84.

ON THE DEVELOPABLE SURFACES WHICH ARISE FROM TWO SURFACES OF THE SECOND ORDER.

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Any two surfaces considered in relation to each other give rise to a curve of intersection, or, as I shall term it, an Intersect and a circumscribed Developable[1] or Envelope. The Intersect is of course the edge of regression of a certain Developable which may be termed the Intersect-Developable, the Envelope has an edge of regression which may be termed the Envelope-Curve. The order of the Intersect is the product of the orders of the two surfaces, the class of the Envelope is the product of the classes of the two surfaces. When neither the Intersect breaks up into curves of lower order, nor the Envelope into Developables of lower class, the two surfaces are said to form a proper system. In the case of two surfaces of the second order (and class) the Intersect is of the fourth order and the Envelope of the fourth class. Every proper system of two surfaces of the second order belongs to one of the following three classes:—*A*. There is no contact between the surfaces; *B*. There is an ordinary contact; *C*. There is a singular contact. Or the three classes may be distinguished by reference to the conjugates (conjugate points or planes) of the system. *A*. The four conjugates are all distinct; *B*. Two conjugates coincide; *C*. Three conjugates coincide.

To explain this it is necessary to remark that in the general case of two surfaces of the second order not in contact (that is for systems of the class *A*) there is a certain tetrahedron such that with respect to either of the surfaces (or more generally with respect to any surface of the second order passing through the Intersect of the system

[1] The term 'Developable' is used as a substantive, as the reciprocal to 'Curve,' which means curve of double curvature. The same remark applies to the use of the term in the compound Intersect-Developable. For the signification of the term 'singular contact,' employed lower down, see Mr Salmon's memoir 'On the Classification of Curves of Double Curvature,' [same volume] p. 32.

or inscribed in the Envelope) the angles and faces of the tetrahedron are reciprocals of each other, each angle of its opposite face, and *vice versâ*. The angles of the tetrahedron are termed the conjugate points of the system, and the faces of the tetrahedron are termed the conjugate planes of the system, and the term conjugates may be used to denote indifferently either the conjugate planes or the conjugate points. A conjugate plane and the conjugate point reciprocal to it are said to correspond to each other. Each conjugate point is evidently the point of intersection of the three conjugate planes to which it does not correspond, and in like manner each conjugate plane is the plane through the three conjugate points to which it does not correspond.

In the case of a system belonging to the class B, two conjugate points coincide together in the point of contact forming what may be termed a double conjugate point, and in like manner two conjugate planes coincide in the plane of contact (that is the tangent plane through the point of contact) forming what may be termed a double conjugate plane. The remaining conjugate points and planes may be distinguished as single conjugate points and single conjugate planes. It is clear that the double conjugate plane passes through the three conjugate points, and that the double conjugate point is the point of intersection of the three conjugate planes: moreover each single conjugate plane passes through the single conjugate point to which it does not correspond and the double conjugate point; and each single conjugate point lies on the line of intersection of the single conjugate plane to which it does not correspond and the double conjugate plane.

In the case of a system belonging to the class (C), three conjugate points coincide together in the point of contact forming what may be termed a triple conjugate point, and three conjugate planes coincide together in the plane of contact forming a triple conjugate plane. The remaining conjugate point and conjugate plane may be distinguished as the single conjugate point and single conjugate plane. The triple conjugate plane passes through the two conjugate points and the triple conjugate point lies on the line of intersection of the two conjugate planes; the single conjugate plane passes through the triple conjugate point and the single conjugate point lies on the triple conjugate plane.

