}^{-1}],$$

$$\epsilon^{\frac{1}{2}ax^2+bx} \div Gx = ib^{-\frac{1}{2}}e^{-\frac{1}{2}}\Sigma\Sigma\,[(-1)^{-mn-\frac{1}{2}m-n}\,\epsilon^{\frac{1}{2}a(m,\,\overline{n})^2+b(m,\,\overline{n})}\, q_1^{\frac{1}{2}m^2}\, q^{\frac{1}{2}(n+\frac{1}{2})^2}\{x - (m,\,\overline{n})\}^{-1}],$$

$$\epsilon^{\frac{1}{2}ax^2+bx} \div \mathfrak{G}x = ic^{-\frac{1}{2}}e^{-\frac{1}{2}}\Sigma\Sigma\,[(-1)^{-(m+\frac{1}{2})(n+\frac{1}{2})}\,\epsilon^{\frac{1}{2}a(\overline{m},\,\overline{n})^2+b(\overline{m},\,\overline{n})}\, q_1^{\frac{1}{2}(m+\frac{1}{2})^2}\, q^{\frac{1}{2}(n+\frac{1}{2})^2}\{x - (\overline{m},\,\overline{n})\}^{-1}],$$

in which the modulus of a must not exceed β: in the limiting cases, for $a=\beta$, b must be entirely impossible, and for $a=-\beta$, b must be entirely real. The formulæ for γx are

$$\epsilon^{\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma\,(-1)^{-m-n}\, q^{n^2}\, \epsilon^{b\,(m,\,n)}\, \{x-(m,\,n)\}^{-1} \quad \ldots\ldots\ldots\ldots (44),$$

$$\epsilon^{-\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma\,(-1)^{m-n}\, q_1^{m^2}\, \epsilon^{b\,(m,\,n)}\, \{x-(m,\,n)\}^{-1};$$

and for $b=0$,

$$\epsilon^{\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma\,(-1)^{-m-n}\, q^{n^2}\, \{x-(m,\,n)\}^{-1} \quad \ldots\ldots\ldots\ldots (45),$$

$$\epsilon^{-\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma\,(-1)^{-m-n}\, q_1^{m^2}\, \{x-(m,\,n)\}^{-1}.$$

Next the system,

$$\mathfrak{G}x \div \gamma x = \Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,\,n)\}^{-1} \quad \ldots\ldots\ldots\ldots (B'),$$

$$gx \div \gamma x = \Sigma\Sigma\,(-1)^{m}\,\{x-(m,\,n)\}^{-1},$$

$$Gx \div \gamma x = \Sigma\Sigma\,(-1)^{n}\,\{x-(m,\,n)\}^{-1};$$

$$\gamma x \div gx = -b^{-1}c^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},\,n)\}^{-1},$$

$$Gx \div gx = -c^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\bar{m},\,n)\}^{-1},$$

$$\mathfrak{G}x \div gx = -b^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(\bar{m},\,n)\}^{-1};$$

$$\gamma x \div Gx = -b^{-1}e^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(m,\,\bar{n})\}^{-1},$$

$$gx \div Gx = ie^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,\,\bar{n})\}^{-1},$$

$$\mathfrak{G}x \div Gx = ib^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(m,\,\bar{n})\}^{-1};$$

$$\gamma x \div \mathfrak{G}x = ic^{-1}e^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\bar{m},\,\bar{n})\}^{-1},$$

$$gx \div \mathfrak{G}x = e^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(\bar{m},\,\bar{n})\}^{-1},$$

$$Gx \div \mathfrak{G}x = ic^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},\,\bar{n})\}^{-1},$$

which is partially given by Abel.

