61.

ON GEOMETRICAL RECIPROCITY.

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THE fundamental theorem of reciprocity in plane geometry may be thus stated.

"The points and lines of a plane P may be considered as corresponding to the lines and points of a plane P' in such a manner that to a set of points in a line in the first figure, there corresponds a set of lines through a point in the second figure, (namely through the point corresponding to the line); and to a set of lines through a point in the first figure, there corresponds a set of points in a line in the second figure, (namely in the line corresponding to the point)."

And from this theorem, without its being in any respect necessary further to particularize the nature of the correspondence, or to consider in any manner the relative position of the two planes, an endless variety of propositions and theories may be deduced, as, for instance, the duality of all theorems which relate to the purely descriptive properties of figures, the theory of the singular points and tangents of curves, &c.

Suppose, however, that the two planes coincide, so that a point may be considered indifferently as belonging to the first or to the second figure: an entirely independent series of propositions (which, properly speaking, form no part of the general theory of reciprocity) result from this particularization. In general, the line in the second figure which corresponds to a point considered as belonging to the first figure, and the line in the first figure which corresponds to the same point considered as belonging to the second figure, will not be identical; neither will the point in the second figure which corresponds with a line considered as belonging to the first figure, and the point in the first figure which corresponds to the same line considered as belonging to the second figure, be identical.

In the particular case where these lines and points are respectively identical (the identity of the lines implies that of the points and *vice versâ*) we have the theory of "reciprocal polars." Here, where it is unnecessary to define whether the points or lines belong to the first or second figures, the line corresponding to a point and the point corresponding to a line are spoken of as the polar of the point and the pole of the line, or as reciprocal polars.

"The points which lie in their respective polars are situated in a conic, to which the polars are tangents." Or, stating the *same* theorem conversely,

"The lines which pass through their respective poles are tangents to a conic, the points of contact being the poles."

To determine the polar of a point, let two tangents be drawn through this point to the conic, the points of contact are the poles of the tangents; hence the line joining them is the polar of the point of intersection of the tangents, that is, "The polar of a point is the line joining the points of contact of the tangents which pass through the point."

Conversely, and by the same reasoning,

"The pole of a line is the intersection of the tangents at the points where the line meets the conic."

The actual geometrical constructions in the several cases where the point is within or without the conic, or the line does or does not intersect the conic, do not enter into the plan of the present memoir.

Passing to the general case where the lines and points in question are not identical, which I should propose to term the theory of "Skew Polars" (Polaires Gauches), we have the theorem,

"Considering the points in the first figure which are situated in their respective corresponding lines in the second figure, or the points in the second figure which are situated in their respective corresponding lines in the first figure, in either case the points are situated in the same conic (which will be spoken of as the 'pole conic'), and the lines are tangents to the same conic (which will be spoken of as the 'polar conic'), and these two conics have a double contact." This theorem is evidently identical with the converse theorem.

The corresponding lines to a point in the pole conic are the tangents through this point to the polar conic; viz. one of these tangents is the corresponding line when the point is considered as belonging to the first figure, and the other tangent is the corresponding line when the point is considered as belonging to the second figure.

The corresponding points to a tangent of the polar conic are the points where this line intersects the pole conic; viz. one of these points is the corresponding point when the line is considered as belonging to the first figure, and the other is the corresponding point when the line is considered as belonging to the second figure.

Let i be a point in the pole conic, and when i is considered as belonging to the first figure, let iI_1 be considered as the corresponding line in the second figure (I_1 being the point of contact on the polar conic).

Then if j be another point in the pole conic, in order to determine which of the tangents is the line in the second figure which corresponds to j considered as a point of the first figure, let iI_2 be the other tangent through I: the points of contact of the tangents through j may be marked with the letters J_1, J_2, in such order that I_1J_2, I_2J_1 meet in the line of contact of the two conics, and then jJ_1 is the required corresponding line. Again, I and i, as before, if B be a tangent to the polar conic, then, marking the point of contact as J_1, let J_2 be so determined that I_1J_2, I_2J_1 meet in the line of contact of the conics: the tangent to the polar conic at J_2 will meet the pole conic in one of the points where it is met by the line B, and calling this point j, B considered as belonging to the second figure will have j for its corresponding point in the first figure. Similarly, if the point of contact had been marked J_2, J_1 would be determined by an analogous construction, and the tangent at J_1 would meet the pole conic in one of the points where it is met by the line B (viz. the other point of intersection); and representing this by j', B considered as belonging to the first figure would have j' for its corresponding point in the second figure, that is, considered as belonging to the second figure, it would have j for its corresponding point in the first figure (the same as before).

Similar considerations apply in the case where a tangent A of the polar conic, considered as belonging to one of the figures, has for its corresponding point in the other figure one of its points of intersection with the polar conic; in fact, if A represents the line iI_1, then A, considered as belonging to the second figure has i for its corresponding point in the first figure, which shows that this question is identical with the former one.

