41.

ON CERTAIN FORMULÆ FOR DIFFERENTIATION WITH APPLICATIONS TO THE EVALUATION OF DEFINITE INTEGRALS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 122—128.]

In attempting to investigate a formula in the theory of multiple definite integrals (which will be noticed in the sequel), I was led to the question of determining the $(i+1)^{\text{th}}$ differential coefficient of the $2i^{\text{th}}$ power of $\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}$; the only way that occurred for effecting this was to find the successive differential coefficients of this quantity, which may be effected as follows. Assume

$$U_{k,i}=\{(x+\lambda)(x+\mu)\}^{\frac{1}{2}k}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i},$$

then

$$\frac{1}{U_{k,i}}\frac{d}{dx}U_{k,i}=\tfrac{1}{2}k\,\frac{2x+\lambda+\mu}{(x+\lambda)(x+\mu)}-\frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}$$

$$=\tfrac{1}{2}k\,\frac{\{\sqrt{(x+\lambda)}+\sqrt{(x+\mu)}\}^2-2\sqrt{\{(x+\lambda)(x+\mu)\}}}{(x+\lambda)(x+\mu)}-\frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}$$

$$=\tfrac{1}{2}k\,\frac{(\lambda-\mu)^2}{\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^2(x+\lambda)(x+\mu)}-\frac{k+i}{\sqrt{\{(x+\lambda)(x+\mu)\}}};$$

or, attending to the signification of $U_{k,i}$,

$$\frac{d}{dx}U_{k,i}=\tfrac{1}{2}k(\lambda-\mu)^2U_{k-2,i-1}-(k+i)U_{k-1,i}.$$

Hence

$$-\frac{1}{i}\frac{d}{dx}U_{0,i}=U_{-1,i}$$

$$\frac{1}{i}\frac{d^2}{dx^2}U_{0,i}=\tfrac{1}{2}(\lambda-\mu)^2U_{-3,i-1}+(i-1)U_{-2,i},$$

&c.

from which the law is easily seen to be of the form

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i} = S_\theta K_{r,\theta}(\lambda-\mu)^{2r-2-2\theta} U_{-2r+1+\theta,i-r+1+\theta}$$

(where the extreme values of θ are 0 and $(r-1)$ respectively) and $K_{r,\theta}$ is determined by

$$K_{r+1,\theta+1} = (r-1-\tfrac{1}{2}\theta) K_{r,\theta+1} + (i-3r+2+2\theta) K_{r,\theta}.$$

This equation is satisfied by

$$K_{r,\theta} = \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma\frac{1}{2}\,\Gamma(\theta+1)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)};$$

for in the first place this gives

$$(r-1-\tfrac{1}{2}\theta) K_{r,\theta+1} = \frac{(r-1-\frac{1}{2}\theta)}{\Gamma\frac{1}{2}\,\Gamma(\theta+2)}\,\frac{\Gamma(r-\frac{3}{2}-\theta)\,\Gamma(2r-2-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(2r-3-2\theta)\,\Gamma(i-r+1)}$$

$$= \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-2-2\theta)\,\Gamma(i-r+1)},$$

and hence the second side of the equation reduces itself to

$$\frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)}\{2(r-1-\theta)(i-r+\theta+1)+(\theta+1)(i-3r+2-2\theta)\},$$

where the quantity within brackets reduces itself to $(i-r)(2r-1-\theta)$, so that the above value reduces itself to $K_{r+1,\theta+1}$, which verifies the equation in question. Also by comparing the first few terms, it is immediately seen that the above is the correct value of $K_{r,\theta}$, so that

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i} = S_\theta \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+1)}(\lambda-\mu)^{2r-2-\theta} U_{-2r+1+\theta,i-r+\theta+1} \;\ldots(1),$$