We may obtain, in like manner, expressions for the functions

$$\frac{1}{\gamma x\, gx},\ \frac{1}{\gamma x\, Gx}, \ldots \text{(six terms of this form)} \ldots\ldots\ldots\ldots (C'),$$

$$\frac{Gx}{\gamma x\, gx}, \ldots \quad \text{(twelve)} \quad \ldots\ldots\ldots\ldots (D'),$$

$$\frac{\gamma x\, gx}{\mathfrak{G}x\, Gx}, \ldots \quad \text{(six)} \quad \ldots\ldots\ldots\ldots (E'),$$

$$\frac{1}{\gamma x\, gx\, Gx}, \ldots \quad \text{(four)} \quad \ldots\ldots\ldots\ldots (F'),$$

$$\frac{\mathfrak{G}x}{\gamma x\, gx\, Gx}, \ldots \quad \text{(four)} \quad \ldots\ldots\ldots\ldots (G'),$$

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}, \ldots \quad \text{(one)} \quad \ldots\ldots\ldots\ldots (H');$$

each of them, except (E'), (the system for which, admitting no exponential, has already been given,) multiplied by an exponential $\epsilon^{\frac{1}{2}ax^2+bx}$, the limits of a being $\pm 2\beta$, $\pm\beta$, $\pm\beta$, $\pm 3\beta$, $\pm 2\beta$, $\pm 4\beta$. For the limiting values, b must be entirely impossible for the superior limit, and entirely possible for the inferior one.

Thus the last case is

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}\,\epsilon^{\frac{1}{2}ax^2+bx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (H'),$$

$$= \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(m,\,n)^2+b\,(m,\,n)}\; q_1^{\;2m^2}\; q^{2n^2}\; \{x-(m,\,n)\}^{-1}]$$

$$- \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(\overline{m},\,n)^2+b\,(\overline{m},\,n)}\; q_1^{\;2\,(m+\frac{1}{2})^2}\; q^{2n^2}\; \{x-(\overline{m},\,n)\}^{-1}]$$

$$+ \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(m,\,\overline{n})^2+b\,(m,\,\overline{n})}\; q_1^{\;2m^2}\; q^{2\,(n+\frac{1}{2})^2}\; \{x-(m,\,\overline{n})\}^{-1}]$$

$$- \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(\overline{m},\,\overline{n})^2+b\,(\overline{m},\,\overline{n})}\; q_1^{\;2\,(m+\frac{1}{2})^2}\; q^{2\,(n+\frac{1}{2})^2}\; \{x-(\overline{m},\,\overline{n})\}^{-1}];$$

in particular

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}\,\epsilon^{2\beta x^2} = \Sigma\Sigma\, q_1^{\;4m^2}\; \{x-(m,\,n)\}^{-1} \ldots\ldots\ldots\ldots (46),$$

$$- \Sigma\Sigma\, q_1^{\;(2m+1)^2}\, \{x-(\overline{m},\,n)\}^{-1}$$

$$+ \Sigma\Sigma\, q_1^{\;4m^2}\; \{x-(m,\,\overline{n})\}^{-1}$$

$$+ \Sigma\Sigma\, q_1^{\;(2m+1)^2}\, \{x-(\overline{m},\,\overline{n})\}^{-1},$$

or the analogous formula obtained by changing β, q_1, m into $-\beta$, q, n.

The function $\phi^2 x$, which is even, and for which $\lambda = 0$, cannot be expanded entirely in a series of partial fractions: but $(x-a)^{-1}\phi^2 x$ may be so expanded. Multiply by $(x-a)$, the second side has for its general term

$$(x-a)\,(Mx+N)\,\{x-(\overline{m},\,\overline{n})\}^{-2},$$

equivalent to

$$K' + (M'x+N')\,\{x-(\overline{m},\,\overline{n})\}^{-2}.$$

Summing all the K''s, we have an equation of the form

$$\phi^2 x = A + \Sigma\Sigma\,[L\,\{x-(\overline{m},\,\overline{n})\}^{-2} + M\,\{x-(\overline{m},\,\overline{n})\}^{-1}] \ldots\ldots\ldots\ldots (47).$$

To determine the coefficients as simply as possible, change x into $x+\frac{1}{2}\omega+\frac{1}{2}n\omega i$,

$$-e^{-2}c^{-2}(\phi x)^{-2} = A + \Sigma\Sigma\,[L\,\{x-(m,\,n)\}^{-2} + M\,\{x-(m,\,n)\}^{-1}] \ldots\ldots (48),$$

$$L = -e^{-2}c^{-2}\,[\{x-(m,\,n)\}^2(\phi x)^{-2}],\; x=(m,\,n) \quad \ldots\ldots\ldots (49),$$

$$M = -e^{-2}c^{-2}\,\partial_x\,[\{x-(m,\,n)\}^2(\phi x)^{-2}],$$

or writing $x+(m,\,n)$ for x, and therefore $x=0$ in the values of L and M,

$$L = -e^{-2}c^{-2}\{x^2(\phi x)^{-2}\} = e^{-2}c^{-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

$$M = -e^{-2}c^{-2}\,\partial_x\,\{x^2(\phi x)^{-2}\} = 0\,;$$

whence

$$\phi^2 x = A - e^{-2}c^{-2}\,\Sigma\Sigma\,\{x-(\overline{m},\,\overline{n})\}^{-2} \ldots\ldots\ldots\ldots\ldots (51).$$

