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NOTE ON A GEOMETRICAL THEOREM CONTAINED IN A PAPER BY SIR W. THOMSON.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 207, 208.]

It is easily shown that if three confocal surfaces of the second order pass through a point P, then the square of the distance of this point from the origin is equal to the sum of the squares of three of the axes, no two of which are parallel or belong to the same surface (the squares of one or two of the axes of the hyperboloids being considered negative); i.e. if

$$\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}+\frac{z^2}{c^2+h}=1,$$

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

$$\frac{x^2}{a^2+l}+\frac{y^2}{b^2+l}+\frac{z^2}{c^2+l}=1;$$

then

$$x^2+y^2+z^2=a^2+b^2+c^2+h+k+l.$$

In fact these equations give

$$x^2=\frac{(a^2+h)(a^2+k)(a^2+l)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2=\frac{(b^2+h)(b^2+k)(b^2+l)}{(b^2-a^2)(b^2-c^2)},$$

$$z^2=\frac{(c^2+h)(c^2+k)(c^2+l)}{(c^2-a^2)(c^2-b^2)},$$

and adding these and reducing, we have the relation in question; which is also immediately obtained by forming the cubic whose roots are h, k, l.

From this property may be deduced the theorem given by Mr Thomson in the preceding memoir ["On the Principal Axes of a Solid Body," pp. 127—133 and 195—206, see p. 205]. In fact, writing

$$r^2 = x^2 + y^2 + z^2,$$

and

$$k = -a^2 - b^2 - c^2 + h,$$

we see that in consequence of these relations the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{r^2+a^2-h} + \frac{y^2}{r^2+b^2-h} + \frac{z^2}{r^2+c^2-h} = 1,$$

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1,$$

are equivalent to two independent equations, i.e. the third can be deduced from the two first. Now the first equation is that of an ellipsoid (or generally a surface of the second order, since a, b, c are not necessarily real). The second is that of what may be called a conjugate equimomental surface, defining the term as follows: "The conjugate equimomental surfaces of an ellipsoid (or surface of the second order) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, are the equimomental surfaces derived in the usual manner from any surface of the second order $\frac{x^2}{h-a^2} + \frac{y^2}{h-b^2} + \frac{z^2}{h-c^2} = 1$, which is confocal with the *conjugate* surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ of the given ellipsoid," viz. by measuring along any line through the centre distances equal to the axes of the section by a plane through the centre perpendicular to this line, and taking the locus of the points so determined for the equimomental surface. The third equation is that of a surface confocal with the given ellipsoid; hence the theorem, "The curves of curvature of a given ellipsoid lie upon a system of conjugate equimomental surfaces."

But since the first and second equations are evidently satisfied by the combination of the first equation with the relation $r^2 = h$, which is that of a sphere, we have also, "The curve of intersection of the ellipsoid with any one of the conjugate equimomental surfaces, is composed of the line of curvature and a spherical conic." And these two curves being each of them of the fourth order make up the complete curve of intersection, which should obviously be of the eighth order.

It would be an interesting question to determine the relations existing between the curve of curvature and the spherical conic, which have been thus brought into connection by means of the conjugate equimomental surfaces; i.e. between the two curves obtained by combining the equation of the ellipsoid with

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1,$$

$$r^2 = a^2 + b^2 + c^2 + k,$$

respectively: but it will be sufficient at present to have suggested the problem.