44.

ON A MULTIPLE INTEGRAL CONNECTED WITH THE THEORY OF ATTRACTIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 219—223.]

MR BOOLE [in the Memoir "On a Certain Multiple definite Integral" Irish Acad. Trans. vol. XXI. (1848), pp. 40—150] has given for the integral with n variables

$$V = \int \frac{\phi\left(\frac{x^2}{f^2} + \frac{y^2}{g^2} + \dots\right) dx\, dy \dots}{[(a-x)^2 + (b-y)^2 \dots + u^2]^{\frac{1}{2}n+q}} \quad \dots\dots\dots\dots\dots\dots(1);$$

limits

$$\frac{x^2}{f^2} + \frac{y^2}{g^2} + \dots \overset{=}{<} 1,$$

the following formula, or one which may readily be reduced to that form[1],

$$V = \frac{fg \dots \pi^{\frac{1}{2}n}}{\Gamma\left(\frac{1}{2}n + q\right)} \int_\eta^\infty \frac{S s^{-q-1}\, ds}{\sqrt{\{(s+f^2)(s+g^2)\dots\}}} \quad \dots\dots\dots\dots\dots\dots(2),$$

where

$$S = \frac{(1-\sigma)^{-q}}{\Gamma(-q)} \int_0^1 t^{-q-1} \phi\{\sigma + t(1-\sigma)\}\, dt \quad \dots\dots\dots\dots\dots\dots(3);$$

in which

$$\sigma = \frac{a^2}{f^2+s} + \frac{b^2}{g^2+s} \dots + \frac{u^2}{s} \quad \dots\dots\dots\dots\dots\dots(4)$$

and η is determined by

$$1 = \frac{a^2}{f^2+\eta} + \frac{b^2}{g^2+\eta} \dots + \frac{u^2}{\eta}.$$

[1] See note at the end of this paper.

Suppose $f=g=\ldots=\infty$; also assume

$$\phi(\lambda)=\frac{1}{(f^2\lambda+v^2)^{\frac{1}{2}n+q'}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

then the integral becomes

$$U=\int\frac{dx\,dy\ldots}{(x^2+y^2\ldots+v^2)^{\frac{1}{2}n+q'}\{(x-a)^2+(y-b^2)\ldots+u^2\}^{\frac{1}{2}n+q}} \quad\ldots\ldots\ldots\ldots(6),$$

the limits for each variable being $-\infty$, ∞.

Now, writing f^2s for s and $f^2\eta$ for η, the new value of η reduces itself to zero, and

$$U=\frac{f^{-2q}\,\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)}\int_0^\infty\frac{Sds}{(1+s)^{\frac{1}{2}n}}.$$

Also $\sigma=0$; but

$$f^2\sigma=\frac{r^2}{1+s}+\frac{u^2}{s},$$

where $r^2=a^2+b^2+\ldots$ whence also putting $\frac{t}{f^2}$ for t, $\phi\{\sigma+t(1-\sigma)\}$ becomes

$$\frac{1}{\{f^2\sigma+t(1-\sigma)+v^2\}^{\frac{1}{2}n+q'}};$$

i.e.

$$\frac{1}{(t+A)^{\frac{1}{2}n+q'}},$$

if for a moment

$$A=\frac{r^2}{1+s}+\frac{u^2}{s}+v^2.$$

Hence

$$S=\frac{f^{2q}}{\Gamma(-q)}\int_0^\infty\frac{t^{-q-1}\,dt}{(t+A)^{\frac{1}{2}n+q'}}$$

$$=\frac{\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q')}\,\frac{f^{2q}}{A^{\frac{1}{2}n+q+q'}};$$

or substituting in U, and replacing A by its value,

$$U=\frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n+q')}\int_0^\infty\frac{s^{-q-1}\,ds}{(1+s)^{\frac{1}{2}n}\left(\frac{r^2}{1+s}+\frac{u^2}{s}+v^2\right)^{\frac{1}{2}n+q+q'}};$$

or, what comes to the same,

$$U=\frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n+q')}\int_0^\infty\frac{s^{\frac{1}{2}n+q'-1}\,(1+s)^{q+q'}\,ds}{(v^2s^2+js+u^2)^{\frac{1}{2}n+q+q'}} \quad\ldots\ldots\ldots\ldots(7)$$

where

$$j=u^2+v^2+r^2.$$

The only practicable case is that of $q' = -q$, for which

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)} \int_0^\infty \frac{s^{\frac{1}{2}n-q-1}\,ds}{(v^2s^2+js+u^2)^{\frac{1}{2}n}} \quad\text{.........................(8).}$$