Suppose now that it is required to find the Intersect-Developable of two surfaces of the second order. If the equations of the surfaces be $\Upsilon = 0$, $\Upsilon' = 0$ (Υ, Υ' being homogeneous functions of the second order of the coordinates ξ, η, ζ, ω), and x, y, z, w represent the coordinates of a point upon the required developable surface: if moreover U, U' are the same functions of x, y, z, w that Υ, Υ' are of ξ, η, ζ, ω and X, Y, Z, W; X', Y', Z', W' denote the differential coefficients of U, U' with respect to x, y, z, w; then it is easy to see that the equation of the Intersect-Developable is obtained by eliminating ξ, η, ζ, ω between the equations

$$\Upsilon = 0, \quad \Upsilon' = 0,$$

$$X\xi + Y\eta + Z\zeta + W\omega = 0,$$

$$X'\xi + Y'\eta + Z'\zeta + W'\omega = 0.$$

If, for shortness, we suppose

$$\overline{F} = YZ' - Y'Z, \quad \overline{L} = XW' - X'W,$$

$$\overline{G} = ZX' - Z'X, \quad \overline{M} = YW' - Y'W,$$

$$\overline{H} = XY' - X'Y, \quad \overline{N} = ZW' - Z'W,$$

(values which give rise to the identical equation

$$\overline{L}\overline{F} + \overline{M}\overline{G} + \overline{N}\overline{H} = 0),$$

then, λ, μ, ν, ρ denoting any indeterminate quantities, the two linear equations in ξ, η, ζ, ω are identically satisfied by assuming

$$\begin{aligned}
\xi &= \quad\cdot\quad \overline{N}\mu - \overline{M}\nu + \overline{F}\rho, \\
\eta &= -\overline{N}\lambda \quad\cdot\quad + \overline{L}\nu + \overline{G}\rho, \\
\zeta &= \overline{M}\lambda - \overline{L}\mu \quad\cdot\quad + \overline{H}\rho, \\
\omega &= -\overline{F}\lambda - \overline{G}\mu - \overline{H}\nu \quad\cdot
\end{aligned}$$

and, substituting these values in the equations $\Upsilon = 0$, $\Upsilon' = 0$, we have two equations:

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu + 2L\lambda\rho + 2M\mu\rho + 2N\nu\rho = 0,$$

$$A'\lambda^2 + B'\mu^2 + C'\nu^2 + 2F'\mu\nu + 2G'\nu\lambda + 2H'\lambda\mu + 2L'\lambda\rho + 2M'\mu\rho + 2N'\nu\rho = 0,$$

which are of course such as to permit the four quantities λ, μ, ν, ρ to be simultaneously eliminated. The coefficients of these equations are obviously of the fourth order in x, y, z, w.

Suppose for a moment that these coefficients (instead of being such as to permit this simultaneous elimination of λ, μ, ν, ρ) denoted any arbitrary quantities, and suppose that the indeterminates λ, μ, ν, ρ were besides connected by two linear equations,

$$a\lambda + b\mu + c\nu + d\rho = 0,$$

$$a'\lambda + b'\mu + c'\nu + d'\rho = 0;$$

then, putting

$$bc' - b'c = f, \quad ad' - a'd = l,$$

$$ca' - c'a = g, \quad bd' - b'd = m,$$

$$ab' - a'b = h, \quad cd' - c'd = n,$$

(values which give rise to the identical equation $lf + mg + nh = 0$), and effecting the elimination of λ, μ, ν, ρ between the four equations, we should obtain a final equation $\square = 0$, in which $\square$ is a homogeneous function of the second order in each of the systems of coefficients A, B, &c., and A', B', &c., and a homogeneous function of the fourth order (indeterminate to a certain extent in its form on account of the identical equation $lf + mg + nh = 0$) in the coefficients f, g, h, l, m, n [1]. But re-establishing the actual values of the coefficients A, B, &c., A', B', &c. (by which means the function $\square$ becomes a function of the sixteenth order in x, y, z, w) the quantities f, g, h, l, m, n ought, it is clear, to disappear of themselves; and the way this happens is that the function $\square$ resolves itself into the product of two factors M and Ψ, the latter of which is independent of f, g, h, l, m, n. The factor M is a function of the fourth order in these quantities, and it is also of the eighth order in the variables x, y, z, w: the factor Ψ is consequently of the eighth order in x, y, z, w. And the result of the elimination being represented by the equation $\Psi = 0$, the Intersect-Developable in the general case, or (what is the same thing) for systems of the class (A), is of the eighth order. In the case of a system of the class (B) the equation obtained as above contains as a factor the square, and in the case of a system of the class (C) the cube, of the linear function which equated to zero is the equation of the plane of contact. The Intersect-Developable of a system of the class (B) is therefore a Developable of the sixth order, and that of a system of the class (C) a Developable of the fifth order. The elimination is in every case most simply effected by supposing two of the quantities λ, μ, ν, ρ to vanish (e.g. ν and ρ): the equations between which the elimination has to be effected then are