θ extending as before from 0 to $(r-1)$. In particular if i be integer and $r=i+1$,

$$\frac{(-)^{i+1}}{i}\left(\frac{d}{dx}\right)^{i+1}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i} = \frac{\Gamma(i+\frac{1}{2})}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i}\frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}} \;\ldots\ldots\ldots(2),$$

(since the factor $\Gamma(i-r+\theta+1) \div \Gamma(i-r+1)$ vanishes except for $\theta=0$ on account of $\Gamma(i-r+1)=\infty$). Thus also, if r be greater than $(i+1)$, $=i+1+s$ suppose, then

$$(-)^s\left(\frac{d}{dx}\right)^s\frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}}$$

$$= S_\theta \frac{\Gamma(i+s+\frac{1}{2}-\theta)\,\Gamma(2i+2s+1-\theta)\,\Gamma(\theta-s)}{\Gamma(i+\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2i+2s+1-2\theta)\,\Gamma(-s)}(\lambda-\mu)^{2s-2\theta} U_{-2i-2s-1+\theta,-s+\theta} \;\ldots\ldots(3),$$

where θ extends only from $\theta=0$ to $\theta=s$, on account of the factor $\Gamma(\theta-s) \div \Gamma(-s)$, which vanishes for greater values of θ: a rather better form is obtained by replacing this factor by

$$(-)^\theta \frac{\Gamma(1+s)}{\Gamma(1+s-\theta)}.$$

The above formulæ have been deduced on the supposition of i being an integer; assuming that they hold generally, the equation (2) gives, by writing $(i-\frac{1}{2})$ for i,

$$\frac{(-)^{i+\frac{1}{2}}}{i-\frac{1}{2}}\left(\frac{d}{dx}\right)^{i+\frac{1}{2}} \{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i-1} = \frac{\Gamma i}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i-1}\frac{1}{\{(x+\lambda)(x+\mu)\}^i},$$

or integrating $(i+\frac{1}{2})$ times by means of the formula

$$\int_0^\infty x^{i-\frac{1}{2}} fx\, dx = \frac{\Gamma(i+\frac{1}{2})}{(-)^{i+\frac{1}{2}}}\left(\int_\infty d\alpha\right)^{i+\frac{1}{2}} f\alpha, \quad \alpha=0;$$

this gives

$$\int_0^\infty \frac{x^{i-\frac{1}{2}} dx}{\{(x+\lambda)(x+\mu)\}^i} = \frac{\Gamma\frac{1}{2}\Gamma(i-\frac{1}{2})}{\Gamma i}\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i-1}} \text{ ....(4),}$$ [1]

whence also

$$\int_0^\infty \frac{x^{i-\frac{1}{2}} dx}{(x+\lambda)^{i+1}(x+\mu)^i} = \frac{\Gamma\frac{1}{2}\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i}\sqrt{\lambda}} \text{ ....(5);}$$

and from these, by simple transformations,

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{1}{2}} dx}{\{(\alpha-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\frac{(\alpha-\beta)^i}{(\sqrt{m}+1)^{2i}} \text{ ....(6),}$$

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{3}{2}} dx}{\{(\alpha-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\Gamma(i-\frac{1}{2})}{\Gamma i}\frac{(\alpha-\beta)^{i-1}}{(\sqrt{m}+1)^{2i-1}} \text{ ....(7).}$$

These last two formulæ are connected also by the following general property:

"If $$(a, b, i) = \int_\beta^\alpha \frac{(\alpha-x)^{a-1}(x-\beta)^{b-1} dx}{\{(\alpha-x)+m(x-\beta)\}^i},$$

then $$(a, b, i) = \frac{\Gamma a \Gamma b}{\Gamma(a+b-i)\Gamma i}(\alpha-\beta)^{b-i}(a+b-i, i, \beta)\text{" ....(8),}$$

which I have proved by means of a multiple[2] integral. From (6) we may obtain for $\gamma<1$,

$$\int_{-1}^1 \frac{(1-x^2)^{i-\frac{1}{2}} dx}{(1-2\gamma x+\gamma^2)^i} = \frac{\Gamma(\frac{1}{2})\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} \text{ ....(9),}$$

[1] This is immediately transformed into

$$\int_0^\infty \frac{x^{i-\frac{1}{2}} dx}{(ax^2+bx+c)^i} = \frac{\Gamma\frac{1}{2}\Gamma(i-\frac{1}{2})}{\Gamma i}\frac{1}{\{b+2\sqrt{(ac)}\}^{i-\frac{1}{2}}},$$

which is a particular case of a formula which will be demonstrated in a subsequent paper. [I am not sure to what this refers.]