To appreciate these constructions it is necessary to bear in mind the following system of theorems, the third and fourth of which are the polar reciprocals of the first and second.

If there be two conics having a double contact, such that K is the line joining the points of contact, and k the point of intersection of the tangents at the points of contact:

1. If two tangents to one of the conics meet the other in i, i_1 and j, j_1 respectively, then, properly selecting the points j, j_1, the lines ij_1, i_1j meet in K. And

2. The line joining the points of intersection of the tangents at i, j_1, and of the tangents at i_1, j passes through k. Also

3. If from two points of one of the conics, tangents be drawn touching the other in the points I, I_1 and J, J_1, then, properly selecting the points J, J_1, the lines IJ_1, I_1J meet in K. And

4. The line joining the points of intersection of the tangents at I_1J_1 and of the tangents at I_1, J passes through k.

These theorems are in fact particular cases of two theorems relating to two conics having a double contact with a given conic.

It may be remarked also that the corresponding points to a tangent of the pole conic are the points of contact of the tangents to the polar conic which pass through the point of contact of the given tangent, and the corresponding lines to a point of the polar conic are the tangents to the pole conic at the points where it is intersected by the tangent at the point in question.

We have now to determine the corresponding lines to a given point and the corresponding points to a given line, which is immediately effected by means of the preceding results.

Thus, if the point be given,

"Through the point draw tangents to the polar conic, meeting the pole conic in A_1, A_2 and B_1, B_2 (so that A_1B_2 and A_2B_1, intersect on the line joining the points of contact of the conics), then A_2B_2 and A_1B_1 are the required lines."

In fact A_1, B_1 and A_2, B_2 are pairs of points corresponding to the two tangents, so that A_1B_1 and A_2B_2 are the lines which correspond to their point of intersection, that is, to the given point, and similarly for the remaining constructions. Again,

"Through the point draw tangents to the pole conic, and from the points of contact draw tangents to the polar conic, touching it in α_1, α_2 and β_1, β_2 (so that $\alpha_1\beta_2$ and $\alpha_2\beta_1$ intersect on the line joining the points of contact of the conic), then $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are the required lines."

So that A_1, B_1, α_1, β_1 are situated in the same line, and also A_2, B_2, α_2, β_2.

Again, if the line be given,

"Through the points where the line meets the pole conic draw tangents to the polar conic C_1, C_2 and D_1, D_2 (so that the points C_1D_2 and C_2D_1 lie on a line passing through the intersection of the tangents at the points of contact of the tangents), then C_1D_1 and C_2D_2 are the required points."

Again,

"At the points where the line meets the polar conic draw tangents meeting the pole conic, and let γ_1, γ_2 and δ_1, δ_2 be the tangents to the pole conic at these points (so that the points $\gamma_1\delta_2$ and $\gamma_2\delta_1$ lie on a line through the intersection of the tangents at the points of contact of the conics), then γ_1, δ_1 and γ_2, δ_2 are the required points"; so that C_1, D_1, γ_1, δ_1 pass through the same point and also C_2, D_2, γ_2, δ_2.

"The preceding constructions have been almost entirely taken from Plücker's "System der Analytischen Geometrie," § 3, Allgemeine Betrachtungen über Coordinatenbestimmung. I subjoin analytical demonstrations of some of the theorems in question.

Using x, y, z to determine the position of a variable point, and putting for shortness

$$\xi = ax + a'y + a''z,$$

$$\eta = bx + b'y + b''z,$$

$$\zeta = cx + c'y + c''z.$$

then if the position of a point be determined by the coordinates α, β, γ, the equation of one of the corresponding lines is

$$\alpha\xi + \beta\eta + \gamma\zeta = 0,$$

(that of the other is obtainable from this by writing a, b, c; a', b', c'; a'', b'', c'', for a, a', a''; b, b', b''; c, c', c''). Hence if the point lies in the corresponding line, this equation must be satisfied by putting α, β, γ for x, y, z; or, substituting x, y, z in the place of α, β, γ, the point must lie in the conic

$$U = ax^2 + b'y^2 + c''z^2 + (b'' + c')\, yz + (c + a'')\, zx + (a' + b)\, xy = 0,$$

(which equation is evidently not altered by interchanging the coefficients, as above). Again, determining the curve traced out by the line $\alpha\xi + \beta\eta + \gamma\zeta = 0$, when α, β, γ are connected by the equation into which $U = 0$ is transformed by the substitution of these letters for x, y, z; we obtain

$$V = -\begin{vmatrix} & \xi\ , & \eta\ , & \zeta \\ \xi, & 2a\ , & a' + b, & a'' + c \\ \eta, & a' + b, & 2b', & b'' + c' \\ \zeta, & a'' + c, & b'' + c', & 2c'' \end{vmatrix} = 0,$$

which is also a conic. It only remains to be seen that the conics $U = 0$, $V = 0$ have a double contact. Writing for shortness

$$\nabla = \begin{vmatrix} a\ , & b\ , & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

it may be seen by expansion that the following equation is identically true,

$$V = 4\nabla U - [x\,(ab'' - a''b + a'c - ac') + y\,(b'c - bc' + b''a' - b'a'') + z\,(c''a' - c'a'' + cb'' - c''b)]^2,$$

which proves the property in question.