$$A\lambda^2 + B\mu^2 + 2H\lambda\mu = 0,$$
$$A'\lambda^2 + B'\mu^2 + 2H'\lambda\mu = 0;$$

and the result may be presented under the equivalent forms

$$(AB' + A'B - 2HH')^2 - 4(AB - H^2)(A'B' - H'^2) = 0,$$

and

$$(AB' - A'B)^2 + 4(AH' - A'H)(BH' - B'H) = 0,$$

the latter of which is the most convenient. These two forms still contain an extraneous factor of the eighth order in x, y, z, w, of which they can only be divested by substituting the actual values of A, B, H, A', B', H'.

A. Two surfaces forming a system belonging to this class may be represented by equations of the form

$$a x^2 + b y^2 + c z^2 + d w^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0,$$

[1] I believe the result of the elimination is

$$\square = 4(PR - Q^2) = 0,$$

where, if we write $uA + u'A' = A$, &c., the quantities P, Q, R are given by the equation (identical with respect to u, u')

$$Pu^2 + 2Quu' + Ru'^2 = (Aa^2 + \ldots)(Aa'^2 + \ldots) - (Aaa' + \ldots)^2$$
$$= u^2\{(BC - F^2)f^2 + \ldots\} + uu'\{(BC' + B'C - 2FF')f^2 + \ldots\} + u'^2\{(B'C' - F'^2)f^2 + \ldots\}$$

a theorem connected with that given in the second part of my memoir 'On Linear Transformations' (*Journal*, vol. I. p. 109) [see 14, p. 100]. I am not in possession of any verification *à posteriori* of what is subsequently stated as to the resolution into factors of the function $\square$ and the forms of these factors.

where $x=0$, $y=0$, $z=0$, and $w=0$, are the equations of the four conjugate planes. There is no particular difficulty in performing the operations indicated by the general process given above; and if we write, in order to abbreviate,

$$\begin{aligned} bc' - b'c &= f, & ad' - a'd &= l\,, \\ ca' - c'a &= g, & bd' - b'd &= m, \\ ab' - a'b &= h, & cd' - c'd &= n\,, \end{aligned}$$

(values which satisfy the identical equation $lf + mg + nh = 0$), the result after all reductions is

$$\begin{aligned} & l^2f^4y^4z^4 + m^2g^4z^4x^4 + n^2h^4x^4y^4 + l^4f^2x^4w^4 + m^4g^2y^4w^4 + n^4h^2z^4w^4 \\ & + 2mng^2h^2x^4y^2z^2 + 2nlh^2f^2y^4z^2x^2 + 2lmf^2g^2z^4x^2y^2 \\ & - 2m^2n^2ghw^4y^2z^2 - 2n^2l^2hfw^4z^2x^2 - 2l^2m^2fgw^4x^2y^2 \\ & + 2fmg^2l^2x^4z^2w^2 + 2gnh^2m^2y^4x^2w^2 + 2hlf^2n^2z^4y^2w^2 \\ & - 2fnh^2l^2x^4y^2w^2 - 2glf^2m^2y^4z^2w^2 - 2hmg^2n^2z^4x^2w^2 \\ & + 2\,(mg - nh)\,(nh - lf)\,(lf - mg)\,x^2y^2z^2w^2 = 0, \end{aligned}$$

which is therefore the equation of the Intersect-Developable for this case. The discussion of the geometrical properties of the surface will be very much facilitated by presenting the equation under the following form, which is evidently one of a system of six different forms,

$$\begin{aligned} & \{m\,(gx^2 + nw^2)\,(hy^2 - gz^2 + lw^2) - l\,(-fy^2 + nw^2)\,(-hx^2 + fz^2 + mw^2)\}^2 \\ & \qquad - 4fglmx^2y^2\,(hy^2 - gz^2 + lw^2)\,(-hx^2 + fz^2 + mw^2) = 0. \end{aligned}$$