Integrating this last equation twice,

$$\int_0 dx \int_0 dx\, \phi^2 x = \tfrac{1}{2}Ax^2 + e^{-2}c^{-2}\,\Sigma\Sigma l\,\{x - (\overline{m}, \overline{n})\} \quad\ldots\ldots\ldots\ldots\ldots (52),$$

or

$$\mathfrak{G}x = \epsilon^{-\frac{1}{2}e^2c^2Ax^2 + e^2c^2\int_0 dx\int_0 dx\,\phi^2 x} \quad\ldots\ldots\ldots\ldots\ldots (53),$$

an equation from which it is easy to determine the coefficient A.

Suppose for a moment $\phi_{,}x = \int_0 \phi^2 x\, dx$, $\phi_{,,}x = \int_0 \phi_{,}x\, dx$; then, since $\phi^2(x+\omega) - \phi^2 x = 0$,

$$\phi_{,}(x+\omega) - \phi x = \phi_{,}\omega, \quad \phi_{,,}(x+\omega) - \phi_{,}x = \phi_{,,}\omega + x\phi_{,}\omega.$$

But similarly $\phi^2 x - \phi^2(\omega - x) = 0$; whence

$$\phi_{,}x + \phi_{,}(\omega - x) = \phi_{,}\omega, \quad \phi_{,,}x - \phi_{,,}(\omega - x) + \phi_{,,}\omega = x\phi_{,}\omega;$$

whence, writing $x = \dfrac{\omega}{2}$,

$$\phi_{,}\omega = 2\phi_{,}\left(\frac{\omega}{2}\right), \quad \phi_{,,}\omega = \omega\phi_{,}\left(\frac{\omega}{2}\right), \quad \text{or} \quad \phi_{,,}(x+\omega) - \phi_{,,}x = \phi_{,}\left(\frac{\omega}{2}\right)(2x+\omega).$$

Hence

$$\mathfrak{G}(x+\omega) = \epsilon^{-\frac{1}{2}e^2c^2\{A - \frac{1}{2}\omega\phi_{,}(\frac{1}{2}\omega)\}(2\omega x + \omega^2)}\,\mathfrak{G}x \quad\ldots\ldots\ldots\ldots\ldots (54).$$

But

$$\mathfrak{G}(x+\omega) = \epsilon^{\beta\omega x}\, q_1^{-\frac{1}{2}}\,\mathfrak{G}x = \epsilon^{\frac{1}{2}\beta(2\omega x + \omega^2)}\,\mathfrak{G}x \quad\ldots\ldots\ldots\ldots\ldots (55);$$

or, comparing these,

$$-e^2c^2\left\{A - \frac{2}{\omega}\,\phi_{,}\left(\frac{\omega}{2}\right)\right\} = \beta \quad\ldots\ldots\ldots\ldots\ldots (56),$$

$$-\tfrac{1}{2}e^2c^2A = \tfrac{1}{2}\beta - \frac{e^2c^2}{\omega}\,\phi_{,}\left(\frac{\omega}{2}\right) \quad\ldots\ldots\ldots\ldots\ldots (57),$$

or writing

$$M = \frac{e^2c^2}{\omega}\int_0^{\frac{1}{2}\omega} \phi^2 x\, dx \quad\ldots\ldots\ldots\ldots\ldots (58),$$

then

$$\mathfrak{G}x = \epsilon^{(\frac{1}{2}\beta - M)x^2 + e^2c^2\int_0 dx\int_0 dx\,\phi^2 x} \quad\ldots\ldots\ldots\ldots\ldots (I');$$

which is the formula corresponding to the one of Jacobi's referred to at the beginning of this paper. Analogous formulæ may be deduced from it by writing $x + \dfrac{\omega}{2}$, or $x + \dfrac{\nu i}{2}$, or $x + \dfrac{\omega}{2} + \dfrac{\nu i}{2}$, instead of x.

