1.

A CONSTRUCTIVE THEORY OF PARTITIONS, ARRANGED IN THREE ACTS, AN INTERACT AND AN EXODION.

[*American Journal of Mathematics*, v. (1882), pp. 251—330; vi. (1884), pp. 334—336.]

Act I. On Partitions Regarded as Entities.

. . . seeming parted,
But yet a union in partition.
Twelfth-night.

(1) In the new method of partitions it is essential to consider a partition as a *definite thing*, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves *per se* as an instrument of transformation.

(2) The most obvious mode of graphically representing a partition is by means of a network or web formed by two systems of parallel lines or filaments. Any continuous portion of such web will serve to represent a partition, as for example the graph

```
    *   *   *
*   *   *   *   *
*   *   *   *   *
*   *   *   *
    *   *   *
```

will represent the partition 3 5 5 4 3 of 20 by reading off the successive numbers of nodes parallel to the horizontal lines of the web. This, however, is not a regularized partition; the partition will be represented in its regularized form by such a graph as the following:

```
*   *   *   *   *
    *   *   *   *   *
    *   *   *   *
            *   *   *
                *   *   *
```

which corresponds to the order 5 5 4 3 3, but it may be represented much more advantageously by the figure

```
*   *   *   *   *
*   *   *   *   *
*   *   *   *
*   *   *
*   *   *
```

which is a portion of the web bounded by lines of nodes belonging to the two systems of parallel filaments. Any such portion becomes then subject to the important condition that the two transverse parallel readings will each give a regularized partition, one being in the present example 5 5 4 3 3, and the other 5 5 5 3 2. Any such graph as this will be termed a *regular* partition-graph, and the two partitions which it represents will be said to be conjugate to one another. The mere conception of a regular graph serves at once by effecting an interchange (so to say) between the warp and the woof, through the principle of correspondence, to establish a well-known fundamental theorem of reciprocity. In the last figure, the extent* of (meaning the number of nodes contained by) the uppermost horizontal line or filament is the maximum magnitude of any element (or part) of the partition, and the extent of the first vertical line is the number of the parts. Hence, every regularized partition beginning with i and containing j parts is conjugate to another beginning with j and containing i parts. The content of the graph (that is, the sum of the parts) of the partition is the same in both cases (it will sometimes be convenient to speak of the *partible number* as the content of the elements of the partition). From the above correspondence it is clear that if two complete partition-systems be formed with the same content in one of which the largest part is i and the number of parts j, and in the other the largest part is j and the number of the parts i, the order (that is, the number of partitions) of the first system will be identical with the order of the second: so that calling the content n, it follows that $n-i$ may be decomposed in as many ways into $j-1$ parts as $n-j$ into $i-1$ parts.

(3) This, however, is not the usual nor the more convenient mode of expressing the reciprocity in question. We may, for the two partition systems spoken of, substitute two others of larger inclusion, taking for the first, all partitions of n in which no one part is greater than i, and the number of parts is not greater than j (that is, is j or fewer), and for the second system, one subject to the same conditions as just stated, but with i and j (as before) interchanged: it is obvious that each regularized partition

* *Extent* may be used to denote the number of nodes on a line or column or angle of a graph; *content* the number of nodes in the graph itself; but I have by inadvertence in what follows frequently applied *content* alike to designate areal and linear numerosity.

of one system will be conjugate to one regularized partition of the other system, and accordingly the order of the two systems will be the same*.

(4) When $i = \infty$ it follows from the general theorem of reciprocity last established, that the number of partitions of n into j parts or fewer will be the same as the number of ways of composing n with the integers $1, 2, \ldots j$, and is therefore the coefficient of x^n in the expansion of

$$\frac{1}{1-x \,.\, 1-x^2 \ldots 1-x^j}.$$

Thus, then, we can at once find the general term in

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots},$$

expanded according to ascending powers of a; for, if the above fraction be regarded as the product of an infinite number of infinite series arising from the expansion of the several factors

$$\frac{1}{1-a}, \frac{1}{1-ax}, \frac{1}{1-ax^2}, \ldots$$

it will readily be seen that the coefficient of $x^n a^j$ will be the number of ways in which n can be resolved into j parts or fewer, that is, by what has been just shown is the coefficient of x^n in

$$\frac{1}{1-x \,.\, 1-x^2 \ldots 1-x^j};$$

and this being true for all values of n, it follows that the entire coefficient of a^j is the fraction last written developed in ascending powers of x; so that

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots}$$
$$= 1 + \frac{1}{1-x} a + \frac{1}{1-x \,.\, 1-x^2} a^2 + \frac{1}{1-x \,.\, 1-x^2 \,.\, 1-x^3} a^3 \ldots$$

as is well known.

The general term in

$$\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$$

is also well known to be

$$\frac{1-x^{i+1} \,.\, 1-x^{i+2} \ldots 1-x^{i+j}}{1-x \,.\, 1-x^2 \ldots 1-x^j} a^j,$$

* The above proof of the theorem of reciprocity is due to Dr Ferrers, the present head of Gonville and Caius College, Cambridge. It possesses the double merit of having set the first example of graphical construction and of putting into salient relief the principle of correspondence, applied to the theory of partitions. It was never made public by its author, but first promulgated by myself in the *Lond. and Edin. Phil. Mag.* for 1853. [Vol. I. of this Reprint, p. 597.]

or in other words, the number of ways of resolving n into j parts none greater than i is the coefficient of x^n in the fraction

$$\frac{1-x^{i+1}\,.\,1-x^{i+2}\,\ldots\,1-x^{i+j}}{1-x\,.\,1-x^2\,\ldots\,1-x^j},$$

which [denoting $1-x^q$ by (q)] is the same as

$$\frac{(1)\,(2)\ldots(i+j)}{(1)\,(2)\ldots(i)\,.\,(1)\,(2)\ldots(j)},$$

and furnishes, if I am not mistaken, Euler's proof of the theorem of reciprocity already established by means of the correspondence of conjugate partitions.

(5) [It may be as well to advert here to the practical method of obtaining the conjugate to a given partition. For this purpose it is only necessary to call a_i the number of parts in the given partition not less than i; a_1, a_2, a_3, ... a_i ... continued to infinity (or which comes to the same thing until i is equal to the maximum part), will be the required conjugate.]

(6) The following very beautiful method of obtaining the general term in question by the constructive method is due to Mr F. Franklin of the Johns Hopkins University* :

He, as it were, interpolates between the theorem to be established in general and the theorem for $i=\infty$, and attaches a definite meaning to the above fraction regarded as a generating function when the factors in the numerator are limited to the first q of them, q being any number not exceeding i, so that in fact the theorem to be proved, according to this view, is only the extreme case of (the last link in the chain to) a new and more general one with which he has enriched the theory of partitions. The method will be most easily understood by means of an example or two: the proof and use to be made of the construction will be given towards the end of the Act.

Let $n=10$, $i=5$, $j=4$.

Write down the indefinite partitions of 10 into 4 or fewer parts, or say rather into 4 parts, among which zeros are admissible: they will be

(1)	10.0.0.0	5.5.0.0
(1)	9.1.0.0	5.4.1.0
(1)	8.2.0.0	5.3.2.0
(1)	8.1.1.0	5.3.1.1
(2)	7.3.0.0	5.2.2.1
(2)	7.2.1.0	4.4.2.0
(1)	7.1.1.1	4.4.1.1
(2)	6.4.0.0	4.3.3.0
(3)	6.3.1.0	4.3.2.1
(3)	6.2.2.0	4.2.2.2
(4)	6.2.1.1	3.3.3.1
		3.3.2.2

* For a vindication of the constructive method applied to this and an allied theorem, see p. [18] *et seq.*

The partitions to which (1) is prefixed are those in which the *first excess*, that is, the excess of the first (the dominant) part over the next is *too great* (meaning greater than i, here 5); those to which (2) is prefixed are those in which after the batch marked with (1) are removed, the second excess, that is, the excess of the first over the third element is "too great"; those to which (3) is prefixed are those in which after the batches marked (1) and (2) are removed, the third excess is "too great," and lastly those (only one as it happens) marked with j (here 4) are those in which, so to say, the *absolute excess* of the dominant, that is its actual value, is "too great," that is, exceeding i (here 5); the partitions that are left over will be the partitions of n (here 10) into 4 parts, none exceeding i (here 5) in magnitude.

It is easy to see from this how to *construct* the partitions which are to be *eliminated* from the indefinite partitions of the n (10) into 4 (j) parts so as to obtain the quaternary partitions in which no part superior to 5 (i) appears. To obtain the first batch we must subtract $i+1$ (6) from n (10) and form the system of indefinite partitions of 4 into four parts, namely:

4 . 0 . 0 . 0
3 . 1 . 0 . 0
2 . 2 . 0 . 0
2 . 1 . 1 . 0
1 . 1 . 1 . 1

and adding to each of these 6 . 0 . 0 . 0 (term-to-term addition) batch (1) will be obtained.

To obtain the second batch, form the quaternary partitions of $n-(i+2)$, that is, 3, namely:

3 . 0 . 0 . 0
2 . 1 . 0 . 0
1 . 1 . 1 . 0

[but omit those in which the first excess is "too great" (greater than i); here there are none such to be omitted] and bring the second element into the first place; thus we shall obtain the system

0 3 0 0
1 2 0 0
1 1 1 0

The *augments* of those obtained by adding 6 . 1 . 0 . 0 to each of them will reproduce batch (2).

Again, form the quaternary partition-system of $n-(i+3)$, rejecting all those (here there are none such) in which the second excess is "too great." We thus obtain

2 0 0 0
1 1 0 0

and now bringing the third element in each of these into the first place so as to obtain

$$\begin{matrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}$$

The augments of these last partitions obtained by adding 6 . 1 . 1 . 0 to each of them will give the third batch, and finally taking the quaternary partition-system to $n-(i+j)$, that is, 1, rejecting (if there should be any such) those in which the third excess is "too great," we obtain 1 . 0 . 0 . 0, and bringing the fourth element to the first place so as to get 0 . 1 . 0 . 0, and adding 6 . 1 . 1 . 1, the fourth batch 6 . 2 . 1 . 1 is reconstructed.

As another example take $n=15$, $i=3$, $j=3$.

The indefinite ternary partitions of 15 are

15 . 0 . 0	(1)	9 . 4 . 2	(1)
14 . 1 . 0	(1)	9 . 3 . 3	(1)
13 . 2 . 0	(1)	8 . 7 . 0	(2)
13 . 1 . 1	(1)	8 . 6 . 1	(2)
12 . 3 . 0	(1)	8 . 5 . 2	(2)
12 . 2 . 1	(1)	8 . 4 . 3	(1)
11 . 4 . 0	(1)	7 . 7 . 1	(2)
11 . 3 . 1	(1)	7 . 6 . 2	(2)
11 . 2 . 2	(1)	7 . 5 . 3	(2)
10 . 5 . 0	(1)	7 . 4 . 4	(3)
10 . 4 . 1	(1)	6 . 6 . 3	(3)
10 . 3 . 2	(1)	6 . 5 . 4	(3)
9 . 6 . 0	(2)	5 . 5 . 5	(3)
9 . 5 . 1	(1)		

There are, of course, no partitions left in which no part exceeds 3, as the maximum content subject to that condition would be only 9.

The partitions marked (1) (2) (3) are those in which the first, second and absolute excess respectively exceed 3.

Firstly, the indefinite ternary partitions of $15-4$ or 11 augmented by 4 . 0 . 0 will obviously reproduce the system of partitions marked (1).

Secondly, taking the indefinite ternary partitions of 10 in which the *first* excess, and those of 9 in which the *second* excess, does not exceed 3, we shall obtain

6 . 4 . 0	and	5 . 2 . 2
6 . 3 . 1		4 . 4 . 1
5 . 5 . 0		4 . 3 . 2
5 . 4 . 1		3 . 3 . 3
5 . 3 . 2		
4 . 4 . 2		
4 . 3 . 3		

which by *metastasis* become

4.6.0	2.5.2
3.6.1	1.4.4
5.5.0	2.4.3
4.5.1	3.3.3
3.5.2	
4.4.2	
3.4.3	

and adding to each term of these two groups 4.1.0 and 4.1.1 respectively, the systems of partitions marked (2) and (3) respectively result.