[2] [The triple integral $\iiint u^{i-1} x^{a-1} y^{\beta-1} e^{-(x+my)u-x-y}\, dx\, dy\, du$.]

which however is only a particular case of

$$\int_{-1}^{1} dx\,(1-x^2)^{i-\frac{1}{2}}\,(1-2\,\gamma x+\gamma^2)^{-i}\,\frac{d}{d\beta}\left[\beta^i\left(1-2\frac{\beta}{\gamma}x+\frac{\beta^2}{\gamma^2}\right)^{-i}\right]$$

$$=\frac{\Gamma\tfrac{1}{2}\,\Gamma(i+\tfrac{1}{2})}{\Gamma(i+1)}\,\beta^{i-1}(1-\beta)^{-2i}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

which supposes γ and $\dfrac{\beta}{\gamma}$ each less than unity. This formula was obtained in the case of $(i+\frac{1}{2})$ an integer, from a theorem, *Leg. Cal. Int.*, tom. II. p. 258, but there is no doubt that it is generally true.

From (9), by writing $x=\cos\theta$, we have

$$\int_0^\pi \frac{\sin^{2i}\theta\,d\theta}{(1-2\gamma\cos\theta+\gamma^2)^i}=\frac{\Gamma\tfrac{1}{2}\,\Gamma(i+\tfrac{1}{2})}{\Gamma(i+1)}\ \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

which may also be demonstrated by the common equation in the theory of elliptic functions $\sin(\phi-\theta)=\gamma\sin\phi$, as was pointed out to me by Mr [Sir W.] Thomson. It may be compared with the following formula of Jacobi's, *Crelle*, tom. XV. [1836] p. 7,

$$\int_0^\pi \frac{\sin^{2i-1}\theta\,d\theta}{(1-2\gamma\cos\theta+\gamma^2)^i}=\frac{1}{\Gamma(i+\frac{1}{2})}\int_0^\pi \frac{\cos(i-\frac{1}{2})\,\theta\,d\theta}{\sqrt{(1-2\gamma\cos\theta+\gamma^2)}}\ \ldots\ldots\ldots(12).$$

Consider the multiple integral

$$W=\int\frac{dx\,dy\ldots}{\{(x-a)^2+\ldots u^2\}^i}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

the number of variables being $(2i+1)$ (not necessarily odd), and the equation of the limits being

$$x^2+y^2\ldots=\xi\,;$$

then, as will presently be shown, W may be expanded in the form

$$W=\pi^{i+\frac{1}{2}}\,S_\lambda\,\frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}\left(\frac{d}{du}\right)^{2\lambda}\int_0 \xi^{i-\frac{1}{2}}\,(\xi+u^2)^{-i}d\xi\ \ldots\ldots(14),$$

where $A=a^2+b^2+\ldots\ldots$ and λ extends from 0 to ∞. Suppose next

$$V=\int\frac{dx\,dy\ldots}{\{(x-a)^2\ldots+u^2\}^i\,(x^2+\ldots v^2)^{i+1}}\ldots\ldots\ldots\ldots\ldots\ldots(15):$$

the number of variables as before, and the limits for each variable being $-\infty$, ∞. We have immediately

$$V=\int_0^\infty\frac{1}{(\xi+v^2)^{i+1}}\,\frac{dW}{d\xi}\,d\xi\,;$$

W as before, i.e.

