39.

ON THE DIAMETRAL PLANES OF A SURFACE OF THE SECOND ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 274—278.]

LET $U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 0$, be the equation of a surface of the second order referred to its centre, and let $\alpha x + \alpha' y + \alpha'' z = 0$ be the equation of one of its diametral planes; then, as usual

$$\begin{aligned}(A-u)\,\alpha + \quad H\alpha' + \quad G\alpha'' &= 0,\\ H\alpha + (B-u)\,\alpha' + \quad F\alpha'' &= 0,\\ G\alpha + \quad F\alpha' + (C-u)\,\alpha'' &= 0,\end{aligned}$$

which are equivalent to two independent equations, and consequently capable of determining the ratios $\alpha : \alpha' : \alpha''$, provided that u satisfy the cubic equation that is obtained by eliminating α, α', α'' from the three equations.

We have from the second and third, from the third and first, and from the first and second equations respectively,

$$\alpha : \alpha' : \alpha'' = \mathfrak{A}_{,} : \mathfrak{H}_{,} : \mathfrak{G}_{,} = \mathfrak{H}_{,} : \mathfrak{B}_{,} : \mathfrak{F}_{,} = \mathfrak{G}_{,} : \mathfrak{F}_{,} : \mathfrak{C}_{,};$$

where, if

$$\begin{aligned}\mathfrak{A} &= BC - F^2,\\ \mathfrak{B} &= CA - G^2,\\ \mathfrak{C} &= AB - H^2,\\ \mathfrak{F} &= GH - AF,\\ \mathfrak{G} &= HF - BG,\\ \mathfrak{H} &= FG - CH,\end{aligned}$$

$\mathfrak{A}_{,}$, $\mathfrak{B}_{,}$, $\mathfrak{C}_{,}$, $\mathfrak{F}_{,}$, $\mathfrak{G}_{,}$, $\mathfrak{H}_{,}$ are what these become when A, B, C are changed into

$A-u$, $B-u$, $C-u$, so that

$$\begin{aligned}\mathfrak{A}_{,} &= \mathfrak{A} - (B+C)\,u + u^2,\\ \mathfrak{B}_{,} &= \mathfrak{B} - (C+A)\,u + u^2,\\ \mathfrak{C}_{,} &= \mathfrak{C} - (A+B)\,u + u^2,\\ \mathfrak{F}_{,} &= \mathfrak{F} + Fu,\\ \mathfrak{G}_{,} &= \mathfrak{G} + Gu,\\ \mathfrak{H}_{,} &= \mathfrak{H} + Hu.\end{aligned}$$

Hence the equation $\alpha x + \alpha' y + \alpha'' z = 0$ may be written in the three forms

$$\begin{aligned}\mathfrak{A}_{,}x + \mathfrak{H}_{,}y + \mathfrak{G}_{,}z &= 0,\\ \mathfrak{H}_{,}x + \mathfrak{B}_{,}y + \mathfrak{F}_{,}z &= 0,\\ \mathfrak{G}_{,}x + \mathfrak{F}_{,}y + \mathfrak{C}_{,}z &= 0;\end{aligned}$$

or, what comes to the same thing, as follows,

$$\begin{aligned}\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z + u\,(Ax + Hy + Gz) + vx &= 0,\\ \mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z + u\,(Hx + By + Fz) + vy &= 0,\\ \mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z + u\,(Gx + Fy + Cz) + vz &= 0,\end{aligned}$$

in which for shortness v has been written instead of

$$u^2 - (A+B+C)\,u.$$

The elimination of u, v from these equations gives a result $\Theta = 0$, where Θ is a homogeneous function of the third order in x, y, z; and this equation, it is evident, must belong to the three diametral planes jointly, i.e. Θ must be the product of three linear factors, each of which equated to zero would correspond to a diametral plane. Thus the system of diametral planes is given by

$$\Theta = \begin{vmatrix}\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z, & Ax + Hy + Gz, & x\\ \mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z, & Hx + By + Fz, & y\\ \mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z, & Gx + Fy + Cz, & z\end{vmatrix} = 0,$$