(7) It may, I think, be desirable to give here my own construction for the case of repeated partitions, which, having regard to its features of resemblance to the one preceding, it is proper to state preceded it in the date of its discovery and promulgation. The problem which I propose to myself is to construct a system of partitions of a given number into parts limited in number and magnitude, by means of partitions of itself and other numbers into parts limited in number but not in magnitude.

As before, let i be the limit of magnitude, j the number of parts (zeros admissible), and n the partible number; form a square matrix of the jth order in which the diagonal elements are all $i+1$, the elements below the diagonal all of them unity, and those above the diagonal all of them zero, say M_1.

From this matrix construct $M_1, M_2, M_3, \ldots M_j$, such that the lines in M_q (q being any integer from 1 to j inclusive) are the sums of those in M_1, added (term-to-term) q and q together.

Let (r, q) be the rth line in M_q and $[r, q]$ the sum of the numbers which it contains.

Form the complete system of the partitions of $n-[r, q]$ into j parts, and to each such add (term-to-term) (r, q).

In this way, by giving r all possible values we shall obtain a system of partitions of n into j parts corresponding to M_q, which may be called P_q. I say that $P_1 - P_2 + P_3 \ldots + (-)^{j-1} P_j$ will be the complete system of partitions of n into j parts in which one element at least exceeds i; where it is to be observed that having regard to the effect of the $-$ and $+$ signs (which are used here to indicate the addition and subtraction, or say rather the adduction and sub-duction not of numbers but of things), each such partition will occur once and once only; so that calling P the complete system of indefinite partitions of n into j parts, the complete system of partitions of n into j parts in which no part exceeds i in magnitude will be

$$P - P_1 + P_2 \ldots + (-)^j P_j *.$$

* It must, however, be understood that the same partition is liable to appear in more than one, and not exclusively in its regularized phase, or as it may be expressed, the regularized partition undergoes *metastasis*.

(8) This construction, which I will illustrate by two examples, proceeds upon the fact which, although confirmed by a multitude of instances, *remains to be proved*, that if $k_1, k_2, \ldots k_j$ be any partition of n into j parts and the number of *unequal* parts greater than i be μ, then the number of times in which this partition, in its regular or any other phase, appears in P_q is $\dfrac{\mu(\mu-1)\ldots(\mu-q+1)}{1.2\ldots q}$ (interpreted to mean 1 when $q=0$), and consequently its total number of appearances in $P - P_1 + P_2 \ldots$ is $(1-1)^\mu$, that is, is 0.

From this it follows that the total number of partitions of n into j parts none exceeding i in magnitude will be $C - C_1 + C_2 - \ldots$, where C_q is the sum of the number of ways in which the various numbers $n_1, n_2, n_3 \ldots$ can be decomposed into j parts, the numbers $n_1, n_2, n_3, \ldots$ being n diminished by the sums of the quantities $i+1, i+2, \ldots, i+j$ added q and q together; C_q is therefore the coefficient of x^n in $\dfrac{x^{n-n_1} + x^{n-n_2} + x^{n-n_3} + \ldots}{(1-x)(1-x^2)\ldots(1-x^j)}$; and consequently the number of partitions of n into j parts none exceeding i in magnitude will be the coefficient of x^n in $\dfrac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{i+j}}{1-x.1-x^2\ldots 1-x^j}$ as was to be shown.

(9) As a first example let $i=2$, $j=3$, $n=12$, the matrices and the partitions corresponding to their several lines will be as underwritten; the indefinite partitions of the reduced contents, $n-[r, q]$, are written opposite to the respective matrix lines to which they correspond, and their augments, found by adding the line to this partition system, are written immediately under them. The zeros are omitted for the sake of brevity.

3.0.0	9	8.1	7.2	7.1.1	6.3	6.2.1	5.4	5.3.1	5.2.2	4.4.1	4.3.2	3.3.3
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.4	8.3.1	8.2.2	7.4.1	7.3.2	6.3.3
1.3.0	8	7.1	6.2	6.1.1	5.3	5.2.1	4.4	4.3.1	4.2.2	3.3.2		
	9.3	8.4	7.5	7.4.1	6.6	6.5.1	5.7	5.6.1	5.5.2	4.6.2		
1.1.3	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2				
	8.1.3	7.2.3	6.3.3	6.2.4	5.4.3	5.3.4	4.4.4	4.3.5				
—												
4.3.0	5	4.1	3.2	3.1.1	2.2.1							
	9.3	8.4	7.5	7.4.1	6.5.1							
4.1.3	4	3.1	2.2	2.1.1								
	8.1.3	7.2.3	6.3.3	6.2.4								
2.4.3	3	2.1	1.1.1									
	5.4.3	4.5.3	3.5.4									
—												
5.4.3	0											
	5.4.3											

In 6.3.3 there are two unlike elements greater than 2; accordingly 6.3.3 occurs 2 times in P_1 and 1 time in P_2.

In 7 . 3 . 2 there are again two unlike elements greater than 2, and 7 . 3 . 2, 7 . 2 . 3 (the metastatic equivalent to the former) are found in P_1 and 7 . 2 . 3 in P_2.

Again, in 5 . 4 . 3 there are 3 unlike elements greater than 2, and we find

5 . 4 . 3	5 . 3 . 4	4 . 3 . 5	in P_1
5 . 4 . 3	4 . 5 . 3	3 . 5 . 4	„ P_2
5 . 4 . 3			„ P_3.

But such terms as 11 . 1 10 . 1 . 1 9 . 2 . 1 8 . 2 . 2 in which there is only one distinct element greater than 2 are found 1 time only in P_1 and not at all in P_2 or P_3.

As another example let $n = 12$, $i = 4$, $j = 3$, then a similarly constructed table to the foregoing will be as follows, in which, however, all matrices or lines of matrices which have a sum too large to give rise to partition systems are omitted.

5 . 0 . 0	7	6 . 1	5 . 2	5 . 1 . 1	4 . 3	4 . 2 . 1	3 . 3 . 1	3 . 2 . 2
	12	11 . 1	10 . 2	10 . 1 . 1	9 . 3	9 . 2 . 1	8 . 3 . 1	8 . 2 . 2
1 . 5 . 0	6	5 . 1	4 . 2	4 . 1 . 1	3 . 3	3 . 2 . 1	2 . 2 . 2	
	7 . 5	6 . 6	5 . 7	5 . 6 . 1	4 . 8	4 . 7 . 1	3 . 7 . 2	
1 . 1 . 5	5	4 . 1	3 . 2	3 . 1 . 1	2 . 2 . 1			
	6 . 1 . 5	5 . 2 . 5	4 . 3 . 5	4 . 2 . 6	3 . 3 . 6			
—								
6 . 5 . 0	1							
	7 . 5							
6 . 1 . 5	0							
	6 . 1 . 5							

7 . 5 and 6 . 5 . 1 are the only two partitions of 12 into 3 parts in which there are two unlike parts greater than 4; each of these accordingly is found twice (in one or another phase) in P_1 and once in P_2. Every other partition of 12 into 3 parts in which one of them at least is greater than 4 will be found exclusively and only once in P_1.

(10) The two expansions for $(1 - ax)(1 - ax^2) \dots (1 - ax^i)$ and its reciprocal may readily be obtained from one another by the method of correspondence.

The coefficient of $x^n a^j$ in the former is the number of partitions of n into j *unequal*, and in the latter into j *equal or unequal* parts none greater than i or less than unity. The correspondence to be established has been given by Euler for the case of $i = \infty$ (*Comm. Arith.*, 1849, Tom. I. p. 88), and is probably known for the general case, but as coming strictly within the purview of the essay, seems to deserve mention here.

If $k_1, k_2, k_3, \ldots, k_j$ be a partition of n into j equal or unequal parts written in ascending order, none exceeding i, on adding to it $0, 1, 2 \ldots (j-1)$, it becomes a partition of $n+\frac{j^2-j}{2}$ into j parts none exceeding $i+j-1$, and conversely, if $\lambda_1, \lambda_2, \ldots, \lambda_j$ be a partition of $n+\frac{j^2-j}{2}$ into j unequal parts none exceeding $i+j-1$, written in ascending order, on subtracting from it $0, 1, 2 \ldots (j-1)$, it becomes a partition of n into equal or unequal (say relatively independent) parts none exceeding i.

Hence the complete system of partitions of n into j unlike parts none exceeding i has a one-to-one correspondence with the complete system of the partitions of $n-\frac{j^2-j}{2}$ into j parts none exceeding $i-j+1$. Consequently the coefficient of a^j in the expansion of $(1-ax)\ldots(1-ax^i)$ may be found from that of a^j in the expansion of its reciprocal by changing i into $i-j+1$ and introducing the factor $x^{\frac{j^2-j}{2}}$.

(11) The expansion of the reciprocal may of course be found algebraically from the multiplication of the expansion which has been given of $\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$ by $(1-a)$, or immediately by the correspondence between partitions into an exact number j of parts limited not to exceed i, and partitions into j or fewer parts so limited.

By subtracting a unit from each term of $k_1, k_2, \ldots, k_j$, a partition of n where no k exceeds i, results a partition $q_1, q_2, \ldots q_j$, a partition of $n-j$ where no q exceeds $i-1$. Hence the coefficient of a^j in

$$\frac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^i}$$

may be found from that in

$$\frac{1}{1-a\,.\,1-ax\ldots 1-ax^i}$$

by introducing the factor x^j and changing i into $i-1$, so that choosing for the latter the alternative form

$$\frac{1-x^{j+1}\,.\,1-x^{j+2}\ldots 1-x^{j+i}}{1-x\,.\,1-x^2\ldots 1-x^i},$$

the former becomes

$$\frac{1-x^{j+1}\,.\,1-x^{j+2}\ldots 1-x^{j+i-1}}{1-x\,.\,1-x^2\ldots 1-x^{i-1}}\,x^j,$$

and consequently the coefficient of a^j in $1-ax\,.\,1-ax^2\ldots 1-ax^i$ will be

$$\frac{1-x^{j+1}\,.\,1-x^{j+2}\ldots 1-x^i}{1-x\,.\,1-x^2\ldots 1-x^{i-j}}\,x^{\frac{j^2+j}{2}}.$$

(12) Before quitting this part of the subject it is desirable to make mention of Dr F. Franklin's remarkable method of proving Euler's celebrated expansion of $(1-x)(1-x^2)(1-x^3)\ldots$ *ad inf.* by the method of correspondence. This has been given by Dr Franklin himself in the *Comptes Rendus* of the Institut (1880), and by myself in some detail in the last February Number of the *J. H. U. Circular**. The method is in its essence absolutely independent of graphical considerations, but as it becomes somewhat easier to apprehend by means of graphical description and nomenclature, I shall avail myself here of graphical terminology to express it.

If a regular graph represent a partition with unequal elements, the lines of magnitude must continually increase or decrease. Let the annexed figures be such graphs written in ascending order from above downwards.

```
          *  *
          *  *  *
          *  *  *  *  *                          (A)
          *  *  *  *  *  *
          *  *  *  *  *  *  *

     *  *                   *  *  *
(B)  *  *  *                *  *  *  *           (C)
     *  *  *  *  *  *       *  *  *  *  *  *
     *  *  *  *  *  *  *    *  *  *  *  *  *  *
```

In A and B the graphs may be transformed without altering their content or regularity by removing the nodes at the summit and substituting for them a new slope line at the base. In C the slope line at the base may be removed and made to form a new summit; the graphs so transformed will be as follows:

```
          *  *  *
          *  *  *  *  *                          (A')
          *  *  *  *  *  *  *
          *  *  *  *  *  *  *  *

                                 *  *
(B')                             *  *  *         (C')
     *  *  *                     *  *  *  *
     *  *  *  *  *  *  *         *  *  *  *  *
     *  *  *  *  *  *  *  *      *  *  *  *  *  *
```

A' and B' may be said to be derived from A, B by a process of contraction, and C' from C by one of protraction.

Contraction could not now be applied to A' and B', nor protraction to C' without destroying the regularity of the graph; but the inverse processes may of course be applied, namely, of protraction to A' and B' and contraction to C', so as to bring back the original graph A, B, C.

