60.

ON THE EXPANSION OF INTEGRAL FUNCTIONS IN A SERIES OF LAPLACE'S COEFFICIENTS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 120, 121.]

SUPPOSE

$$S = A\mu^s + A_1\mu^{s-1} + \dots$$

$$= \alpha Q_s + \alpha_1 Q_{s-1} + \dots \qquad (1),$$

where, as usual, Q_0, Q_1, &c. are the coefficients of the successive powers of p in $(1-2\mu p + p^2)^{-\frac{1}{2}}$. Assume

$$S = \frac{\sqrt{(1+\Delta^2)}}{\sqrt{(1-2\mu\Delta+\Delta^2)}}\, C \qquad (2),$$

where Δ refers to C; then expanding this expression, first in powers of μ, and then in a series of terms of the form $\sqrt{(1+\Delta^2)}\,.\,Q$, and comparing these with the preceding values of S,

$$A_{s-q} = \frac{1\,.\,3 \dots 2q-1}{2\,.\,4 \dots 2q}\left(\frac{2\Delta}{1+\Delta^2}\right)^q . \, C,$$

$$\alpha_{s-r} = \sqrt{(1+\Delta^2)}\,.\,\Delta^r\,.\,C \qquad (3).$$

Assume $\dfrac{2\Delta}{1+\Delta^2} = \delta$, that is,

$$\Delta = \frac{1}{\delta}\{1-\sqrt{(1-\delta^2)}\}, \quad \sqrt{(1+\Delta)^2} = \frac{\sqrt{2}}{\delta}\{1-\sqrt{(1-\delta^2)}\}^{\frac{1}{2}} \qquad (4).$$

Then

$$\delta^q\, C = \frac{2\,.\,4 \dots 2q}{1\,.\,3 \dots 2q-1} A_{s-q},$$

$$\alpha_{s-r} = \frac{\sqrt{2}}{\delta^{r+1}}\{1-\sqrt{(1-\delta^2)}\}^{r+\frac{1}{2}}\,.\,C \qquad (5);$$

or, expanding this last equation in powers of δ,

$$\alpha_{s-r} = \frac{(2r+1)}{2^r}\,\delta^r\left(\frac{1}{2r+1} + \frac{1}{2}\frac{\delta^2}{2^2} + \frac{(2r+7)}{2\,.\,4}\frac{\delta^4}{2^4} + \frac{(2r+9)(2r+11)}{2\,.\,4\,.\,6}\frac{\delta^4}{2^6} + \ldots\right) C \ldots (6);$$

or, replacing the successive terms of the form $\delta^q\,.\,C$ by their respective values,

$$\alpha_{s-r} = (2r+1)\left\{\frac{2\,.\,4 \ldots \quad 2r}{3\,.\,5 \ldots (2r+1)\,2^r} A_{s-r} + \frac{4\,.\,6 \ldots (2r+4)}{3\,.\,5 \ldots (2r+3)\,2^{r+2}} A_{s-r-2} \ldots\right.$$

$$\left. + \frac{(2k+2)}{3}\frac{(2k+4)}{5}\ldots\frac{(2r+4k)}{\ldots(2r+2k+1)\,2^{r+2k}} A_{s-r-2k} \ldots\right\} \ldots\ldots\ldots\ldots (7).$$

Thus, if $S = \mu^s$, so that $A_{s-r} = 0$, except in the particular case $A = 1$,

$$\alpha_{2k-1} = 0,$$

$$\alpha_{2k} = \frac{2s-4k+1}{2^s}\,\frac{(2k+2)}{3}\frac{(2k+4)}{5}\ldots\frac{2s}{\ldots(2s-2k+1)} \ldots\ldots\ldots\ldots (8),$$

or $$\mu^s = \frac{1}{2^s}\Sigma\left\{(2s-4k+1)\frac{(2k+2)}{3}\frac{(2k+4)}{5}\ldots\frac{2s}{\ldots(2s-2k+1)} Q_{s-2k}\right\} \ldots\ldots\ldots (9),$$

which of course includes the preceding case. By substituting the expanded values of the coefficients Q, or again, by determining the value of $(1-\mu)^s$ in terms of these coefficients, and equating it with that given in Murphy's *Electricity*, [8°. Cambridge, 1833], p. 10, or in a variety of other ways, a series of identical equations involving sums of factorials may readily be obtained. The mode of employing the general theory of the separation of symbols made use of in the preceding example, may easily be applied to the solution of analogous questions.