or developing the determinant, as follows,

$$\begin{aligned}\Theta = {} & (G\mathfrak{H} - H\mathfrak{G})\,x^3 + (H\mathfrak{F} - F\mathfrak{H})\,y^3 + (F\mathfrak{G} - G\mathfrak{F})\,z^3\\ & + \{\ \ G\,(\mathfrak{C} - \mathfrak{B}) - \mathfrak{G}\,(C - B) - (H\mathfrak{F} - F\mathfrak{H})\}\,yz^2\\ & + \{\ \ H\,(\mathfrak{A} - \mathfrak{C}) - \mathfrak{H}\,(A - C) - (F\mathfrak{G} - G\mathfrak{F})\}\,zx^2\\ & + \{\ \ F\,(\mathfrak{B} - \mathfrak{A}) - \mathfrak{F}\,(B - A) - (G\mathfrak{H} - H\mathfrak{G})\}\,xy^2\\ & + \{-H\,(\mathfrak{C} - \mathfrak{B}) + \mathfrak{H}\,(C - B) + (F\mathfrak{G} - G\mathfrak{F})\}\,y^2z\\ & + \{-F\,(\mathfrak{A} - \mathfrak{C}) + \mathfrak{F}\,(A - C) + (G\mathfrak{H} - H\mathfrak{G})\}\,z^2x\\ & + \{-G\,(\mathfrak{B} - \mathfrak{A}) + \mathfrak{C}\,(B - A) + (H\mathfrak{F} - F\mathfrak{H})\}\,x^2y\\ & + (C\mathfrak{B} - B\mathfrak{A} + \mathfrak{A}\mathfrak{C} - C\mathfrak{A} + B\mathfrak{A} - A\mathfrak{B})\,xyz;\end{aligned}$$

or reducing

$$\begin{aligned}\Theta = \; & \{F(G^2-H^2)-GH(C-B)\}\,x^3\\ & +\{G(H^2-F^2)-HF(A-C)\}\,y^3\\ & +\{H(F^2-G^2)-FG(B-A)\}\,z^3\\ & +\{G(A-B)(B-C)+FH(A+B-2C)+G(F^2+G^2-2H^2)\}\,yz^2\\ & +\{H(B-C)(C-A)+GF(B+C-2A)+H(G^2+H^2-2F^2)\}\,zx^2\\ & +\{F(C-A)(A-B)+GH(C+A-2B)+F(H^2+F^2-2G^2)\}\,xy^2\\ & +\{H(B-C)(C-A)+FG(C+A-2B)+H(H^2+F^2-2G^2)\}\,y^2z\\ & +\{F(C-A)(A-B)+GH(A+B-2C)+F(F^2+G^2-2H^2)\}\,z^2x\\ & +\{G(A-B)(B-C)+HF(B+C-2A)+G(G^2+H^2-2F^2)\}\,x^2y\\ & -\{(A-B)(B-C)(C-A)+(B-C)F^2+(C-A)G^2+(A-B)H^2\}\,xyz.\end{aligned}$$

In the case of *curves* of the second order, the result is much more simple; we have

$$\Theta = \begin{vmatrix} Ax+Hy, & x \\ Hx+By, & y \end{vmatrix} = 0,$$

i.e. $$\Theta = H(y^2-x^2)+(A-B)\,xy = 0,$$

for the equation of the two diameters.

The above formulæ may be applied to the question of finding the diametral planes of the cone circumscribed about a given surface of the second order, (or of the lines bisecting the angles made by two tangents of a curve of the second order). Considering the latter question first: if

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

be the equation of the curve, and α, β the coordinates of the point of intersection of the two tangents, the equation of the pair of tangents is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2=0;$$

or making the point of intersection the origin,

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}\right)^2=0,$$

i.e. $$(\beta x-\alpha y)^2-(b^2x^2+a^2y^2)=0;$$

whence $A=\beta^2-b^2$, $B=\alpha^2-a^2$, $H=-\alpha\beta$, and the equation to the lines bisecting the angles formed by the tangents is

$$\alpha\beta\,(x^2-y^2)-\{\alpha^2-\beta^2-(a^2-b^2)\}\,xy=0,$$

which is the same for all confocal ellipses; whence the known theorem,

"If there be two confocal ellipses, and tangents be drawn to the second from any point P of the first, the tangent and normal of the first conic at the point P, bisect the angles formed by the two tangents in question."