Suppose the equations of the two conics to be given, and let it be required to determine the corresponding lines to the point defined by the coordinates α, β, γ.

Writing, to abbreviate,

$$\left\{\begin{aligned} U &= Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy, \\ U_0 &= A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta, \\ W &= A\alpha x + B\beta y + C\gamma z + F\,(\beta z + \gamma y) + G\,(\gamma x + \alpha z) + H\,(\alpha y + \beta x), \\ P &= lx + my + nz, \\ P_0 &= l\alpha + m\beta + n\gamma, \\ K &= ABC - AF^2 - BG^2 - CH^2 + 2FGH, \end{aligned}\right.$$

$$\mathfrak{A} = BC - F^2, \quad \mathfrak{B} = CA - G^2, \quad \mathfrak{C} = AB - H^2,$$

$$\mathfrak{F} = GH - AF, \quad \mathfrak{G} = HF - BG, \quad \mathfrak{H} = FG - CH,$$

$$\Theta = \mathfrak{A}l^2 + \mathfrak{B}m^2 + \mathfrak{C}n^2 + 2\mathfrak{F}mn + 2\mathfrak{G}nl + 2\mathfrak{H}lm,$$

$$\begin{aligned}\square = &(\mathfrak{A}l + \mathfrak{H}m + \mathfrak{G}n)(\gamma y - \beta z)\\ &+ (\mathfrak{H}l + \mathfrak{B}m + \mathfrak{F}n)(\alpha z - \gamma x)\\ &+ (\mathfrak{G}l + \mathfrak{F}m + \mathfrak{C}n)(\beta x - \alpha y);\end{aligned}$$

suppose $U = 0$ represents the equation of the polar conic, $U - P^2 = 0$ that of the pole conic. The two tangents drawn to the polar conic are represented by $UU_0 - W^2 = 0$, and by determining k in such a way that $UU_0 - W^2 - k(U - P^2)$ may divide into factors the equation

$$UU_0 - W^2 - k(U - P^2) = 0,$$

represents the lines passing through the points of intersection of the tangents with the pole conic. Thus if $k = U_0$, the equation reduces itself to $U_0P^2 - W^2 = 0$, or $W = \pm\sqrt{(U_0)}\,P$, the equation of two straight lines each of which passes through the point of intersection of the lines $P = 0$, $W = 0$, (that is, of the line of contact of the conics, and the ordinary polar of the point with respect to the polar conic); these are in fact the lines A_1B_2, A_2B_1 intersecting in the line of contact. The remaining value of k is not easily determined, but by a somewhat tedious process I have found it to be

$$= K(U_0 - P_0^2) \div (\Theta - K).$$

In fact, substituting the value, it may be shown that

$$\square^2 + K(PP_0 - W)^2 = K(U_0 - P_0^2)(U - P^2) + (\Theta - K)(UU_0 - W^2),$$

which is an equation of the required form. To verify this, we have, by a simple reduction,

$$\Theta(UU_0 - W^2) - \square^2 = K(UP_0^2 - 2WPP_0 + U_0P^2);$$

or, writing for shortness

$$\gamma y - \beta z = \xi, \quad \alpha z - \gamma x = \eta, \quad \beta x - \alpha y = \zeta,$$

$$\begin{aligned}&(\mathfrak{A}l^2 + \mathfrak{B}m^2 + \mathfrak{C}n^2 + 2\mathfrak{F}mn + 2\mathfrak{G}nl + \mathfrak{H}lm)(\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\zeta^2 + 2\mathfrak{F}\eta\zeta + 2\mathfrak{G}\zeta\xi + 2\mathfrak{H}\xi\eta)\\ &\qquad - [\mathfrak{A}l\xi + \mathfrak{B}m\eta + \mathfrak{C}n\zeta + \mathfrak{F}(n\eta + m\zeta) + \mathfrak{G}(l\zeta + n\xi) + \mathfrak{H}(m\xi + l\eta)]^2\\ &= K\{A(m\zeta - n\eta)^2 + B(n\xi - l\zeta)^2 + C(l\eta - m\xi)^2\\ &\qquad + 2F(n\xi - l\zeta)(l\eta - m\xi) + 2G(l\eta - m\xi)(m\zeta - n\eta) + 2H(m\zeta - n\eta)(n\xi - l\zeta)\},\end{aligned}$$

which is easily seen to be identically true.