B. Two surfaces forming a system belonging to this class may be represented by equations such as

$$\begin{aligned} a\,x^2 + b\,y^2 + c\,z^2 + 2n\,zw &= 0, \\ a'x^2 + b'y^2 + c'z^2 + 2n'zw &= 0, \end{aligned}$$

in which $x=0$, $y=0$ are the equations of the single conjugate planes, $z=0$ that of the double conjugate plane or plane of contact, $w=0$ that of an indeterminate plane through the two single conjugate points. If we write

$$\begin{aligned} bc' - b'c &= f, & an' - a'n &= p, \\ ca' - c'a &= g, & bn' - b'n &= q, \\ ab' - a'b &= h, & cn' - c'n &= r, \end{aligned}$$

(values which satisfy the identical equation $pf + qg + rh = 0$), the result after all reductions is

$$\begin{aligned} & r^4h^2z^6 + 2pr^2h\,(rh - qg)\,z^4x^2 - 2qr^2h\,(pf - rh)\,z^4y^2 \\ & + 4p^2qr^2hz^3x^2w - 4pq^2r^2hz^3y^2w \\ & + p^2\,(rh - qg)^2\,z^2x^4 + q^2\,(pf - rh)^2\,z^2y^4 + 2pq\,(4r^2h^2 - fgpq)\,z^2x^2y^2 \\ & + 4p^3q\,(rh - qg)\,zx^4w + 4pq^3zy^4w - 4p^2q^2\,(qg - pf)\,zx^2y^2w \\ & + 4p^4q^2x^4w^2 + 4p^2q^4y^4w^2 + 8p^3q^3x^2y^2w^2 + 4p^2qrh^2x^4y^2 + 4pq^2rh^2x^2y^4 = 0, \end{aligned}$$

which is therefore the equation of the Intersect-Developable for systems of the case in question. The equation may also be presented under the form

$$\{q(px^2+rz^2)(hy^2-gz^2+2pzw)-p(qy^2+rz^2)(-hx^2+fz^2+2qzw)\}^2$$
$$+4p^2q^2x^2y^2(hy^2-gz^2+2pzw)(-hx^2+fz^2+2qzw)=0,$$

which it is to be remarked contains the extraneous factor z^2. The following is also a form of the same equation,

$$\{r(qg-pf)z^3-fp^2zx^2+gq^2zy^2+2pq(px^2+qy^2)w\}^2$$
$$-4pq(px^2+qy^2+rz^2)\{r(hy^2-gz^2)(fz^2-hx^2)+2pq(gx^2-fy^2)zw\}=0.$$

C. Two surfaces forming a system belonging to this class may be represented by equations of the form

$$a\,x^2+b\,y^2+2f\,yz+2n\,zw=0,$$
$$a'x^2+b'y^2+2f'yz+2n'zw=0,$$

in which $bn'-b'n=0$, $af'-a'f=0$. In these equations $x=0$ is the equation of a properly chosen plane passing through the two conjugate points, $y=0$ is the equation of the single conjugate plane, $z=0$ that of the triple conjugate plane, and $w=0$ is the equation of a properly chosen plane passing through the single conjugate point. Or without loss of generality, we may write

$$\alpha\,(x^2-2yz)+\beta\,(y^2-2zw)=0,$$
$$\alpha'(x^2-2yz)+\beta'(y^2-2zw)=0,$$

where x, y, z and w have the same signification as before[1]. The result after all reductions is

$$4z^3w^2+12y^2z^2w+9y^4z-24x^2yzw-4x^2y^3+8x^4w=0,$$

which may also be presented under the forms

$$z(y^2-2zw)^2-4y(y^2-2zw)(x^2-2yz)+8w(x^2-2yz)^2=0,$$

and
$$z(3y^2+2zw)^2-4x^2(y^3-2x^2w+6yzw)=0.$$

[In these three equations and in the last two equations of p. 495 as originally printed, there was by mistake, an interchange of the letters x and y.]