The following formulæ, making the necessary changes of notation, are taken from Jacobi. We have

$$\phi^2(x+a) - \phi^2(x-a) = \frac{4\phi a\, fa\, Fa\, \phi x fx Fx}{(1 + e^2c^2\phi^2 a\,\phi^2 x)^2} \quad\ldots\ldots\ldots\ldots\ldots (59),$$

whence $$\int_0 \{\phi^2(x+a)-\phi^2(x-a)\}\,dx = \frac{2\phi a\, fa\, Fa\, \phi^2 x}{1+e^2c^2\phi^2 a\phi^2 x} \quad\text{.................}(60),$$

the first side of which is

$$\int_{-a}\phi^2(x+a)\,dx - \int_a \phi^2(x-a)\,dx - 2\int_0 \phi^2 a\, da \quad\text{.................}(61).$$

Hence, multiplying by e^2c^2, and observing the value of $\mathfrak{G}x$,

$$\frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} - \frac{\mathfrak{G}'(x-a)}{\mathfrak{G}(x-a)} - 2\frac{\mathfrak{G}'a}{\mathfrak{G}a} = \frac{2e^2c^2 fa\, Fa\, \phi a\, \phi^2 x}{1+e^2c^2\phi^2 a\phi^2 x} \quad\text{..............}(62).$$

If in this case we interchange x, a and add,

$$\frac{\mathfrak{G}'x}{\mathfrak{G}x} + \frac{\mathfrak{G}'a}{\mathfrak{G}a} - \frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} = e^2c^2\phi a\,\phi x\,\phi(a+x) \quad\text{................}(63).$$

[By subtracting, we should have obtained an equation only differing from the above in the sign of a.]

Integrating the last equation but one, with respect to a,

$$l\mathfrak{G}(x+a) + l\mathfrak{G}(x-a) - 2l\mathfrak{G}x - 2l\mathfrak{G}a = l(1+e^2c^2\phi^2 x\phi^2 a),$$

the integral being taken from $a=0$. Hence

$$\mathfrak{G}(x+a)\,\mathfrak{G}(x-a) = \mathfrak{G}^2x\mathfrak{G}^2a\,(1+e^2c^2\phi^2x\phi^2a) \quad\text{...........}(64);$$

or

$$\left.\begin{aligned} \mathfrak{G}(x+a)\,\mathfrak{G}(x-a) &= \mathfrak{G}^2x\mathfrak{G}^2a + e^2c^2\gamma^2x\gamma^2a,\\ \gamma(x+a)\;\;\gamma(x-a) &= \gamma^2x\mathfrak{G}^2a - \;\gamma^2a\;\mathfrak{G}^2x,\\ g(x+a)\;\;g(x-a) &= g^2x\mathfrak{G}^2a - c^2g^2a\;\mathfrak{G}^2x,\\ G(x+a)\;\;G(x-a) &= G^2x\mathfrak{G}^2a + e^2G^2a\mathfrak{G}^2x, \end{aligned}\right\} \quad\text{.............}(J),$$

whence also

these equations being obtained from the first by the change of x into $x+\frac{\omega}{2}$, $x+\frac{\upsilon i}{2}$, $x+\frac{\omega}{2}+\frac{\upsilon i}{2}$. They form a most important group of formulæ in the present theory. By integrating the same formulæ with respect to x, and representing by $\Pi(x,\,a)$ the integral $\int_0 \frac{-e^2c^2\,\phi a\, fa\, Fa\, \phi^2 x dx}{1+e^2c^2\phi^2a\phi^2x}$, Jacobi obtains

$$\Pi(x,\,a) = \tfrac{1}{2}l\frac{\mathfrak{G}(x-a)}{\mathfrak{G}(x+a)} + x\frac{\mathfrak{G}'a}{\mathfrak{G}a}:$$

an equation which conducts him almost immediately to the formulæ for the addition of the argument or of the parameter in the function Π. This, however, is not very closely connected with the present subject. For some formulæ also deduced from (63), by which $\dfrac{\mathfrak{G}(x-a)\,\mathfrak{G}(y-a)\,\mathfrak{G}(x+y+a)}{\mathfrak{G}(x+a)\,\mathfrak{G}(y+a)\,\mathfrak{G}(x+y-a)}$ is expressed in terms of the function ϕ, see Jacobi.