In general (but as will be seen not universally), it is obvious that when the number of nodes in the summit is inferior or equal to the number in the base-slope, contraction may be applied, and when superior to that number, protraction: each process alike will alter the number of parts from even to

[* Vol. III. of this Reprint, p. 664.]

odd or from odd to even, so that barring the exceptional cases which remain to be considered where neither protraction nor contraction is feasible, there will be a one-to-one correspondence between the partitions of n into an odd number and the partitions of n into an even number of unrepeated parts; the exceptional cases are those shown below where the summit meets the base-slope line, and contains either the same number or one more than the number of nodes in that line; in which case neither protraction nor contraction will be possible, as seen in the annexed figures which are written in regular order of succession, but may be indefinitely continued :

```
*    * *    * *      * * *      * * *
            * * *    * * * *    * * * *
                                * * * * *

     * * * *          * * * *
     * * * * *        * * * * *
     * * * * * *      * * * * * *
                      * * * * * * *

            * * * * *
            * * * * * *
            * * * * * * *
            * * * * * * * *
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for the protraction process which *ought*, for example, according to the general rule, to be applicable to the last of the above graphs, cannot be applied to it, because on removing the nodes in the slope line and laying them on the summit, in the very act of so doing the summit undergoes the loss of a node and is thereby incapacitated to be surmounted by the nodes in the slope, which will have not now a less, but the same number of nodes as itself; and in like manner, in the last graph but one, the nodes in the summit cannot be removed and a slope line be added on containing the same number of nodes without the transformed graph ceasing to be regular, in fact it would take the form

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      * *
* * * * * *
* * * * * * *
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and so the last graph transformed according to rule [by protraction] would become :

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which, although regular, would cease to represent a partition into unlike numbers.

The excepted cases then or unconjugate partitions are those where the number of parts being j, the successive parts form one or the other of the two arithmetical series

$$j, j+1, j+2, \dots 2j-1 \text{ or } j+1, j+2, \dots 2j,$$

in which cases the contents are $\frac{3j^2-j}{2}$ and $\frac{3j^2+j}{2}$ respectively, and consequently since in the product of $1-x \,.\, 1-x^2 \,.\, 1-x^3 \ldots$ the coefficient of x^n is the number of ways of composing n with an even less the number of ways of composing it with an odd number of parts, the product will be completely represented by $\sum\limits_{j=+\infty}^{j=-\infty} (-)^j x^{\frac{3j^2+j}{2}}$ *.

(13) It has been well remarked by Prof. Cayley that barring the unconjugate partitions, the rest really constitute 4 classes, which using c and x to signify contractile and extensile and e and o to signify of-an-even or of-an-odd order, may be denoted by

$$\begin{matrix} c\,.\,e & c\,.\,o \\ x\,.\,e & x\,.\,o. \end{matrix}$$

Hence as each $c\,.\,e$ is conjugate to an $x \quad o$ and *vice versâ*, and each $c\,.\,o$ to an $x\,.\,e$ and *vice versâ*, the theorem established really splits up into two, one affirming that the number of contractile partitions of an odd order is the same as the number of extensile ones of an even order, the other that the number of contractiles of an even is equal to the number of extensiles of an odd order. It might possibly be worth while to investigate the difference between the number of partitions which each set of one couple and the number of partitions which each set of the sub-contrary couple contain: the sets which belong to the same couple and contain the same number of partitions being those *both* of whose characters are dissimilar.

(14) There are one or two other simple cases of correspondence which are interesting, inasmuch as the construction employed to effect the correspondence involves the operations of division and multiplication, which have not occurred previously.

If $$fx = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ldots$$
and $$\phi x = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$
$$fx\,.\,\phi x = 1,$$
from which we obtain $\phi x = 1/fx$ and $1/\phi x = fx$.

The first of these equations has been noticed by Euler as involving the elegant theorem that a number may be partitioned in as many ways into only-once-occurring odd-or-even integers as into any-number-of-times-occurring only-odd integers.

* Another proof of this theorem, deduced as an immediate algebraical consequence of a more general one, obtained by graphical dissection, will be given in Act 2; and in the Exodion I furnish a purely arithmetical proof by the method of correspondence of Jacobi's series for

$$(1 \pm x^{n-m})(1 \pm x^{n+m})(1-x^{2n})(1 \pm x^{3n-m})(1 \pm x^{3n+m})(1-x^{4n})\ldots$$

(which includes Euler's theorem as a particular case). I prove this theorem in a more extended sense than was probably intended by its immortal author, inasmuch as I regard m and n as absolutely general symbols.

The second, which I think he does not dwell upon, expresses that the difference between the number of partitions with an even number of parts and that of partitions with an odd number of parts of the same number n is the same as the number of partitions of n into exclusively odd [unrepeated] numbers (such difference being in favour of the partitions of even or of odd order, according as the partible number is even or odd).

This latter theorem brings out a point of analogy between repetitional and non-repetitional partition systems which appears to me worthy of notice.

Any one of the former contains a class of what may be termed singular partitions, in the sense that they are their own associates, or more briefly, *self-conjugate* in respect to the Ferrers transformation. Any one system of the latter may also be *said* to contain a set of singular partitions (0 or 1 in number) in the sense of being *unconjugate* in respect to the Franklin process of transformation. Since then in this case the difference between the number of partitions of an odd and those of an even order of the same number is equal to the number (1 or 0) of singular partitions of that number, so we might anticipate as not improbable that the like difference for the repetitional partitions of a number should be equal to the number of singular partitions of that number—and such is actually the case; for it will be shown in a future section that the number of self-conjugate partitions of a number is the same as the number of ways in which it can be composed with odd integers.

(15) The correspondence indicated by the equation $\phi x = 1/fx$ can be established as follows:

Let $2^\lambda . l$, $2^\mu . m$, $2^\nu . n$, ... be any partition of unrepeated general numbers, where l, m, n ... are any odd integers not exceeding unity; and let $k^{[q]}$ in general denote q parts k, then without changing its content the above partition can be converted into $l^{[2^\lambda]}$, $m^{[2^\mu]}$, $n^{[2^\nu]}$, ... which consists exclusively of odd numbers.

It will of course be understood that the original partition may contain any the same odd number as l multiplied by different powers 2^λ, $2^{\lambda'}$, $2^{\lambda''}$ ... of 2, with the sole restriction that the λ, λ', λ'', ... must be all unequal.

Conversely, any such partitions as $l^{[\sigma]}$, $m^{[\tau]}$, $n^{[v]}$ may be converted back into one and only one partition of the former kind. For there will be one and but one way of resolving σ into the sum of powers of 2 (the zero power not excluded), and supposing σ to be equal to $2^\lambda + 2^{\lambda'} + 2^{\lambda''} + \ldots$, $l^{[\sigma]}$ may be replaced by $2^\lambda l$, $2^{\lambda'} l$, $2^{\lambda''} l$, and the same process of conversion may be simultaneously applied to each of the other products $m^{[\tau]}$, $n^{[v]}$, ...

Hence each partition of either one kind is conjugate to one of the other, and the number of partitions in the two systems will be the same, as was to be shown.

(16) But we have here another example of the fact that the theory of correspondence reaches far deeper than that of mere numerical congruity with which it is associated as the substance with the shadow. For a correspondence exists of a much more refined nature than that above demonstrated between the two systems, and which, moreover (it is important to notice) does not bring the same individuals into correlation as does the former method.

The partition system made up of unrepeated general numbers may be divided into groups of the first, second, ... ith ... class respectively, those of the ith class containing i distinct sequences of consecutive numbers having no term in common, with the understanding that no two sequences must form part of a single sequence (so that the largest term of one sequence and the smallest one of the next sequence must differ by more than a single unit), and that a single number unpreceded and unfollowed by a consecutive number is to count as a sequence.

The partition system, made up of repeatable odd numbers may, in like manner, be resolved into groups of the 1st, 2nd, ... ith, ... class respectively, those of the ith class containing i distinct numbers; and the new theorem of correspondence is that there is a correlation between the numbers of the ith class of one system and the ith class of the other; so that the number of partitions in a class of the same name must be the same to whichever system it belongs; and thus Euler's theorem becomes a corollary to this deeper-reaching one, obtained from it by *adding together* the number of partitions in all the several classes in the one system and in the other.

(17) As regards the first class, the theorem amounts to the statement that the number of single sequences of consecutive numbers into which n may be resolved is equal to the number of odd factors which n contains; so that if $N = 2^c . l^\lambda . m^\mu . n^\nu \dots$ where $l, m, n, \dots$ are odd numbers, N can be represented by $(\lambda+1)(\mu+1)(\nu+1)\dots$ such sequences; thus, for example, if $N = 15 = 3.5$ we have

$$1+2+3+4+5 = 4+5+6 = 7+8 = 15.$$

So
$$30 = 4+5+6+7+8 = 6+7+8+9 = 9+10+11,$$
$$27 = 2+3+4+5+6+7 = 8+9+10 = 13+14,$$
$$45 = 1+2+3+\dots+9 = 5+6+7+8+9+10$$
$$= 7+8+9+10+11 = 14+15+16 = 22+23.$$

So too if N is a prime number it can only be resolved into the two sequences $\frac{N-1}{2} + \frac{N+1}{2}$ and N. More generally N can be resolved into as many different sets of i distinct sequences as there are solutions in positive integers of the equation $2(x_1y_1 + x_2y_2 + \ldots + x_iy_i) + x_1 + x_2 + \ldots + x_i = N$, of the truth of which remarkable theorem, in its general form, I have for the present only obtained empirical evidence, but may possibly be able to discover the proof in time to annex it in the form of a note at the end, so as not to keep the press waiting*.

(18) The proof for the case of the first class and the mode of establishing the correspondence between the partitions of this class of the two kinds is not far to seek. I use as previously $a^{(b)}$ to signify a repeated b times.

Consider then any sequence of consecutive numbers for the cases where the number of terms is odd and where it is even separately, calling s the sum of the first and last terms, and i the number of terms; where i is odd, so that s is even, the sequence may be replaced by $i^{\left(\frac{s}{2}\right)}$, and where i is even (so that s is odd) by $s^{\left(\frac{i}{2}\right)}$. Hence each partition of the first class of the first kind may be transformed into one of the first class of the second kind.

It is necessary to show the converse of this, which may be done as follows: Let λ^{μ} be any partition of the second kind so that λ is necessarily odd. I say that this must be transformable into one or the other (but not into both) of two sequences, namely, one of λ terms of which the sum of the first and last is 2μ, the other of which the sum of the first and last terms is λ and the number of terms 2μ. The former supposition is admissible if 2μ is equal to or greater than $\lambda + 1$, inadmissible if 2μ is less than $\lambda + 1$. The second supposition is admissible if λ is equal to or greater than $2\mu + 1$, inadmissible if λ is less than $2\mu + 1$.

The two conditions of admissibility coexisting would imply that 2μ is equal to or greater than $2\mu + 2$; the two conditions of inadmissibility the one that 2μ is equal to or less than $\lambda - 1$, the other that λ is equal to or less than $2\mu - 1$, that is, $\lambda - 1$ equal to or less than $2\mu - 2$, which are inconsistent. Hence one of the two transformations is always possible and the other impossible to be effected; which proves the correlation that was to be established. A single example will serve to show that this correspondence is entirely different from that offered by the first and (so to say) grosser method; suppose $N = 15$, then $1 . 2 . 3 . 4 . 5$ will be a partition of the first kind and will be converted by the new rule into $5 . 5 . 5$, whereas, by the former rule, it would be inverted into $1 . 1 . 1 . 3 . 1 . 1 . 1 . 1 . 5$, that is, into $1^7 . 3 . 5$ belonging to the third class instead of to the first.

(19) I will now pass on to the conjugate theorem corresponding to $fx = 1/\phi x$.

* A complete proof of the general theorem will be given in the 3rd Act.

It may be well here to recall that this identity essentially depends upon the identity $1-x=1/(1+x)(1+x^2)(1+x^4)\ldots$ which, interpreted*, signifies that any number greater than unity may be made up in as many ways with an odd as with an even number of numbers restricted to the geometrical progression 1, 2, 4, 8 .... This may be called, for brevity, a geometric partition. The correspondence to which this points is itself worthy of notice; one mode of establishing it would be to proceed to decompose N into such parts in regular dictionary order—it would easily be seen that each pair of partitions thus deduced would be of contrary *parities*, but it would not be easy, or at all events evident, how to determine at once the conjugate to a given partition by reference to this principle; but if we observe that it is possible to pass from the geometric partitions of n immediately to those of $n+1$ by the addition of a unit to each of the former, and consequently to those of $n+2$ from the partitions of $E\frac{n}{2}, E\frac{n-2}{2}, E\frac{n-4}{2}, \ldots 2, 1$, by an obvious process of doubling and adding complementary units, another rule or law of correspondence, which proves itself as soon as stated (and is not identical in effect with that supplied by the dictionary-order method), looms into the field of vision, than which nothing can be simpler. Hence we may derive a transcendental equation in differences for u_n, the number of geometric partitions (with radix 2) to n, namely, to find the conjugate of any geometric partition, look at its greatest part—if it is repeated add two of them together: if it is unrepeated split it into two equal parts; these processes are obviously reversible, just as in Dr Franklin's method of correspondence for the pentagonal-series-theorem; and the method is equally open to the remark made thereon by Prof. Cayley; that is to say, there will be four classes, extensile even, extensile odd, contractile even and contractile odd, and the number of partitions in any class will be the same as in the class in which both the characters are reversed.