Consider the more general expression

$$\Theta = \int_0^\infty s^{-q-1}\,\phi\left(\frac{v^2s^2+js+u^2}{s}\right) ds \quad\text{..............................(9);}$$

by writing

$$2u\sqrt{s} = \sqrt{(s'+4uv)} \pm \sqrt{s'},$$

the upper sign from $s=\infty$ to $s=\frac{u}{v}$, and the lower one from $s=\frac{u}{v}$ to $s=0$, it is easy to derive

$$\Theta = (2v)^{2q} \int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,\phi(s+j+2uv)\,ds \quad\text{...... ..(10).}$$

Now, by a formula which will presently be demonstrated,

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,e^{-\theta s}\,ds$$

$$= \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\,\theta^{-q} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,e^{-\theta s}\,ds \quad\text{..................(11);}$$

whence

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,Fs\,.\,ds$$

$$= \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q} Fs\,.\,ds \quad\text{........(12).}$$

Thus, by merely changing the function,

$$\Theta = \frac{2^{2q+1}\,v^{2q}\,\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q} \phi(s+j+2uv)\,ds \quad\text{........(13);}$$

and hence in the particular case in question

$$U = \frac{2^{2q+1}\,v^{2q}\,\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,(s+j+2uv)^{-\frac{1}{2}n+q}\,ds \quad\text{........(14),}$$

by means of the formula

$$\left(-\frac{d}{ds}\right)^{-q}(s+\alpha)^{-\frac{1}{2}n} = \frac{\Gamma(\frac{1}{2}n-q)}{\Gamma(\frac{1}{2}n)}\,(s+\alpha)^{-\frac{1}{2}n+q}.$$

But as there may be some doubt about this formula, which is not exactly equivalent either to Liouville's or Peacock's expression for the general differential coefficient of a power, it is worth while to remark that, by first transforming the $\frac{1}{2}n^{\text{th}}$ power into an exponential, and then reducing as above (thus avoiding the general differentiation), we should have obtained

$$U = \frac{2^{2q+1} v^{2q} \pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)} \int_0^\infty d\theta \int_0^\infty ds\, \theta^{\frac{1}{2}n-q-1} e^{-\theta(s+j+2uv)} s^{-\frac{1}{2}-q} (s+4uv)^{-\frac{1}{2}-q} e^{-\theta s},$$

which reduces itself to the equation (14) by simply performing the integration with respect to θ; thus establishing the formula beyond doubt[1]. The integral may evidently be effected in finite terms when either q or $q-\frac{1}{2}$ is integral. Thus for instance in the simplest case of all, or when $q = -\frac{1}{2}$,

$$U = \frac{\pi^{\frac{1}{2}(n-1)}}{v\Gamma\frac{1}{2}(n+1)} \frac{1}{(j+2uv)^{\frac{1}{2}(n-1)}} = \int_{-\infty}^{\infty} \frac{dx\, dy \ldots}{(x^2+y^2 \ldots + v^2)^{\frac{1}{2}(n+1)} \{(x-a)^2 + \ldots u^2\}^{\frac{1}{2}(n-1)}},$$

a formula of which several demonstrations have already been given in the *Journal*.

The following is a demonstration, though an indirect one, of the formula (11): in the first place

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q} + \{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}} e^{-\theta s} ds$$

$$= \frac{2\Gamma(\frac{1}{2}-q)\,\theta^q e^{2uv\theta}}{\sqrt{\pi}\,(4uv)^{2q}} \int_0^\infty (4u^2v^2+x^2)^{q-\frac{1}{2}} e^{i\theta x} dx \quad \ldots\ldots\ldots\ldots(16),$$

(where as usual $i = \sqrt{-1}$): to prove this, we have

$$\int_{-\infty}^{\infty} (4u^2v^2+x^2)^{q-\frac{1}{2}} e^{i\theta x} dx = \frac{1}{\Gamma(\frac{1}{2}-q)} \int_{-\infty}^{\infty} dx \int_0^\infty dt\, t^{-\frac{1}{2}-q} e^{-t(4u^2v^2+x^2)+i\theta x}$$

$$= \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)} \int_0^\infty dt\, t^{-1-q} e^{-4u^2v^2t - \frac{\theta^2}{4t}};$$

or, putting $4uv\sqrt{t} = \sqrt{(s+4uv)} \pm \sqrt{s}$ (which is a transformation already employed in the present paper), the formula required follows immediately.