[1] Of course in working out the equation of the Intersect-Developable, it is simpler to employ the equations $x^2-2yz=0$, $y^2-2zw=0$. These equations belong to two cones which pass through the Intersect and have their vertices in the triple conjugate point and single conjugate point respectively. I have not alluded to these cones in the text, as the theory of them does not come within the plan of the present memoir, the immediate object of which is to exhibit the equations of certain developable surfaces—but these cones are convenient in the present case as furnishing the easiest means of defining the planes $x=0$, $w=0$. If we represent for a moment the single conjugate point by S and the triple conjugate point by T (and the cones through these points by the same letters), then the point T is a point upon the cone S, and the triple conjugate plane which touches the cone S along the line TS touches the cone T along some generating line TM. Let the other tangent plane through the line TS to the cone T be TM', where M' may represent the point where the generating line in question meets the cone S; and we may consider M as the point of intersection of the line TM with the tangent plane through the line SM' to the cone S: then the plane TMM' is the plane represented by the equation $x=0$, and the plane SMM' is that represented by the equation $w=0$. We may add that $y=0$ is the equation of the plane TSM', and $z=0$ that of the plane TSM.

Proceeding next to the problem of finding the envelope of two surfaces of the second order, this is most readily effected by the following method communicated to me by Mr Salmon. Retaining the preceding notation, the equation $U + kU' = 0$ belongs to a surface of the second order passing through the Intersect of the two surfaces $U = 0$, $U' = 0$. The polar reciprocal of this surface $U + kU' = 0$ is therefore a surface inscribed in the envelope of the reciprocals of the two surfaces $U = 0$, $U' = 0$, and consequently this envelope is the envelope (in the ordinary sense of the word) of the reciprocal of the surface $U + kU' = 0$, k being considered as a variable parameter. It is easily seen that the reciprocal of the surface $U + kU' = 0$ is given by an equation of the form

$$\mathrm{A} + 3\mathrm{B}k + 3\mathrm{C}k^2 + \mathrm{D}k^3 = 0,$$

in which A, B, C, D are homogeneous functions of the second order in the coordinates x, y, z, w. Differentiating with respect to k, and performing the elimination, we have for the equation of the envelope in question,

$$(\mathrm{AD} - \mathrm{BC})^2 - 4(\mathrm{AC} - \mathrm{B}^2)(\mathrm{BD} - \mathrm{C}^2) = 0;$$

or the envelope is, in general or (what is the same thing) for a system of the class (A), a developable of the eighth order. For a system of the class (B) the equation contains as a factor, the square of the linear function which equated to zero is the equation of the plane of contact; or the envelope is in this case a Developable of the sixth order. And in the case of a system of the class (C) the equation contains as a factor the cube of this linear function; or the envelope is a developable of the fifth order only.

A. We may take for the two surfaces the reciprocals (with respect to $x^2 + y^2 + z^2 + w^2 = 0$) of the equations made use of in determining the Intersect-Developable. The equations of these reciprocals are

$$bcd\,x^2 + cda\,y^2 + dab\,z^2 + abc\,w^2 = 0,$$
$$b'c'd'x^2 + c'd'a'y^2 + d'a'b'z^2 + a'b'c'w^2 = 0;$$

and it is clear from the form of them (as compared with the equations of the surfaces of which they are the reciprocals) that $x = 0$, $y = 0$, $z = 0$, $w = 0$, are still the equations of the conjugate planes. We have, introducing the numerical factor 3 to avoid fractions,

$$3\{(b + kb')(c + kc')(d + kd')x^2 + (c + kc')(d + kd')(a + ka')y^2$$
$$+ (d + kd')(a + ka')(b + kb')z^2 + (a + ka')(b + kb')(c + kc')w^2\}$$

$= \mathrm{A} + 3\mathrm{B}k + 3\mathrm{C}k^2 + \mathrm{D}k^3$, which determine the values of A, B, C, D.