$$V=\pi^{i+\frac{1}{2}}\,S_\lambda\,\frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}\left(\frac{d}{du}\right)^{2\lambda}\int_0^\infty\frac{\xi^{i-\frac{1}{2}}\,d\xi}{(\xi+u^2)^i\,(\xi+v^2)^{i+1}}\,.$$

But writing u^2, v^2 for λ, μ in the formula (5) (u and v being supposed positive), the integral in this formula is

$$\frac{\sqrt{\pi}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\frac{1}{v\,(u+v)^{2i}};$$

hence, after a slight reduction,

$$V=\frac{\pi^{i+1}}{v\Gamma(i+1)}\,S\,\frac{(-)^\lambda\,\Gamma(i+\lambda+1)}{\Gamma(i+1)\,\Gamma(\lambda+1)}\,\frac{A^\lambda}{\{(u+v)^2\}^\lambda};$$

or finally $$V=\frac{\pi^{i+1}}{\Gamma(i+1)}\,\frac{1}{v\,\{(u+v)^2+A\}^i}\;\text{..........................(16)},$$

a remarkable formula, the discovery of which is due to Mr Thomson. It only remains to prove the formula for W. Out of the variety of ways in which this may be accomplished, the following is a tolerably simple one. In the first place, by a linear transformation corresponding to that between two sets of rectangular axes, we have

$$W=\int\frac{dx\,dy\,\ldots}{\{(x-\sqrt{A})^2+y^2\ldots+u^2\}^i};$$

or expanding in powers of A, and putting for shortness $R=x^2+y^2\ldots+u^2$, the general term of W is

$$(-)^\sigma A^\lambda\,\frac{\Gamma(i+\lambda+\sigma)}{\Gamma i\,\Gamma(\lambda-\sigma+1)\,\Gamma(2\sigma+1)}\,2^{2\sigma}\int x^{2\sigma}\,R^{-i-\lambda-\sigma}\,dx\,dy\,\ldots$$

the limits being as before $x^2+y^2+\ldots=\xi$. To effect the integrations, write $\sqrt{\xi}\,\sqrt{x}$, $\sqrt{\xi}\,\sqrt{y}$, &c. for x, y ... so that the equation of the limits becomes $x+y+\ldots=1$. Also restricting the integral to positive values, we must multiply it by 2^{2i+1}: the integral thus becomes

$$\xi^{\sigma+i+\frac{1}{2}}\int x^{\sigma-\frac{1}{2}}\,y^{-\frac{1}{2}}\ldots\{\xi\,(x+y\,\ldots)+u^2\}^{-i-\lambda-\sigma}\,dx\,dy\,\ldots$$

equivalent to

$$\xi^{\sigma+i+\frac{1}{2}}\,\frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})}\int_0^1\theta^{i+\sigma-\frac{1}{2}}\,(\xi\theta+u^2)^{-i-\lambda-\sigma}\,d\theta;$$

i.e. to $$\frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})}\int_0\xi^{i+\sigma-\frac{1}{2}}\,(\xi+u^2)^{-i-\lambda-\sigma}\,d\xi.$$

Hence, after a slight reduction, the general term of W is

$$\frac{\pi^{i+\frac{1}{2}}}{\Gamma i}\,(-)^{\lambda+\sigma}A^\lambda\,\frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})}\int_0\xi^{i+\sigma-\frac{1}{2}}\,(\xi+u^2)^{-i-\lambda-\sigma}\,d\xi,$$

where σ may be considered as extending from 0 to λ inclusively, and then λ from 0 to ∞. But by a formula easily proved

$$\left(\frac{d}{du}\right)^{2\lambda}(\xi+u^2)^{-1}=\frac{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}{\Gamma i}\times$$

$$S\,(-)^\sigma\,\frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})}\,\xi^\sigma\,(\xi+u^2)^{-i-\lambda-\sigma},$$

where σ extends from 0 to λ. Hence, substituting and prefixing the summatory sign,

$$W = \pi^{i+\frac{1}{2}} S \frac{(-)^\lambda A^\lambda}{2^{2\lambda}\Gamma(\lambda+1)\Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0 \xi^{i-\frac{1}{2}} (\xi + u^2)\, d\xi,$$

where λ extends from 0 to ∞, the formula required.