The application of this transformation to the construction indicated by the equation $fx=1/\phi x$ will be obvious. Let any partition containing only unrepeated numbers consist of odd numbers $p, q, r, \ldots t$, each multiplied by one or more powers of 2; form batches of these terms which have the same greatest odd divisor $(p, q, r, \ldots t)$, and arrange those batches in a line according to the order of magnitude of $p, q, r, \ldots t$. Then we may agree to proceed either from left to right or from right to left in reading off the batches, and that convention being established once for all, as soon as a batch is reached which does not consist of a single odd term, if it contain one term larger than all the rest that term is to be split into two equal parts, but if it contain two terms not less than any

* Just so the equation $1/(1-x)=(1+x)(1+x^2)(1+x^4)\ldots$ teaches that there is one and only one way of effecting the unrepetitional geometric partition of any number—a theorem which has been applied in the preceding theory.

others in the batch, those two are to be amalgamated into one. In this way the order of a partition consisting of terms not all of them distinct odd numbers, will have its *parity* (quality of being odd or even) reversed, and it is obvious that if A has been under the operation of the rule converted into B, B by the operation of the same rule will be converted back into A. Hence it follows that (making abstraction of the partitions consisting exclusively of unrepeated odd numbers) all the rest will be separable into as many contractile of an odd as into extensile of an even order, and into as many extensile of an odd as into contractile of an even order, so that the difference between the entire number of the partitions of N into an odd and those of an even order of repeatable numbers (odd or even) will be the number of partitions of N into unrepeated odd numbers, and those of an odd or of an even order will be in the majority according as N itself is odd or even*.

It will be convenient to interpolate here Dr F. Franklin's constructive proof of the theorems referred to in p. [4] of what precedes, as there will be frequent occasion to refer to them in what follows. The theory is thus made completely self-contained. I give the proofs in the author's own words, which I think cannot be bettered.

(20) *Constructive Proof of the Formula for Partitions into Repeatable Parts, limited in Number and Magnitude.* The partitions herein spoken of are always partitions into a fixed number, j, of parts, *written in descending order.*

Take any partition of w in which the first excess† is greater than i; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; and conversely if to the first part in a partition of $w-(i+1)$ we add $i+1$ we get a partition of w in which the first excess is greater than i. Hence the number of partitions of w in which the first excess is greater than i is equal to the whole number of partitions of $w-(i+1)$; so that if the generating

* Dr F. Franklin has remarked that "the theorem admits of the following extensions," which the method employed in the text naturally suggests, and "which are very easily obtained either by the constructive proof or by generating functions":

1. The number of ways in which w can be made up of any number of odd and k distinct even parts is equal to the number of ways in which it can be made up of any number of unrepeated and k distinct repeated parts.

2. The number of ways in which w can be made up of parts not divisible by m is equal to the number of ways in which it can be made up of parts not occurring as many as m times.

3. The number of ways in which w can be made up of an infinite number of parts not divisible by m, together with k parts divisible by m, is equal to the number of ways in which it can be made up of an indefinite number of parts occurring less than m times, together with k parts occurring m or more times. (3) of course comprehends (1) and (2) as special cases.

Dr Franklin adds, "another extension is naturally contained in the mode of proof, which it is perhaps not worth while to state." See *Johns Hopkins Circular* for March, 1883.

† The first excess signifies the excess of the largest part over the next largest; the second excess the excess of the largest over the next part but one, and so on.

function for the partitions of w is $f(x)$, that for those partitions in which the first excess is *not greater* than i is $(1-x^{i+1})f(x)$. Confining ourselves now to this class of partitions, consider any one of them in which the second excess is greater than i; subtracting $i+1$ from the first part and 1 from the next, and putting the reduced first part into the second place we have a partition of $w-(i+2)$ in which the first excess is not greater than i; and conversely if in any partition of $w-(i+2)$ in which the first excess is not greater than i, we add $i+1$ to the second part and 1 to the first part and transfer the augmented second part to the first place, we get a partition of w in which the first excess is not greater than i and the second excess *is* greater than i. Hence the generating function for those partitions in which the second excess is *not* greater than i is $(1-x^{i+1})(1-x^{i+2})f(x)$. Considering now exclusively the partitions last mentioned, any one of them in which the third excess is greater than i may be converted into a partition of $w-(i+3)$ in which the second excess is not greater than i, by subtracting $i+1$ from the first part, 1 from the second part, and 1 from the third part, and removing the reduced first part to the third place, and, as before, by the reverse operation, the latter class of partitions are converted into the former. Hence the generating function for the partitions in which the third excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})f(x).$$

So in like manner, the generating function for the partitions in which the k-th excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+k})f(x);$$

and for the partitions in which the j-th or absolute excess is not greater than i, that is in which the greatest part does not exceed i, the generating function is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+j})f(x).$$

(21) *Constructive Proof of the Formula for Partitions into Unrepeated Parts, limited in Number and Magnitude.* All the partitions to be considered consist of a fixed number, j, of *unrepeated* parts, written in *descending order.*

Take any partition of w in which the first excess is greater than $i+1$; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; conversely, if to the first part in any partition of $w-(i+1)$ we add $i+1$, we get a partition of w in which the first excess is greater than $i+1$; hence the number of partitions of w in which the first excess is greater than $i+1$ is equal to the whole number of partitions of $w-(i+1)$; so that, if the generating function for all the partitions is $\phi(x)$, the generating function for partitions whose first excess is *not* greater than $i+1$ is $(1-x^{i+1})\phi(x)$.

Considering now only partitions subject to this condition, if in any such partition of w the second excess is greater than $i+2$, we obtain by subtracting $i+2$ from the first part and removing the part so diminished to the second place a partition of $w-(i+2)$ subject to the condition; and conversely from any partition of $w-(i+2)$ in which the first excess is not greater than $i+1$, we obtain, by adding $i+2$ to the second part and removing the augmented part to the first place, a partition of w, in which the first excess is not greater than $i+1$ and the second excess *is* greater than $i+2$; hence the generating function for the partitions in which the second excess is *not* greater than $i+2$ (which restriction includes the condition that the first excess is not greater than $i+1$) is

$$(1-x^{i+1})(1-x^{i+2})\,\phi(x).$$

Confining ourselves now to this class of partitions, and taking any partition of w in which the third excess is greater than $i+3$, we obtain, by subtracting $i+3$ from the first part and removing the diminished part to the third place, a partition of $w-(i+3)$ belonging to the class now under consideration; and reversely. Hence the number of partitions in which the third excess is not greater than $i+3$ is given by the generating function

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\,\phi(x).$$

Proceeding in this manner, we have finally that the generating function giving the number of partitions into j unrepeated parts, in which the absolute excess, that is the magnitude of the greatest part, is not greater than $i+j$, is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\ldots(1-x^{i+j})\,\phi(x).$$

For example, if $w=18$, $j=3$, $i=4$, the partitions

15, 2, 1   14, 3, 1   13, 4, 1   13, 3, 2   12, 5, 1   12, 4, 2   11, 5, 2   11, 4, 3

in which the first excess is greater than 5, become by subtraction of 5 from their first part,

10, 2, 1   9, 3, 1   8, 4, 1   8 3, 2   7, 5, 1   7, 4, 2   6, 5, 2   6, 4, 3

which are *all* the partitions of 13; the partitions

11, 6, 1   10, 7, 1   10, 6, 2   10, 5, 3   9, 8, 1   9, 7, 2

in which the first excess is not greater than 5, but the second excess is greater than 6 become, by the subtraction of 6 from the first part and its removal to the second place,

6, 5, 1   7, 4, 1   6, 4, 2   5, 4, 3   8, 3, 1   7, 3, 2

which are all the partitions of 12 whose first excess is not greater than 5; the partitions

9, 6, 3   9, 5, 4   8, 7, 3   8, 6, 4

in which the second excess is not greater than 6, but the third excess (the greatest part) is greater than 7, become, by the subtraction of 7 from the first part and its removal to the last place,

$$6, 3, 2 \quad 5, 4, 2 \quad 7, 3, 1 \quad 6, 4, 1$$

which are all partitions of 11 whose second excess is not greater than 6. The only remaining partition of 18 is 7, 6, 5.

INTERACT.

Notes on certain Generating Functions and their Properties.

(22) (A) It may be as well to reproduce here (so as to keep the whole subject together) the entire proof of the well-known expansions of

$$1 + ax \,.\, 1 + ax^2 \,.\, 1 + ax^3 \ldots 1 + ax^i,$$

and of the reciprocal of

$$1 - a \,.\, 1 - ax \,.\, 1 - ax^2 \,.\, 1 - ax^3 \ldots 1 - ax^i,$$

which appeared in *part* in the *Johns Hopkins Circular* for February* last. This is, I think, distinguishable from the ordinary proofs as being, so to say, *classical* in form (using the word in an algebraical sense), inasmuch as it establishes the identity of two rational integral functions, one explicitly, the other implicitly given, by comparison of their zeros.

Let the coefficient of a^j in the expansion of

$$(1 + ax)(1 + ax^2) \ldots (1 + ax^i),$$

say in the expansion of $F(x, a)$, be called J_x, and

$$\frac{1 - x^i \,.\, 1 - x^{i-1} \ldots 1 - x^{i-j+1}}{1 - x \,.\, 1 - x^2 \ldots 1 - x^j}$$

be called X_j.

J_x being the sum of the j-ary combinations of $x, x^2, \ldots x^i$ will necessarily contain $x^{1+2+\ldots+j}$, that is $x^{\frac{j^2+j}{2}}$, and will be of the degree

$$i + (i - 1) + \ldots + (i - j + 1)$$

in x, and therefore of the same degree as $X_j x^{\frac{j^2+j}{2}}$.

All the linear factors of X_j are obviously of the form $x - \rho$, where $x - \rho$ is a primitive factor of some binomial expression $x^r - 1$: the number of times that any $x - \rho$ occurs in X_j will obviously be equal to $E\frac{i}{r} - E\frac{j}{r} - E\frac{i-j}{r}$ which is either 1 or 0. Now consider $F(\rho, a)$, the value of $F(x, a)$ when x becomes ρ. Let $i = kr + \delta$, where $\delta < r$; then $F(\rho, a) = (1 \pm a^r)^k$ multiplied

[* Vol. III. of this Reprint, p. 677.]

by δ linear functions of a, and consequently if $j = k'r + \delta'$, where $\delta' < r$, J_x vanishes when $\delta' > \delta$, in which case

$$E\frac{i}{r} - E\frac{j}{r} - E\frac{i-j}{r} = 1.$$

Hence any linear factor $x - \rho$ of X_j possesses the two-fold property of being unrepeated and of being contained in J_x. Hence J_x must contain $X_j x^{\frac{j^2+j}{2}}$, and being of the same degree as it is in x must bear to it a constant ratio, which, by making $x = 1$, is seen to be that of the coefficient of a^j in $(1+a)^i$, that is of $\frac{i(i-1)(i-2)\dots(i-j+1)}{1.2.3\dots j}$ to the product of the fractions in their vanishing state

$$\frac{1-x^i}{1-x}, \frac{1-x^{i-1}}{1-x^2}, \dots, \frac{1-x^{i-j+1}}{1-x^j},$$

that is, is a ratio of equality, so that $J_x = X_j x^{\frac{j^2+j}{2}}$. Q.E.D.

(23) Again let X_j and J_x now stand respectively for

$$\frac{1-x^{i+1}.1-x^{i+2}\dots 1-x^{i+j}}{1-x.1-x^2\dots 1-x^j}$$

and the coefficient of a^j in the reciprocal of $1-a.1-ax\dots 1-ax^i$ (say $F(x,a)$); this latter is the sum of homogeneous products of the jth order of $1, x, x^2, \dots x^i$, and is therefore of the degree ij which is also the degree (as is obvious) of X_j in x. For like reason as in what precedes $x-\rho$, any linear factor of $x^r - 1$, is contained 1 or 0 times in X_j according as

$$E\frac{i+j}{r} - E\frac{i}{r} - E\frac{j}{r} = 1 \text{ or } 0.$$

Let the minimum negative residue of $i+1$ to modulus r be $-\delta$; $F(\rho, a)$ may be expressed as the product of δ linear functions of a, divided by a power of $1-a^r$, and the only power of a (say a^θ) which appears in its development will accordingly be those for which the residue of θ in respect to r is $0, 1, 2, \dots \delta$, and consequently if a^θ appears in the development

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 0,$$

or conversely if $x-\rho$ is a factor of X_j so that

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 1,$$

J_x vanishes. Hence J_x contains each linear factor of X_j, and these being simple, contains X_j itself, and on account of their degrees in x being the same must bear to it a ratio independent of x, which, by making $x = 1$, so that the things to be compared are the coefficient of a^j in $\dfrac{1}{(1-a)^{i+1}}$ and the product of the vanishing fractions $\dfrac{1-x^{i+1}}{1-x}, \dfrac{1-x^{i+2}}{1-x^2}, \ldots, \dfrac{1-x^{i+j}}{1-x^j}$, is readily seen to be a ratio of equality, so that $J_x = X_j$. Q.E.D.

(24) (B) *On the General Term in the Generating Function to Partitions into parts limited in number and magnitude*, by Dr F. FRANKLIN.