We have in fact

$$\mathrm{A} = 3(bcdx^2 + cday^2 + dabz^2 + abcw^2)$$
$$\mathrm{B} = (b'cd + bc'd + bcd')x^2 + \ldots\ldots$$
$$\mathrm{C} = (bc'd' + b'cd' + b'c'd)x^2 + \ldots\ldots$$
$$\mathrm{D} = 3(b'c'd'x^2 + c'd'a'y^2 + d'a'b'z^2 + a'b'c'w^2)$$

and these values give (with the same signification as before of f, g, h, l, m, n)

$$2(\mathrm{AC}-\mathrm{B}^2) = Aa^2 + Bb^2 + Cc^2 + 2Fbc + 2Gca + 2Hab + 2Lad + 2Mbd + 2Ncd + Dd^2,$$

$$2(\mathrm{BD}-\mathrm{C}^2) = Aa'^2 + Bb'^2 + Cc'^2 + 2Fb'c' + 2Gc'a' + 2Ha'b' + 2La'd' + 2Mb'd' + 2Nc'd' + Dd'^2,$$

$$\begin{aligned}\mathrm{AD}-\mathrm{BC} = {} & Aaa' + Bbb' + Ccc' + F(bc'+b'c) + G(ca'+c'a) + H(ab'+a'b) \\ & + L(ad'+a'd) + M(bd'+b'd) + N(cd'+c'd) + Ddd',\end{aligned}$$

where

$$\begin{aligned}
\mathrm{A} &= n^2y^4 + m^2x^4 + f^2w^4 + 2fmz^2w^2 - 2fny^2w^2 + 2nmy^2z^2,\\
\mathrm{B} &= l^2z^4 + n^2x^4 + g^2w^4 + 2gnx^2w^2 - 2glz^2w^2 + 2lnz^2x^2,\\
\mathrm{C} &= m^2x^4 + l^2y^4 + h^2w^4 + 2hly^2w^2 - 2hmx^2w^2 + 2mlx^2y^2,\\
\mathrm{D} &= f^2x^4 + g^2y^4 + h^2z^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgy^2z^2,\\
\mathrm{F} &= l^2y^2z^2,\\
\mathrm{G} &= m^2z^2x^2,\\
\mathrm{H} &= n^2x^2y^2,\\
\mathrm{L} &= f^2x^2w^2,\\
\mathrm{M} &= g^2y^2w^2,\\
\mathrm{N} &= h^2z^2w^2;
\end{aligned}$$

and then $4(\mathrm{AC}-\mathrm{B}^2)(\mathrm{BD}-\mathrm{C}^2)-(\mathrm{AD}-\mathrm{BC})^2=$

$$\begin{aligned}
& (BC-F^2)f^2 + (CA-G^2)g^2 + (AB-H^2)h^2 + (AD-L^2)l^2 + (BD-M^2)m^2 + (CD-N^2)n^2\\
&\quad + 2(GH-AF)gh + 2(HF-BG)hf + 2(FG-CH)fg\\
&\quad - 2(MN-DF)mn - 2(NL-DG)nl - 2(LM-DH)lm\\
&\quad + 2(AM-LH)lh + 2(BN-NF)mf + 2(CL-NG)ng\\
&\quad - 2(AN-LG)lg - 2(BL-MH)mh - 2(CM-NF)nf\\
&\quad + 2(NH-MG)lf + 2(LF-NH)mg + 2(MG-LF)nh.
\end{aligned}$$

Substituting the values of A, B, &c., in this expression, the result after all reductions is