NOTE.—We have

$$\gamma_\beta x = x\Pi\Pi\left(1+\frac{x}{(m,\ n)}\right).$$

$$g_\beta x = \Pi\Pi\left(1+\frac{x}{(\overline{m},\ n)}\right),$$

the limits of n being $\pm q$, and those of m being $\pm p$, in the first case, and p, $-p-1$, in the second case. Also $\dfrac{p}{q}=\infty$.

We deduce immediately

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)=\left(x+\frac{\omega}{2}\right)\Pi\Pi\left\{1+\frac{\left(x+\frac{\omega}{2}\right)}{(m,\ n)}\right\}=\Pi\Pi\left(1+\frac{x}{(\overline{m},\ n)}\right)\div\frac{\omega}{2}\,\Pi\Pi\,\frac{(m,\ n)}{(\overline{m},\ n)}$$

(paying attention to the omission of $(m=0,\ n=0)$ in $\gamma_\beta x$, and supposing that this value enters into the numerator of the expression just obtained, but not into its denominator). This is of the form

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)=A\Pi\Pi\left(1+\frac{x}{(\overline{m},\ n)}\right);$$

but the limits are not the same in this product and in $g_\beta x$. In the latter m assumes the value $-p-1$, which it does not in the former; hence

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)\div g_\beta x = A\div\Pi_n\left(1+\frac{x}{-(p+\frac{1}{2})\,\omega+n\upsilon i}\right),$$

and the above product reduces itself to unity in consequence of all the values assumed by n being indefinitely small compared with the quantity $(p+\frac{1}{2})$; we have therefore

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)=Ag_\beta x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(65),$$

and similar expressions for the remaining functions. To illustrate this further, suppose we had been considering, instead of $\gamma_\beta x$, the function $\gamma_{-\beta} x$, given by the same formula, but with $\dfrac{p}{q}=0$, instead of $\dfrac{p}{q}=\infty$. We have in this case also

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right)\div g_{-\beta}\,x = A'\div\Pi_n\left(1+\frac{x}{(-p+\frac{1}{2})\,\omega+n\upsilon i}\right),$$

A' different from A on account of the different limits. The divisor of the second side takes the form

$$\{x-(p+\tfrac{1}{2})\,\omega\}\,\Pi\left(1+\frac{x-(p+\frac{1}{2})\,\omega}{n\upsilon i}\right)\div(-p+\tfrac{1}{2})\,\omega\Pi\left(1+\frac{(p+\frac{1}{2})\,\omega}{n\upsilon i}\right),$$

where the extreme values of n are infinite as compared with p. This may be reduced to

$$-\sin\frac{\pi}{\nu i}\{x-(p+\tfrac{1}{2})\,\omega\} \div \sin\,(p+\tfrac{1}{2})\,\omega,$$

$$= \epsilon^{\frac{\pi}{\nu}[(p+\frac{1}{2})\,\omega - x]} \div \epsilon^{\frac{\pi}{\nu}[(p+\frac{1}{2})\,\omega]} = \epsilon^{-\frac{\pi x}{\nu}},$$

neglecting the exponentials whose indices are infinitely great and negative. Observing the value of β this becomes $\epsilon^{-\omega\beta x}$, and we have

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right) = \epsilon^{\beta\omega x}\,.\,A'g_{-\beta}\,.\,x:$$

a result of the form of that which would be deduced from the equations $\gamma_{-\beta}\,x = \epsilon^{\beta x^2}\gamma_\beta\,x$, $g_{-\beta}x = \epsilon^{\beta x^2}g_\beta x$, $\gamma_\beta\left(x+\frac{\omega}{2}\right) = Ag_\beta\,x$. It is scarcely necessary to remark that $\gamma_{-\beta}\,x$ has the same relations to the change of x into $x+\frac{\nu i}{2}$ as $\gamma_\beta\,x$ has to that of x into $x+\frac{\omega}{2}$.