To prove that the coefficient of a^j in the development of

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots(1-ax^i)} \text{ is } \frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

I showed that the number of partitions of w into i or fewer parts, subject to the condition that the first excess (the excess of the first part over the second) is not greater than j, is the coefficient of x^w in

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\ldots(1-x^i)},$$

and in general that the number of partitions in which the rth excess (the excess of the first part over the $(r-1)$th) is not greater than j, is the coefficient in

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}.$$

If we look at the question reversely, namely, the coefficient of a^j in

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots(1-ax^i)}$$

being known to be

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

if we ask what is the significance of the fractions

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\ldots(1-x^i)}, \ldots, \frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

the answer is immediately given by the generating function itself. For

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\ldots(1-x^i)}$$

$$= \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^i)} \cdot \frac{1-x^{j+1}}{1-x}$$

$$= \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^i)} \left(\text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)}\right)$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)(1-x^2)(1-x^3)\ldots(1-x^i)}.$$

But the coefficient of $a^j x^w$ in the last written fraction is obviously the number of ways in which w can be composed of the numbers $1, 2, 3, \ldots i$, using not more than j 1's. And the number of 1's in a given partition is equal to the excess of the first part over the second part in its conjugate. In like manner

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)\ldots(1-ax^r)(1-x^{r+1})\ldots(1-x^i)};$$

and the coefficient of $a^j x^w$ in the fraction on the right is the number of ways in which w can be composed of the parts $1, 2, 3, \ldots i$, not more than j of the parts being as small as r. But the number of 1's in a given partition is equal to the excess of the first part over the second in its conjugate; the number of 2's to the excess of the second part over the third, and so on. Hence the number of 1's plus the number of 2's ... plus the number of r's in a given partition is equal to the excess of the first part over the rth part in its conjugate; and we have thus proved that the coefficient of x^w in the development of

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

may be indifferently regarded as the number of partitions of w into parts none greater than i and not more than j of them as small as r or as the number of partitions of w into j or fewer parts, the excess of the first part over the rth part being as small as j. These results may obviously be extended by introducing the a in non-consecutive factors of the product

$$(1-x)(1-x^2)\ldots(1-x^i).$$

(25) (C) *On the theorem of one-to-one and class-to-class correspondence between partitions of* n *into uneven and its partitions into unequal parts*, by Dr F. Franklin.

The number of partitions of w into k distinct odd numbers, each repeated an indefinite number of times, is evidently the coefficient of $a^k x^w$ in the development of

$$\left(1+a\frac{x}{1-x}\right)\left(1+a\frac{x^3}{1-x^3}\right)\left(1+a\frac{x^5}{1-x^5}\right)\ldots.$$

It is not easy to form the generating function for the number of partitions containing k sequences, but it is plain that the number of partitions of w containing *one* sequence is the coefficient of x^w in

$$S_1 + S_2 + S_3 + \ldots,$$

where

$$S_1 = x + x^2 + x^3 + x^4 + x^5 + \ldots = \frac{x}{1-x}$$

$$S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \ldots = \frac{x^3}{1-x^2}$$

$$S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \ldots = \frac{x^6}{1-x^3}$$

$$S_4 = x^{10} + x^{14} + x^{18} + x^{22} + x^{26} + \ldots = \frac{x^{10}}{1-x^4}$$

$$S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \ldots = \frac{x^{15}}{1-x^5},$$

and in general

$$S_r = x^{1+2+3+\ldots+r} + x^{2+3+4+\ldots+(r+1)} + \ldots = \frac{x^{\frac{1}{2}r(r+1)}}{1-x^r}.$$

So much of Prof. Sylvester's theorem as relates to a single sequence follows from inspection of the above scheme. For $S_1 = \frac{x}{1-x}$; adding to S_3 the first term of S_2, we get $\frac{x^3}{1-x^3}$; adding to S_5 the first term of S_4 and the second term of S_2, we get $\frac{x^5}{1-x^5}$; adding to S_{2m+1} the first term of S_{2m}, the second term of $S_{2(m-1)}$, the third term of $S_{2(m-2)}, \ldots$, and the mth term of S_1, we get $\frac{x^{2m+1}}{1-x^{2m+1}}$; thus the proposition is proved. The fact is made more evident to the eye if we write the scheme as follows:

$$\begin{array}{ll}
S_1 = x + x^2 + x^3 + x^4 + x^5 + \ldots & S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \ldots \\
S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \ldots & S_4 = \quad x^{10} + x^{14} + x^{18} + x^{22} + \ldots \\
S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \ldots & S_6 = \qquad x^{21} + x^{27} + x^{33} + \ldots \\
S_7 = x^{28} + x^{35} + x^{42} + x^{49} + x^{56} + \ldots & S_8 = \qquad\quad x^{36} + x^{44} + \ldots \\
S_9 = x^{45} + x^{54} + x^{63} + x^{72} + x^{81} + \ldots & S_{10} = \qquad\qquad x^{55} + \ldots
\end{array}$$

Here $\frac{x^9}{1-x^9}$, for instance, is obtained by adding the fourth column on the right to the fifth row on the left.

It may be noted that we have thus found that

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \ldots + \frac{x^{2m+1}}{1-x^{2m+1}} + \ldots$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \ldots + \frac{x^{\frac{1}{2}n(n+1)}}{1-x^n} + \ldots.$$

(26) [Compare Jacobi's theorem contained in the last-but-one two lines of the last but one page of the *Fundamenta Nova*, which may be easily reduced to the form

$$\frac{x}{1+x}-\frac{x^3}{1+x^3}+\frac{x^5}{1+x^5}\ldots=\frac{x}{1+x}-\frac{x^3}{1+x^2}+\frac{x^5}{1+x^3}-\ldots. \quad \text{J. J. S.]}$$

Act II. On the Graphical Conversion of Continued Products into Series.

Naturelly, by composicionns
Of anglis, and slie reflexiouns.
The Squieres Tale.

(27) The method about to be explained of representing the elements of partitions by means of a succession of angles fitting into one another arose out of an investigation (instituted for the purpose of facilitating the arrangement of tables of symmetric functions)* as to the number of partitions of n which are their own conjugates. The ordinary graphs to such partitions must obviously be symmetrical in respect to the two nodal boundaries, as seen below.

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Let the above figure be any such graph; it may be dissected into a square (which may contain one or any greater square number) of say i^2 nodes, and two perfectly similar appended graphs, each having the content $\frac{n-i^2}{2}$, and subject to the sole condition that the number of its lines (or columns), that is that the number (or magnitude) of the parts in the partition which it represents, shall be i or less; such number is the coefficient of $x^{\frac{n-i^2}{2}}$ in $\frac{1}{1-x.1-x^2\ldots 1-x^i}$, which is the same as that of x^{n-i^2} in

$$\frac{1}{1-x^2.1-x^4\ldots 1-x^{2i}}$$

or of x^n in

$$\frac{x^{i^2}}{1-x^2.1-x^4\ldots 1-x^{2i}}.$$

* By Mr Durfee, of California (Fellow of the Johns Hopkins University), to whom I suggested the desirability of investigating more completely than had been done the method of arrangement of such tables founded upon the notion of self-conjugate partitions, which M. Faà de Bruno had the honour of initiating. The very valuable results of Mr Durfee's inquiries, embodying, systematising and completing the theory of arrangement originated by Professor Cayley, and further illustrated by the labours of Professors Betti and De Bruno, will probably appear in the next number of the *Journal*.

Hence giving i all possible values we see that the coefficient of x^n in the infinite series

$$1+\frac{x}{1-x^2}+\frac{x^4}{1-x^2.1-x^4}+\frac{x^9}{1-x^2.1-x^4.1-x^6}+\dots$$

is the number of self-conjugate partitions of n, or which is the same thing of symmetrical groups whose content is n.

(28) But any such graph, in which there is a square of i^2 nodes with its two appendices, may be dissected in another manner into i angles or bends, each containing any continually decreasing odd number of nodes, and *vice versâ*, any set of equilateral angles of nodes continually decreasing in number (which condition is necessary in order that the lower lines and posterior columns may not protrude beyond the upper lines and anterior columns) when fitted into one another in the order of their magnitudes will form a regular graph. Thus the actual figure (where there is a square of 9 nodes) formed by the intersections of the lines and columns may be dissected into 3 angles containing respectively 13, 11, 3 nodes; and so in general the number of ways in which n can be made up of odd and unrepeated parts will be the same as the number of ways in which $\frac{n-j^2}{2}$ can be partitioned into not more than j parts; hence we see that the coefficients of $x^n a^j$ in

$$(1+ax)(1+ax^3)\dots(1+ax^{2j-1})\dots$$

and in

$$\frac{x^{j^2}}{1-x^2.1-x^4\dots1-x^{2j}}$$

are the same, so that the continued product above written is equal to

$$1+\frac{x}{1-x^2}a+\dots+\frac{x^{j^2}}{1-x^2.1-x^6\dots1-x^{2j}}a^j+\dots$$

as is well known.

(29) In like manner if the expansion in a series of ascending powers of a of the finite continued product

$$(1+ax)(1+ax^3)\dots(1+ax^{2i-1})$$

be required, the coefficient of x^n in the coefficient of a^j will be the number of ways in which n can be made up with j of the unrepeated numbers $1, 3, \dots 2i-1$, and as $2i-1$ is the number of nodes in an equilateral angle whose sides contain i nodes, it follows that this coefficient will be the number of ways in which $\frac{n-j^2}{2}$ can be composed with parts none *exceeding* $i-j$ in magnitude, and will therefore be the same as the coefficient of $x^{\frac{n-j^2}{2}}$ in

$$\frac{1-x^{i-j+1}.1-x^{i-j+2}\dots1-x^i}{1-x.1-x^2\dots1-x^j},$$

and consequently the finite continued product above written is equal to

$$1+\dots+\frac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\dots1-x^{2i}}{1-x^2.1-x^4\dots1-x^{2j}}x^{j^2}a^j+\dots$$

(30) If it be required to ascertain how many self-conjugate partitions of n there are containing exactly i parts, this may be found by giving j all possible values and making p_j equal to the number of ways in which $\frac{n-j^2}{2}$ can be composed with j or fewer parts the greatest of which is $i-j$, that is $(n-j^2+2j-2i)/2$ with $j-1$ or fewer parts none greater than $i-j$, so that p_j will be the coefficient of $x^{(n-j^2+2j-2i)/2}$ in

$$\frac{1-x^{i-j+1}\,.\,1-x^{i-j+2}\ldots 1-x^{i-1}}{1-x\,.\,1-x^2\ldots 1-x^{j-1}}$$

or of x^n in

$$\frac{1-x^{2i-2j+2}\,.\,1-x^{2i-2j+4}\ldots 1-x^{2i-2}}{1-x^2\,.\,1-x^4\ldots 1-x^{2j-2}}\,x^{j^2-2j+2i};$$

the sum of the values of p_j for all values of j will be the number required: this number, therefore, writing ω for $2i-1$, will be the coefficient of x^n in

$$x^\omega+\frac{1-x^{\omega-1}}{1-x^2}\,x^{\omega+1}+\frac{1-x^{\omega-1}\,.\,1-x^{\omega-3}}{1-x^2\,.\,1-x^4}\,x^{\omega+4}+\text{etc.};$$

the outstanding factor in the qth term in this series being $x^{\omega+(q-1)^2}$ we may suppose q the least integer number not less than $1+\sqrt{(n-\omega)}$ and then the subsequent term to the $(q+1)$th being inoperative may be neglected.

(31) In order to see how any self-conjugate graph may be recovered, so to say, from the corresponding partition consisting of unrepeated odd numbers, consider the diagrammatic case of the partition 17, 9, 5, 1 represented by the angles of the graph below written

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The number of angles is the number of the given parts, that is 4, and the first four lines of the graph will be obtained by adding 0, 1, 2, 3 to the major half (meaning the integer next above the half) of 17, 9, 5, 1, that is will be 9, 6, 5, 4, the total number of lines will be the major half of the highest term (17) and the remaining lines will have the same contents, namely 3, 2, 1, 1, 1, as the columns of the graph found by subtracting 4 (the number of the parts) from the numbers last found, that is will be the lines of the graph which is conjugate to 5, 2, 1. And so in general the self-conjugate graph corresponding to any partition of unrepeated odd numbers $q_1, q_2, \ldots q_j$ will be found by the following rule:

Let P be the system of partitions $k_1, k_2, \dots k_j$, in which any term k_θ is the major half of q_θ augmented by $\theta - 1$, and P' another system of $k_1', k_2', \dots k_j'$, obtained by subtracting j from each term in P, then P and the conjugate to P' will be the self-conjugate partition corresponding to the given q partition. Thus as an example, 19, 11, 7, 5 being given, P, P' will be 10, 7, 6, 6; 6, 3, 2, 2 respectively, and the self-conjugate system required will be 10, 7, 6, 6, 4, 4, 2, 1, 1, 1. Of course P' might also be obtained by taking the minor halves of the given parts in inverse (ascending) order and subtracting from them the numbers 0, 1, 2, ... respectively.

To pass from a given self-conjugate to the corresponding unrepeated odd numbers-partition is a much simpler process, the rule being to take the numbers in descending order and from their doubles subtract the successive odd numbers in the natural scale until the point is reached at which the difference is about to become negative; thus the partition 6 6 5 4 3 2 is self-conjugate, and the correspondent to it is 11 9 5 1.

(32) The expansion of the reciprocal to $(1-ax)(1-ax^3)\dots(1-ax^{2i-1})$ may be read off with the same facility as the direct product. In this case we are concerned with partitions of odd numbers capable of being repeated in the same partition; now, therefore, if we use the same method of equilateral angles as before, and fit them into one another in regular order of magnitude, it will no longer be the case that their sum will form a regular graph, for if there be θ parts alike, each line and column which ranges with either side of any (but the first one) of these will jut out one step beyond the anterior line and column (respectively), so that the line joining the extremities of the lines or columns will be parallel to the axis of symmetry. The figure then corresponding to i odd parts can no longer be dissected into a *square* of nodes and two equal regular graphs, but it may be dissected into a *line* of nodes lying in the axis of symmetry, and two regular graphs one of which has for its boundaries one of the original boundaries and a line of nodes parallel to the axis of symmetry, and the other one the other original boundary and a line of nodes parallel to the same axis, as seen in the annexed figure, where the axial points are distinguished by being made larger than the rest.