Now, by a formula due to M. Catalan, but first rigorously demonstrated by M. Serret,

$$\int_0^\infty \frac{\cos ax\, dx}{(1+x^2)^n} = \frac{\pi}{(\Gamma n)^2} \int_0^\infty e^{-(a+2z)} (z+a)^{n-1} z^{n-1} dz,$$

(*Liouville*, t. VIII. [1843] p. 1), and by a slight modification in the form of this equation

$$\int_{-\infty}^{\infty} (4u^2v^2+x^2)^{q-\frac{1}{2}} e^{i\theta x} dx = \frac{\pi e^{-2uv\theta}(4uv)^{2q}}{\theta^{2q}\,\Gamma^2(\frac{1}{2}-q)} \int_0^\infty s^{-q-\frac{1}{2}} (s+4uv)^{-q-\frac{1}{2}} e^{-\theta s} ds,$$

which, compared with (16), gives the required equation.

[1] A paper by M. Schlömilch "Note sur la variation des constantes arbitraires d'une Integrale definie," *Crelle*, t. XXXIII. [1846], pp. 268—280, will be found to contain formulæ analogous to some of the preceding ones.

NOTE.—One of the intermediate formulæ of Mr Boole [in the Memoir referred to] may be written as follows:

$$S=\frac{1}{\pi}\int_0^1 d\alpha \int_0^\infty dv\, v^q \cos\left[(\alpha-\sigma)\, v+\tfrac{1}{2}q\pi\right] \phi\alpha,$$

or what comes to the same thing, putting $i=\sqrt{-1}$, and rejecting the impossible part of the integral,

$$S=\frac{1}{\pi}\, e^{\frac{1}{2}q\pi i}\int_0^1 d\alpha \int_0^\infty dv\, v^q\, e^{2v(\alpha-\sigma)}\, \phi\alpha,$$

i.e.
$$S=\int_0^1 I\phi\alpha\, d\alpha, \quad I=\frac{1}{\pi}\, e^{\frac{1}{2}q\pi i}\int_0^\infty dv\, v^q\, e^{iv(\alpha-\sigma)}.$$

Now $(\alpha-\sigma)$ being positive,

$$I=\frac{1}{\pi}\, e^{\frac{1}{2}q\pi i}\,\Gamma(q+1)\, e^{\frac{1}{2}(q+1)\pi i}\,(\alpha-\sigma)^{-q-1};$$

i.e.
$$I=\frac{1}{\pi}\, e^{(q+\frac{1}{2})\pi i}\,\Gamma(q+1)\,(\alpha-\sigma)^{-q-1},$$

or, retaining the real part only,

$$I=-\frac{1}{\pi}\sin q\pi\,\Gamma(q+1)\,(\alpha-\sigma)^{-q-1};$$

i.e.
$$I=\frac{1}{\Gamma(-q)}(\alpha-\sigma)^{-q-1}.$$

But $(\alpha-\sigma)$ being negative,

$$I=\frac{1}{\pi}\, e^{\frac{1}{2}q\pi i}\,\Gamma(q+1)\, e^{-\frac{1}{2}(q+1)\pi i}\,(\sigma-\alpha)^{-q-1};$$

i.e.
$$I=\frac{1}{\pi}\, e^{-\frac{1}{2}\pi i}\,\Gamma(q+1)\,(\sigma-\alpha)^{-q-1},$$

or, retaining the real part only, $I=0$.

Hence

$$S=\frac{1}{\Gamma(-q)}\int_\sigma^1 (\alpha-\sigma)^{-q-1}\,\phi\alpha\, d\alpha;$$

or putting $\alpha=\sigma+t(1-\sigma)$, or $\alpha-\sigma=t(1-\sigma)$,

$$S=\frac{(1-\sigma)^{-q}}{\Gamma(-q)}\int_0^1 t^{-q-1}\,\phi\left[\sigma+t(1-\sigma)\right] dt;$$

the expression in the text. Mr Boole's final value is

$$S=\left(-\frac{d}{d\sigma}\right)^q \phi(\sigma),$$

which, though simpler, appears to me to be in some respects less convenient.