[I annex the following Note added in MS. in my copy of the *Journal*, and referring to the formula, *ante* p. 267; a is written to denote $\lambda - \mu$.

N.B.—It would be worth while to find the general differential coefficient of $U_{k,i}$.

$$\partial_x U_{k,i} = -(k+i)\, U_{k-1,i} + \tfrac{1}{2} k a^2\, U_{k-2,i-1},$$

from which it is easy to see that

$$\begin{aligned} \partial_x^r U_{k,i} = (-)^r \quad & K_{r,0} \quad U_{k-r,i} \\ & \vdots \\ + (-)^{r-\theta} & K_{r,\theta}\, a^{2\theta}\, U_{k-r-\theta,\, i-\theta} \\ & \vdots \\ + \quad & K_{r,r}\, a^{2r}\, U_{k-2r,\, i-r}. \end{aligned}$$

The general term of $\partial_x^{r+1} U_{k,i}$ is

$$\begin{aligned} & (-)^{r-\theta} \quad K_{r,\theta}\, a^{2\theta} \left[\tfrac{1}{2}(k-r-\theta)\, a^2 U_{k-r-\theta-2,\, i-\theta-1}\right] \\ & + (-)^{r-\theta-1} K_{r+\theta+1}\, a^{2\theta+2} \left[-(k+i-2\theta-2-r)\, U_{k-r-\theta-2,\, i-\theta-1}\right], \end{aligned}$$

which must be equal to

$$(-)^{r-\theta} \quad K_{r+1,\theta+1}\, a^{2\theta+2}\, U_{k-r-\theta-2,\, i-\theta-1},$$

therefore

$$K_{r+1,\theta+1} = (k+i-r-2\theta-2)\, K_{r,\theta+1} + \tfrac{1}{2}(k-r-\theta)\, K_{r,\theta}.$$

In particular

$$\begin{aligned} & K_{r+1,0} \quad -(k+i-r) \quad K_{r,0} = 0, \\ & K_{r+1,1} \quad -(k+i-r-2)\, K_{r,1} = \tfrac{1}{2}(k-r)\, K_{r,0}, \\ & \vdots \\ & K_{r+1,r+1} - (\tfrac{1}{2}k - r) \quad K_{r,r} = 0, \end{aligned}$$

whence

$$\begin{aligned} K_{r,0} &= [k+i]^r \\ K_{r,1} &= \tfrac{1}{2} r \left\{k^2 + (i-r)\, k - \tfrac{1}{2}(r-1)\, i\right\} [k+i-2]^{r-2}, \\ & \vdots \\ K_{r,r} &= [\tfrac{1}{2}k]^r, \end{aligned}$$

which appears to indicate a complicated general law.

Even the verification of $K_{r,1}$ is long, thus the equation becomes

$$\overline{r+1}\, [k+i-2]^{r-1} \left\{k^2 + (i-r-1)\, k - \tfrac{1}{2} ri\right\} - (k+i-r-2)\, r \left\{k^2 + (i-r)\, k - \tfrac{1}{2}(r-1)\, i\right\} [k+i-2]^{r-2} = (k-r)\, [k+i]^r,$$

or

$$\overline{r+1}\, (k+i-r) \left\{k^2 + (i-r-1)\, k - \tfrac{1}{2} ri\right\} - r\, (k+i-r-2) \left\{k^2 + (i-r)\, k - \tfrac{1}{2}(r-1)\, i\right\} = (k-r)(k+i)(k+i-1),$$

which is identical, as may be most easily seen by taking first the coefficient of k^3, and then writing $k=r$, $k=-i$, $k=-i-1$.]