```
✱   *   *   *   *   *
*   ✱   *   *   *   *   *
*   *   ✱   *   *   *   *   *
*   *   *   ✱   *   *   *
*   *   *   *   ✱   *
*   *   *   *   *   ✱   *
    *   *   *       *
        *
```

The graph read off in angles represents the partition 11 11 11 7 3 3. On removing the six diagonal nodes it breaks up into two regular graphs, of which one is 5 5 5 3 1 1, and the other the conjugate thereto; hence the coefficient of x^n in the coefficient of a^j in the expansion of the reciprocal of $1-ax\,.\,1-ax^3\,\ldots\,.\,1-ax^{2i-1}$ in ascending powers of a is the number of ways in which $\frac{n-j}{2}$ can be resolved into j parts limited not to exceed $i-1$, which is the coefficient of $x^{\frac{n-j}{2}}$ in

$$\frac{1-x^i\,.\,1-x^{i+1}\,\ldots\,1-x^{i+j-1}}{1-x\,.\,1-x^2\,\ldots\,1-x^j}$$

or of x^n in
$$\frac{1-x^{2i}\,.\,1-x^{2i+2}\,\ldots\,1-x^{2i+2j-2}}{1-x^2\,.\,1-x^4\,\ldots\,1-x^{2j}}x^j.$$

(33) Although I shall not require any intermediate expansion whatever in order to obtain the transcendant $\Theta_1 x$ product in the form of a series, I will give another of those which are sometimes employed together in combination (see Cayley, *Elliptic Functions*, pp. 296—7) to obtain this result: thus to prove that the continued product of the reciprocal of

$$(1-ax)(1-ax^2)(1-ax^3)\ldots$$

is identical with

$$1+\frac{x}{1-x}\cdot\frac{a}{1-xa}+\frac{x^4}{1-x\,.\,1-x^2}\cdot\frac{a^2}{1-xa\,.\,1-x^2a}$$
$$+\frac{x^9}{1-x\,.\,1-x^2\,.\,1-x^3}\cdot\frac{a^3}{1-xa\,.\,1-x^2a\,.\,1-x^3a}+\text{etc.}$$

if n is partitioned into j parts, the regular graph which represents the result of any such partition must consist either of 1, 2, 3, ... $j-1$ or of not less than j columns, and its graph may accordingly in these several cases be dissected into a square of 1, 4, 9, ... j^2 nodes; suppose that such square consists of θ parts, then there will be $n-\theta^2$ nodes remaining over subject to distribution into two groups limited by the condition as to one of the groups that it may contain an unlimited number of parts none exceeding θ in magnitude, and as to the other that it must contain $j-\theta$ parts none exceeding θ in magnitude, as seen in the following diagrams:

$$\begin{array}{ccccccc}
* & * & * & * & * & * & * \\
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* & * & * & * & * & * & \\
\times & \times & & & & & \\
\times & \times & & & & & \\
\times & & & & & & \\
\times & & & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & & & & \\
* & * & * & & & & & & \\
\times & \times & \times & & & & & & \\
\times & \times & \times & & & & & & \\
\times & \times & & & & & & & \\
\times & & & & & & & &
\end{array}$$

$$\begin{array}{ccc}
* & * & * \\
* & * & * \\
* & * & * \\
\times & \times & \times \\
\times & \times & \times \\
\times & & \\
\times & &
\end{array}
\qquad
\begin{array}{ccc}
* & * & * \\
* & * & * \\
* & * &
\end{array}$$

in all of which the partible number is 26, and j and θ are 7 and 3 respectively. Now the number of such distributions is the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1}{1-x\,.\,1-x^2\ldots 1-x^\theta}\cdot\frac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta},$$

that is of $x^n a^j$ in

$$\frac{x^{\theta^2}}{1-x\,.\,1-x^2\ldots 1-x^\theta}\cdot\frac{a^\theta}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta},$$

and consequently giving θ all values from 1 to ∞, the proposed equation is verified.

(34) It may be desired to apply the same method to obtain a similar development for the reciprocal of the limited product

$$(1-ax)(1-ax^2)\ldots(1-ax^i);$$

the construction will be the same as in the last case; the distribution into two groups can be made as before; the second group will remain subject to the same condition as in the preceding case (seeing that the number of parts being less than $j-\theta$, will necessarily be less than $i-\theta$, for j cannot exceed i), but the first group will be subject to the condition of being partitioned not now into an unlimited but into $i-\theta$ (or fewer) parts none exceeding θ in magnitude, and the number of such distributions into the two groups will accordingly become the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1-x^{i-\theta+1}\,.\,1-x^{i-\theta+2}\ldots 1-x^i}{1-x\,.\,1-x^2\ldots 1-x^\theta}\cdot\frac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta}$$

or of $x^n a^j$ in the last written fraction multiplied by $x^{\theta^2}\,.\,a^\theta$, so that the required expansion will be

$$1+\frac{1-x^i}{1-x}\cdot\frac{xa}{1-ax}+\frac{1-x^i\,.\,1-x^{i-1}}{1-x\,.\,1-x^2}\cdot\frac{x^4a^2}{1-ax\,.\,1-ax^2}$$
$$+\frac{1-x^i\,.\,1-x^{i-1}\,.\,1-x^{i-2}}{1-x\,.\,1-x^2\,.\,1-x^3}\cdot\frac{x^9a^3}{1-ax\,.\,1-ax^2\,.\,1-ax^3}+\ldots.$$

(35) It is interesting to investigate what will be the form of the mixed development resulting from an application of the same method to the *direct* product

$$1+ax\,.\,1+ax^2\ldots 1+ax^i.$$

For greater clearness I shall first suppose i indefinitely great. Consider the diagram:

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In the above graph j and θ used in the same sense as *ante* are 5 and 3 respectively, so that there is a square of 9 points; an appendage to the right of and another appendage below the square, which I shall call the lateral and subjacent appendages respectively. The content of the graph being 25, there are 16 points to be distributed between these two appendages. What now are the conditions of the distribution of the $n-\theta^2$ points between them?

I say that there will be two sorts of such distribution—one in which the lateral appendage will consist of θ unrepeated parts, none of them zero, as in the graph above, and the subjacent appendage of $j-\theta$ unrepeated parts, limited not to exceed θ in magnitude, and another sort as in the graph below written,

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*  *  *  *  *  *  *  *  *  *  *  *
*  *  *  *  *  *  *
*  *  *
*  *
*
```