$$\begin{aligned}
& f^2m^2n^2x^8 + g^2n^2l^2y^8 + h^2l^2m^2z^8 + f^2g^2h^2w^8\\
& + 2gl^2n(mg-nh)y^6z^2 + 2hm^2l(nh-lf)z^6x^2 + 2fn^2m(lf-mg)x^6y^2\\
& - 2hl^2m(mg-nh)y^2z^6 - 2fm^2n(nh-lf)z^2x^6 - 2gn^2l(lf-mg)x^2y^6\\
& + 2f^2mn(mg-nh)x^6w^2 + 2g^2nl(nh-lf)y^6w^2 + 2h^2lm(lf-mg)z^6w^2\\
& - 2f^2gh(mg-nh)x^2w^6 - 2fg^2h(nh-lf)y^2w^6 - 2fgh^2(lf-mg)z^2w^6\\
& + f^2(l^2f^2-6ghmn)w^4x^4 + g^2(m^2g^2-6hfnl)w^4y^4 + h^2(n^2h^2-6lmfg)w^4z^4\\
& + l^2(l^2f^2-6ghmn)y^4z^4 + m^2(m^2g^2-6hfnl)z^4x^4 + n^2(n^2h^2-6lmfg)x^4y^4\\
& + 2gh(ghmn-3f^2l^2)w^4y^2z^2 + 2hf(hfnl-3g^2m^2)w^4z^2x^2 + 2fg(fglm-3h^2n^2)w^4x^2y^2\\
& + 2hm(ghmn-3f^2l^2)z^4x^2w^2 + 2fn(hfnl-3g^2m^2)x^4y^2w^2 + 2gl(fglm-3h^2n^2)y^4z^2w^2\\
& - 2gn(ghmn-3f^2l^2)y^4x^2w^2 - 2hl(hfnl-3g^2m^2)z^4y^2w^2 - 2fm(fglm-3h^2n^2)x^4z^2w^2\\
& - 2mn(ghmn-3f^2l^2)x^4y^2z^2 - 2nl(hfnl-3g^2m^2)y^4z^2x^2 - 2lm(fglm-3h^2n^2)z^4x^2y^2\\
& - 2(mg-nh)(nh-lf)(lf-mg)x^2y^2z^2w^2 = 0,
\end{aligned}$$

which is therefore the equation of the envelope for this case. The equation may also be presented under the form

$$w^2\Theta + (mnx^2 + nly^2 + lmz^2)^2 (f^2x^4 + g^2y^4 + h^2x^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgx^2y^2) = 0\,;$$

and there are probably other forms proper to exhibit the different geometrical properties of the surface, but with which I am not yet acquainted.

B. Here taking for the two surfaces the reciprocals of the equations made use of in determining the Intersect-Developable, the equations of these reciprocals are

$$n^2\,b\,x^2 + n^2\,a\,y^2 - a\,b\,c\,w^2 + 2n\,a\,b\,zw = 0,$$

$$n'^2b'x^2 + n'^2a'y^2 - a'b'c'w^2 + 2n'a'b'zw = 0,$$

which are similar to the equations of the surfaces of which they are reciprocal, only z and w are interchanged, so that here $x = 0$, $y = 0$ are the single conjugate planes, $z = 0$ is an indeterminate plane passing through the single conjugate points, and $w = 0$ is the equation of the double conjugate plane or plane of contact.

The values of A, B, C, D are

$$\begin{aligned}
A &= 3\,(n^2bx^2 + n^2ay^2 - abcw^2 + 2nabzw),\\
B &= (2nn'b + n^2b')\,x^2 + \ldots\ldots\\
C &= (2nn'b' + n'^2b)\,x^2 + \ldots\ldots\\
D &= 3\,(n'^2b'x^2 + n'^2a'y^2 - a'b'c'w^2 + 2n'a'b'zw).
\end{aligned}$$

Hence, using f, g, h, p, q, r in the same sense as before, we have for $2\,(AC - B^2)$, $2\,(BD - C^2)$, $(AD - BC)$ expressions of the same form as in the last case (p, q, r being written for l, m, n), but in which

$$\begin{aligned}
A &= f^2w^4 + 4q^2z^2w^2 + 8qry^2w^2 - 4fqzw^3,\\
B &= g^2w^4 + 4p^2z^2w^2 + 8prx^2w^2 + gpzw^3,\\
C &= h^2w^4,\\
D &= 2q^2x^4 + 2p^2y^4 + 4h^2z^2w^2 + 4pqx^2y^2 - 8qhx^2zw + 8phy^2zw + 2ghy^2w^2 + 2fhx^2w^2,\\
F &= -\,p^2y^2w^2,\\
G &= -\,q^2x^2w^2,\\
H &= 0,\\
L &= 2pqy^2zw,\\
M &= 2pqx^2zw,\\
N &= -\,2h^2zw^3\,.
\end{aligned}$$