In the case of surfaces, the equation of the circumscribing cone referred to its vertex as origin, is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}\right)^2=0\,;$$

whence

$$\begin{aligned} A &= \beta^2c^2+\gamma^2b^2-b^2c^2,\\ B &= \gamma^2a^2+\alpha^2c^2-a^2c^2,\\ C &= \alpha^2b^2\ +\beta^2a^2-b^2a^2,\\ F &= -\,a^2\beta\gamma,\\ G &= -\,b^2\gamma\alpha\,,\\ H &= -\,c^2\alpha\beta\,. \end{aligned}$$

Hence, omitting the factor $b^2c^2\alpha^2+c^2a^2\beta^2+a^2b^2\gamma^2-a^2b^2c^2$, we have

$$\begin{aligned} \mathfrak{A} &= \alpha^2-a^2,\\ \mathfrak{B} &= \beta^2-b^2,\\ \mathfrak{C} &= \gamma^2-c^2\,,\\ \mathfrak{F} &= \beta\gamma,\\ \mathfrak{G} &= \gamma\alpha\,,\\ \mathfrak{H} &= \alpha\beta\,; \end{aligned}$$

and the equation of the system of diametral planes becomes

$$\begin{aligned} \Theta=0= \ & \alpha^2\beta\gamma\,(c^2-b^2)\,x^3+\beta^2\gamma\alpha\,(a^2-c^2)\,y^3+\gamma^2\alpha\beta\,(b^2-a^2)\,z^3\\ &+\ \gamma\alpha\,\{\alpha^2\,(c^2-b^2)+\beta^2\,(b^2+c^2-2a^2)-\gamma^2\,(b^2-a^2)+(b^2-a^2)\,(c^2-b^2)\}\,yz^2\\ &+\ \alpha\beta\,\{-\,\alpha^2\,(c^2-b^2)+\beta^2\,(a^2-c^2)+\gamma^2\,(c^2+a^2-2b^2)+(c^2-b^2)(a^2-c^2)\}\,zx^2\\ &+\ \gamma\alpha\,\{\alpha^2\,(a^2+b^2-2c^2)-\beta^2\,(a^2-c^2)+\gamma^2\,(b^2-a^2)+(a^2-c^2)\,(b^2-a^2)\}\,xy^2\\ &-\ \alpha\beta\,\{\alpha^2\,(c^2-b^2)-\beta^2\,(a^2-c^2)-\gamma^2\,(b^2+c^2-2a^2)-(a^2-c^2)\,(c^2-b^2)\}\,y^2z\\ &-\ \beta\gamma\,\{-\,\alpha^2\,(c^2+a^2-2b^2)+\beta^2\,(a^2-c^2)-\gamma^2\,(b^2-a^2)-(b^2-a^2)\,(a^2-c^2)\}\,z^2x\\ &-\ \gamma\alpha\,\{-\,\alpha^2\,(c^2-b^2)-\beta^2\,(a^2+b^2-2c^2)+\gamma^2\,(b^2-a^2)-(c^2-b^2)\,(b^2-a^2)\}\,x^2y\\ &+\{(a^2-b^2)\,(b^2-c^2)\,(c^2-a^2)\,+\\ &\quad(\alpha^4+\beta^2\gamma^2)\,(b^2-c^2)-(\beta^4+\gamma^2\alpha^2)\,(c^2-a^2)-(\gamma^4+\alpha^2\beta^2)\,(a^2-b^2)\,+\\ &\quad\alpha^2\,(b^2-c^2)\,(2a^2-b^2-c^2)+\beta^2\,(c^2-a^2)\,(2b^2-c^2-a^2)+\gamma^2\,(a^2-b^2)\,(2c^2-a^2-b^2)\}\,xyz\,; \end{aligned}$$

and since this is a function of a^2-b^2, b^2-c^2, and c^2-a^2, the equation is the same for all confocal ellipsoids; whence the known theorem, "The axes of the circumscribing cone having its vertex in a given point P, are tangents to the curves of intersection of the three surfaces, confocal with the given surface, which pass through the point P."