in which the jth line of the lateral appendage is missing, and consequently the subjacent graph will consist of $j-\theta$ unrepeated parts limited not to exceed $\theta-1$ in magnitude, for there could not be a part so great as θ without the last line of the square having the same content as the first line of the subjacent appendage.

It should be observed that only the *last* admissible line of the lateral appendage can be wanting, for if more than this were wanting, two lines of the square would belong to the graph, and consequently there would be two equal parts θ.

Hence there are two kinds of association of the appendages, one leading to a distribution of $n-\theta^2$ between one group of θ unrepeated but unlimited parts, and another of $j-\theta$ unrepeated parts limited not to exceed θ; the other to a distribution of $n-\theta^2$ between one group of $\theta-1$ unrepeated but unlimited parts, and another of $j-\theta$ unrepeated parts limited not to exceed $\theta-1$.

The number of distributions of the first kind is the coefficient of $x^{n-\theta^2}.\,a^{j-\theta}$ in

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1-x\,.\,1-x^2\ldots 1-x^\theta}\,.\,(1+ax)(1+ax^2)\ldots(1+ax^\theta),$$

the other of $x^{n-\theta^2}.\,a^{j-\theta}$ in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x\,.\,1-x^2\ldots 1-x^{\theta-1}}\,.\,(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1});$$

hence the sum of the distributions of the two kinds is the coefficient of the same argument in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x\,.\,1-x^2\,\ldots\,1-x^\theta}\{x^\theta(1+ax^\theta)+(1-x^\theta)\}\{1+ax\,.\,1+ax^2\,\ldots\,1+ax^{\theta-1}\},$$

that is of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^\theta\left(\frac{1+ax\,.\,1+ax^2\,\ldots\,1+ax^{\theta-1}}{1-x\,.\,1-x^2\,\ldots\,1-x^{\theta-1}}\cdot\frac{1+ax^{2\theta}}{1-x^\theta}\right),$$

and consequently we obtain the equation

$$1+ax\,.\,1+ax^2\,.\,1+ax^3\,\ldots=1+\frac{1+ax^2}{1-x}xa+\frac{1+ax\,.\,1+ax^4}{1-x\,.\,1-x^2}x^5a^2+\ldots$$
$$+\frac{1+ax\,.\,1+ax^2\,\ldots\,1+ax^{j-1}\,.\,1+ax^{2j}}{1-x\,.\,1-x^2\,\ldots\,1-x^{j-1}\,.\,1-x^j}x^{\frac{3j^2-j}{2}}a^j+\ldots,$$

and thus by a very unexpected route we arrive at a proof of Euler's celebrated pentagonal-number theorem; for on making $a=-1$ the above equation becomes

$$1-x\,.\,1-x^2\,.\,1-x^3\,\ldots=1-(1+x)\,x+(1+x^2)\,x^5\ldots+(-)^j(1+x^j)\,x^{\frac{3j^2-j}{2}}+\ldots.$$

Such is one of the fruits among a multitude arising out of Mr Durfee's ever-memorable example of the dissection of a graph (in the case of a symmetrical one) into a square, and two regular graph appendages.

Even the trifling algebraical operation above employed to arrive at the result might have been spared by expressing the continued product as the sum of the two series (which flow immediately from the graphical dissection process), left uncombined, namely,

$$1+\frac{1+ax}{1-x}x^2a+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}x^7a^2+\frac{1+ax\,.\,1+ax^2\,.\,1+ax^3}{1-x\,.\,1-x^2\,.\,1-x^3}x^{15}a^3+\ldots,$$

together with

$$+xa+\frac{1+ax}{1-x}x^5a^2+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}x^{12}a^3+\ldots,$$

which for $a=-1$ unite into the single series

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}\text{ etc.}$$

(36) I will now proceed to find the expression in a mixed series of the limited product

$$1+ax\,.\,1+ax^2\,\ldots\,1+ax^i.$$

In each of the two systems of distribution (as shown already in the theory of the reciprocal of such product) the second group will remain unaffected by the new limitation, but the first group will now consist of partitions (limited in number as before), but in magnitude instead of being unlimited, limited not to exceed $(i-\theta)$, so that we will have to take the coefficient of $x^{n-\theta^2}.\,a^{j-\theta}$ in the sum of

$$x^{\frac{\theta^2+\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+1}}{1-x.1-x^2\ldots 1-x^{\theta}}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta})$$

and

$$x^{\frac{\theta^2-\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1}).$$

This will be the same as the coefficient of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^{\theta}(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1})\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}.1-x^{\theta}}$$
$$\times\{1-x^{\theta}+(1-x^{i-2\theta+1})(x^{\theta}+ax^{2\theta})\},$$

where the quantity within the final bracket is equal to

$$1-x^{i+1}a-x^{i-\theta+1}+x^{2\theta}a.$$

Hence the required series is

$$\left\{1+\frac{1-x^i}{1-x}xa+\frac{1-x^{i-1}.1-x^{i-2}}{1-x.1-x^2}(1+ax)x^5a^2\right.$$
$$\left.+\frac{1-x^{i-2}.1-x^{i-3}.1-x^{i-4}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{12}a^3+\ldots\right\}$$
$$+\left\{\frac{1-x^{i-1}}{1-x}x^3a^2+\frac{1-x^{i-2}.1-x^{i-3}}{1-x.1-x^2}(1+ax)x^9a^3\right.$$
$$\left.+\frac{1-x^{i-3}.1-x^{i-4}.1-x^{i-5}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{18}a^4+\ldots\right\},$$

the indices in the outstanding powers of x being the pentagonal numbers in the first, and the triangular numbers trebled, in the second of the above series.

In obtaining in the preceding articles mixed series for continued products, it will be noticed that the graphical method has been employed, not to exhibit correspondence, but as an instrument of transformation. The graphs are virtually segregated into classes, and the number of them contained in each class separately determined. (The magnitude of the square in the Durfee-dissection serves as the basis of the classification.)

(37) Now let us consider the famous double product of

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

by

$$(1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\ldots.$$

Here it will be expedient to introduce a new term and to explain the meaning of a bi-partition and a system of parallel bi-partitions of a number. The former indicates that the elements are to be distributed into two groups, say into a left and right-hand group: the latter that the number of the elements (on one, say) on the left-hand side of each bi-partition of the system is to be equal to or exceed by a constant difference the number (on the other, say) on the right-hand side of the same bi-partition. If we use dots, regularly spaced, to represent the elements (themselves numbers and not units), we get a figure or pair of figures such as the following:

```
*  *  *  *  *  *      *  *  *  *
*  *  *  *  *  *      *  *  *  *
*  *  *  *  *         *  *  *
*  *  *  *            *  *
```

for which the corresponding lines of the contour are respectively parallel—hence the name. When the numbers of elements on the two sides are identical, I call the system an equi-bi-partition-system—in the general case, a parallel bi-partition-system to a constant difference j, where j is the excess of the number of elements in the left-hand over that in the right-hand part of any of the bi-partitions.

(38) Consider now the given double product—it is obvious that it may be expanded in terms of paired powers $a^j + a^{-j}$ of a, and the coefficient of x^n in the term not involving a will evidently be the number of equi-bi-partitions of n that can be formed with unrepeated odd numbers; and so the coefficient of x^n associated with a^j or a^{-j} will be the number of parallel bi-partitions of n to the constant difference j that can be so formed.

For the equi-bi-partitions; suppose $l_1, l_2 \dots l_i, \lambda_1, \lambda_2 \dots \lambda_i$ is an equi-bi-partition, all the elements being odd and unrepeated; take successive angles whose (say horizontal and vertical) sides are the major halves of l_1, λ_1; $l_2, \lambda_2 \dots$; l_i, λ_i; these angles will fit on to one another so as to form a regular graph by reason of the relations

$$l_1 > l_2 + 1, \quad l_2 > l_3 + 1 \dots l_{i-1} > l_i + 1,$$
$$\lambda_1 > \lambda_2 + 1, \quad \lambda_2 > \lambda_3 + 1 \dots \lambda_{i-1} > \lambda_i + 1.$$

Conversely any regular graph may be resolved into angles whose horizontal sides shall be the major halves of one set of odd numbers, and their vertical sides the major halves of another set of as many odd numbers, and these two sets of odd numbers will each form a decreasing series; hence there is a one-to-one conjugate correspondence between any bi-partition of n written in regular order, and the totality of regular graphs whose content is $\frac{n}{2}$, so that the number of the equi-bi-partitions of n will be the coefficient of $x^{\frac{n}{2}}$ in

$$\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3\dots},$$

that is of x^n in

$$\frac{1}{1-x^2\,.\,1-x^4\,.\,1-x^6\dots},$$

which fraction is therefore equal to the totality of the terms not involving a.

(39) Next for the coefficient of a^j.

Let $l_1, l_2, \ldots l_j, l_{j+1}, l_{j+2}, \ldots l_{j+\theta}; \lambda_1, \lambda_2, \ldots \lambda_\theta$ be an equi-parallel bi-partition to the difference j (with the elements on each side written in descending order); with the equi-bi-partition $l_{j+1}, l_{j+2}, \ldots l_{j+\theta}; \lambda_1, \lambda_2, \ldots \lambda_\theta$, form a graph, as in the preceding case; say, for distinctness, with major halves of the l series horizontal and of the λ series vertical; over the highest horizontal line the successive quantities*

$$\frac{l_j-1}{2}, \ \frac{l_{j-1}-3}{2}, \ \frac{l_{j-2}-5}{2}, \ldots \frac{l_1-(2j-1)}{2}$$

may be laid so as to form a regular graph of which the content will be $\frac{n-j^2}{2}$.

Conversely every regular graph whose content is $\frac{n-j^2}{2}$ will correspond to a parallel bi-partition of unrepeated odd numbers to a difference j; to obtain the bi-partition the first j lines of the graph must be abstracted†, and the graph thus diminished resolved into angles; the doubles of the contents of each vertical side of these angles diminished by unity will constitute the right-hand side of the bi-partition, and the doubles of the contents of each horizontal side preceded by the doubles of the lines of the abstracted portion of the graph increased by $1, 3, 5, \ldots 2j-1$ respectively, will form the left-hand portion. Hence the number of such bi-partitions will be the number of ways of resolving $\frac{n-j^2}{2}$ into unrestricted parts, that is, will be the coefficient of x^n in

$$\frac{1}{1-x^2 \,.\, 1-x^4 \,.\, 1-x^6 \ldots} x^{j^2},$$

and this being true for all values of n and j, we see that the double product in question will be identical with the infinite series

$$\frac{1}{1-x^2 \,.\, 1-x^4 \,.\, 1-x^6 \ldots}\{1+x(a+a^{-1})+x^4(a^2+a^{-2})+x^9(a^3+a^{-3})+\ldots\}.$$

(40) To expand the limited double product

$$(1+ax)(1+ax^3)\ldots(1+ax^{2i-1})$$

into

$$(1+a^{-1}x)(1+a^{-1}x^3)\ldots(1+a^{-1}x^{2i-1})$$

the procedure and reasoning will be precisely the same as in the extreme case of i infinite, the only difference being that the elements of the bi-partition instead of being unlimited odd numbers will be limited not to exceed $2i-1$. In the case of $j=0$ the equi-bi-partition will furnish a series of nodal angles in which neither side can exceed the major half of $2i-1$,

* Any number of these quantities may happen to become zero.

† If the actual number of horizontal lines in the graph is less than j, it must be made to count as j, by understanding lines of zero content to be supplied underneath the graph.

that is i, and the coefficient of x^n in the term not containing any power of a will consequently be the number of ways in which n can be divided into parts limited as well in number as in magnitude not to exceed i, and will therefore be the same as the coefficient of $x^{\frac{1}{2}n}$ in the development of

$$\frac{1-x^{i+1}\,.\,1-x^{i+2}\ldots 1-x^{2i}}{1-x\,.\,1-x^2\ldots 1-x^i},$$

or, which is the same thing, of x^n in the development of

$$\frac{1-x^{2i+2}\,.\,1-x^{2i+4}\ldots 1-x^{4i}}{1-x^2\,.\,1-x^4\ldots 1-x^{2i}},$$

and when the bi-partition system has a constant difference j, the corresponding graph will be of the same form, except that it will be overlaid with j lines, obtained as in the preceding case by subtracting $1, 3, \ldots 2j-1$ from the first j left-hand elements, and taking the halves of the remainders; the graphs thus formed will be subject to the condition of having a content $\frac{n-j^2}{2}$, and parts limited not to exceed $i-j$ in magnitude nor $i+j$ in number [$i-j$ in magnitude because the topmost line cannot exceed $\frac{(2i-1)-(2j-1)}{2}$ in content; $i+j$ in number because without reckoning the j superimposed lines the subjacent portion of the graph cannot contain more than i lines]. The converse that out of every regular graph fulfilling these conditions may be spelled out a parallel bi-partition with a difference j, and containing only unrepeated odd numbers limited not to exceed $2i-1$ in magnitude may be shown as in the preceding case. Hence the coefficient of x^n in the coefficient of a^j+a^{-j} in the expansion, is the number of ways of resolving $\frac{n-j^2}{2}$ into parts none exceeding $i-j$ in magnitude nor $i+j$ in number, that is, is the coefficient of x^n in

$$\frac{1-x^{2i+2j+2}\,.\,1-x^{2i+2j+4}\ldots 1-x^{4i}}{1-x^2\,.\,1-x^4\ldots 1-x^{2i-2j}}\,x^{j^2}.$$

Hence by the process of reasoning, which has been so often applied, we see that the finite double product

$$1+ax\,.\,1+ax^3\ldots 1+ax^{2i-1}$$

into

$$1+a^{-1}x\,.\,1+a^{-1}x^3\ldots 1+a^{-1}x^{2i-1}$$

$$=\frac{1-x^{2i+2}\,.\,1-x^{2i+4}\ldots 1-x^{4i}}{1-x^2\,.\,1-x^4\ldots 1-x^{2i}}\left\{1+\frac{1-x^{2i}}{1-x^{2i+2}}x+\frac{1-x^{2i}\,.\,1-x^{2i-2}}{1-x^{2i+2}\,.\,1-x^{2i+4}}x^4\right.$$
$$\left.+\frac{1-x^{2i}\,.\,1-x^{2i-2}\,.\,1-x^{2i-4}}{1-x^{2i+2}\,.\,1-x^{2i+4}\,.\,1-x^{2i+6}}x^9+\ldots\right\}.$$

Compare Hermite, *Note sur les fonctions elliptiques*, p. 35, where Cauchy's method is given of arriving at this and the preceding identity.