The substitution of these values gives after all reductions the result

$$
\begin{aligned}
& f^2g^2h^2w^6 + 4\,(pf-qg)\,fgh^2zw^5 \\
& + 4\,(r^2h^2 - 6pqgf)\,h^2z^2w^4 + 2\,(q^2g^2 + 2prfh)\,fhx^2w^4 + 2\,(p^2f^2 + 2qrgh)\,ghy^2w^4 \\
& - 16\,(pf-qg)\,z^3w^3 - 4\,(q^2g^2 - 4p^2f^2 - 6pqfg)\,qhx^2zw^3 - 4\,(p^2f^2 - 4q^2g^2 - 6pqfg)\,phy^2zw^3 \\
& + 16p^2q^2h^2z^4w^2 - 8\,(pf+4qg)\,q^2phx^2z^2w^2 - 8\,(qg+4pf)\,pq^2hy^2z^2w^2 \\
& + (q^2g^2 + 8prfh)\,q^2x^4w^2 + (p^2f^2 + 8qrgh)\,p^2y^4w^2 + 2\,(10r^2h^2 - pqfg)\,pqx^2y^2w^2 \\
& - 16p^2q^3hx^2z^3w + 16p^3q^2hy^2z^3w \\
& + 4\,(4pf + 5qg)\,pq^3x^4zw - 4\,(4qg + 5pf)\,p^3qy^4zw - 4\,(pf-qg)\,p^2q^2x^2y^2zw \\
& + 4p^2q^4x^4z^2 + 4p^4q^2y^4z^2 + 8p^3q^3x^2y^2z^2 + 4pq^4rx^6 + 4p^4qry^6 + 12p^2q^3rx^4y^2 + 12p^3q^2rx^2y^4 = 0\,;
\end{aligned}
$$

which is therefore the equation of the envelope for this case. This equation may be presented under the form

$$w\Psi + 4pq\,(qx^2 + py^2)^2\,(qrx^2 + rpy^2 + pqz^2) = 0,$$

and there are probably other forms which I am not yet acquainted with.

C. The reciprocals of the two surfaces made use of in determining the Intersect-Developable, although in reality a system of the same nature with the surfaces of which they are reciprocals, are represented by equations of a somewhat different form. There is no real loss of generality in replacing the two surfaces by the reciprocals of the cones $x^2 = 2yz$, $y^2 = 2zw$; or we may take the two conics

$$(x^2 - 2yz = 0,\ w = 0) \text{ and } (y^2 - 2zw = 0,\ x = 0),$$

for the surfaces of which the envelope has to be found, these conics being, it is evident, the sections by the planes $w = 0$ and $x = 0$ respectively of the cones the Intersect-Developable of which was before determined. The process of determining the envelope is however essentially different: supposing the plane $\xi x + \eta y + \zeta z + \omega w = 0$ to be the equation of a tangent plane to the two conics (that is, of a plane passing through a tangent of each of the conics) then the condition of touching the first conic gives $\xi^2 - 2\eta\zeta = 0$, and that of touching the second conic gives $\eta^2 - 2\zeta\omega = 0$. We have therefore to find the envelope (in the ordinary sense of the word) of the plane $\xi x + \eta y + \zeta z + \omega w = 0$, in which the coefficients ξ, η, ζ, ω are variable quantities subject to the conditions

$$\xi^2 - 2\eta\zeta = 0,\quad \eta^2 - 2\zeta\omega = 0.$$

The result which is obtained without difficulty by the method of indeterminate multipliers, [or more easily by writing $\xi : \eta : \zeta : w = 2\theta^3 : 2\theta^2 : \theta^4 : 2$] is

$$8y^4z - 32y^2z^2w + 32z^3w^2 - 27x^4w + 27x^2yzw - 4x^2y^3 = 0,$$

which may also be written under the form

$$8z\,(y^2 - 2zw)^2 - x^2\,\{4y^3 + 9\,(3x^2 - 8yz)\,w\} = 0.$$

[Another form, containing the factor w, is $4\,(y^2 + 2zw)^3 - (2y^3 + 27x^2w - 36yzw)^2 = 0$.]