Act III. On the One-to-one and Class-to-class Correspondence between Partitions into Uneven and Partitions into Unequal Parts.

. . . mazes intricate,
Eccentric, intervolved, yet regular
Then most, when most irregular they seem.
Paradise Lost, v. 622.

(41) It has been already shown that any partition of n into unequal parts may be converted into a partition consisting of odd numbers equal or unequal by, first, expressing any even part by its longest odd divisor, say its nucleus and a power of 2, and, second, adding together the powers of 2 belonging to the same nucleus, so that there will result a sum of odd nuclei, each occurring one or more times; a like process is obviously applicable to convert a partition in which any number occurs 1, 2,... or $(r-1)$ times into one in which only numbers not divisible by r occur with unrestricted liberty of recurrence. The nuclei will here be numbers not divisible by r multiplied by powers of r, and by adding together the powers of r belonging to the same nucleus there results a series of nuclei, each occurring one or more times. Conversely when the nuclei and the number of occurrences of each are given, there being only one way in which any such number can be expressed in the scale whose radix is r, it follows that there is but one partition of the previous kind in which one of the latter kind can originate, and there is thus a one-to-one correspondence, and consequently equality of content between the two systems of partitions.

(42) To return to the case of $r = 2$, with which alone we shall be here occupied, we see that the number of parts in the unequal partition which corresponds after this fashion with a partition made up of given odd numbers depends on the sum of the places occupied when the number of occurrences of each of the odd numbers is expressed in the notation of dual arithmetic. Such correspondence then is eminently arithmetical and transcendental in its nature, depending as it does on the forms of the numbers of repetitions of each different integer with reference to the number 2.

Very different is the kind of correspondence which we are now about to consider between the self-same two systems, as well in its nature, which is essentially graphical, as in its operation, which is to bring into correspondence the two systems, not as wholes but as separated each of them into distinct classes; and it is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated*.

* Just so it is possible for two triangles to stand in a treble perspective relation to each other, as I have had previous occasion to notice in this *Journal*.

(43) I shall call the totality of the partitions of n consisting of odd numbers the U, and that consisting of unequal numbers the V system.

I say that any U may be converted into a V by the following rule: Let each part of the given U be converted into an equilateral bend, and these bends fitted into one another as was done in the problem of converting the reciprocal of

$$(1-ax)(1-ax^3)(1-ax^5)\dots$$

into an infinite series, considered in the preceding section. We thus form what may be called a bent graph. Then, as there shown, such graph may be dissected into a diagonal line of points and two precisely similar regular graphs. The graph compounded of the diagonal and one of these, it is obvious, will also be regular, and I shall call it the major component of the bent graph; the remaining portion may be called the minor component. Each of these graphs will be bounded by lines inclined to each other at an angle one-half of that contained between the original bounding lines, and each may be regarded as made up of bends fitting into one another. The contents of these bends taken in alternate succession, commencing with the major graph, will form a series of continually decreasing numbers, that is to say, a V partition. As an example let 11 11 9 5 5 5 be the given U partition; this gives rise to the graph

```
A                   D
*   *   *   *   *   *
A'
*   *   *   *   *   *   *
*   *   *   *   *   *   *
*   *   *   *   *   *
*   *   *   *   *   *   *
E                   B
*   *   *   *   *   *   *   *
                    C
    *   *       *   *
                    *
```

Reading off the bends on the major and minor graphs alternately, commencing with BAD, $CA'E$ respectively, there results the regularized partition into unequal numbers

11 10 9 8 6 2.

(44) The application of the rule is facilitated to the eye by at once constructing a graph, the number of points in whose horizontal lines are the major halves of the given parts, and construing this to signify two graphs, one the graph actually written down, the other the same graph with its first column omitted; for instance in the case before us the graph will be*

```
*   *   *   *   *   *
*   *   *   *   *   *
*   *   *   *   *
*   *   *
*   *   *
*   *   *
```

* This may be regarded as a parallel-ruler form of dislocation of the figure produced by making the portion to the right of the diagonal of larger asterisks revolve about that diagonal

If we call the lines and columns in the directions of the lines and columns of the Durfee-square appurtenant to the graph $a_1 a_2 \ldots a_i$, $\alpha_1 \alpha_2 \ldots \alpha_i$ [i (here 3) being the extent of the side of the square], the partition given by the rule will be

$$a_1 + \alpha_1 - 1, \quad a_1 + \alpha_2 - 2, \quad a_2 + \alpha_2 - 3, \quad a_2 + \alpha_3 - 4, \quad a_3 + \alpha_3 - 5, \ldots$$

$$\ldots [a_{i-1} + \alpha_{i-1} - (2i-3)], \quad [a_{i-1} + \alpha_i - (2i-2)], \quad [a_i + \alpha_i - (2i-1)], \quad [a_i - i],$$

and inasmuch as

$$a_1 = \text{ or } > a_2 = \text{ or } > a_3 \ldots \quad \text{and} \quad \alpha_1 = \text{ or } > \alpha_2 = \text{ or } > \alpha_3 \ldots$$

the above series is necessarily made up of continually decreasing numbers, at all events until the last term is reached. But this term will form no exception, for the fact of i being the content of the side of the square belonging to the transverse graph $\alpha_1, \alpha_2 \ldots, \alpha_i, \alpha_{i+1} \ldots$ implies that $\alpha_i =$ or $> i$, hence

$$[a_i + \alpha_i - (2i-1)] - (a_i - i) = \alpha_i - i + 1 > 0.$$

In the above example the side of the square *nucleus* in the original total graph was supposed to be the same for the major and minor graphs of which it is composed. If we suppose that graph to contain only i nodes in the ith line, then the side of the square to the minor graph which it contains will be $i-1$, and the number of parts given by the angular readings of the two graphs combined will be $2i-1$ instead of $2i$, as for example if the 3rd line in the graph above written be 3 instead of 5, the resulting partition will be 11 10 9 8 2, but we may, if we please, regard this as 11 10 9 8 2 0 and the last term will then still be $a_i - i$, and the general expression will remain unchanged from what it was before.

Next I proceed to the converse of what has been established, namely, that every U may be transformed by the rule into a V, and shall show that any V may be derived from some one (and only one) U.

Whether the number of effective parts in the given V be odd or even, we may always suppose it to be even by supplying a zero part if necessary, and may call the parts $l_1, \lambda_1, l_2, \lambda_2 \ldots, l_i, \lambda_i$. Suppose that it is capable of being derived from a certain U: form with the parts of U a graph expressed in the usual way by equilateral bends or elbows, then the side of the square appurtenant to the regular graph formed by the major half of this, say G, must have for content the given number i.

---

until it coincides with the portion to the left of the diagonal; the graph thus formed (merely as a matter of convenience to the eye) may be then made to revolve about an axis perpendicular to the plane, so as to bring the diagonal out of its oblique into the more usual horizontal position. All this trouble of description might have been saved by beginning not with a bent graph but with a graph formed with *straight* lines of points written *symmetrically* under each other, which is made possible by the fact of there being an *odd* number of points in each line. The graph so formed then resolves itself naturally into a major and minor regular graph.

Let $a_1, a_2 \ldots a_i$, $\alpha_1, \alpha_2 \ldots \alpha_i$ be the contents of the first i rows and first i columns respectively of G, then the equations to be satisfied are

$$a_1 + \alpha_1 - 1 = l_1, \quad a_2 + \alpha_2 - 3 = l_2, \quad a_3 + \alpha_3 - 5 = l_3 \ldots, \quad a_i + \alpha_i - (2i - 1) = l_i,$$

$$a_1 + \alpha_2 - 2 = \lambda_1, \quad a_2 + \alpha_3 - 4 = \lambda_2, \quad a_3 + \alpha_4 - 6 = \lambda_3 \ldots, \quad a_i \qquad - i \qquad = \lambda_i.$$

Hence

$$a_1 - a_2 = \lambda_1 - l_2 - 1 \quad a_2 - a_3 = \lambda_2 - l_3 - 1 \ldots$$

$$a_{i-1} - a_i = \lambda_{i-1} - l_i - 1 \quad a_i = \lambda_i + i,$$

$$\alpha_1 - \alpha_2 = l_1 - \lambda_1 - 1 \quad \alpha_2 - \alpha_3 = l_2 - \lambda_2 - 1 \ldots$$

$$\alpha_{i-1} - \alpha_i = l_{i-1} - \lambda_{i-1} - 1 \quad \alpha_i = l_i - \lambda_i + i - 1,$$

and for all values of θ,

$$l_\theta > \lambda_\theta > l_{\theta+1}.$$

Hence $a_1, a_2 \ldots a_i$ are all positive, and $\alpha_1, \alpha_2 \ldots \alpha_i$ are all at least equal to i. There will therefore be one and only one graph G satisfying the required conditions, namely a graph the contents of whose lines are

$$a_1, a_2, \ldots a_i, \quad A_1, A_2, \ldots A_{\alpha_i} - i$$

[where $A_1, A_2, \ldots A_{\alpha_i} - i$ is the conjugate partition to $\alpha_1 - i, \alpha_2 - i, \ldots \alpha_i - i$]; the partition U will be found by subtracting unity from the doubles of each of those parts. Thus then it has been shown that every U will give rise to some one V, and every V be derived from a determinate U; hence there must exist a one-to-one correspondence between the U and V systems. In a certain sense it is a work of supererogation to show that there is a U corresponding to each V; it would have been sufficient to infer from the linear form of the equations that there could not be more than one U transformable into a V; for each U being associated with a distinct V it would follow that there could be no V's not associated with a U, since otherwise there would be more V's than U's, which we know *aliunde* is impossible.

As an example of what precedes let the partible number be 12. The U system computed exhaustively will be

$11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.3.1^2 \quad 7.1^5 \quad 5^2.1^2 \quad 5.3.1^4$

$5.3^2.1 \quad 5.1^7 \quad 3^4 \quad 3^3.1^3 \quad 3^2.1^6 \quad 3.1^9 \quad 1^{12}$

Underneath of these partitions I will write the major component graph, and underneath this again the corresponding V; we shall thus have the table

11.1	9.3	9.1^3	7.5	$7.3.1^2$	7.1^5
* * * * * * *	* * * * * * *	* * * * * * * *	* * * * * * *	* * * * * * * *	* * * * * * * * *
7.5	$6.5.1$	8.4	$5.4.2.1$	$7.4.1$	9.3

$5^2.1^2$	$5.3.1^4$	$5.3^2.1$	5.1^7	3^4	$3^3.1^3$	$3^2.1^6$	3.1^9	1^{12}
* * *	* * *	* * *	* * *	* *	* *	* *	* *	$(*)^{12}$
* * *	* *	* *	$(*)^7$	* *	* *	* *	$(*)^9$	
*	*	* *		* *	* *	$(*)^6$		
*	*	*		* *	$(*)^3$			
	*							
	*							
$6.3.2.1$	$8.3.1$	$6.4.2$	10.2	$5.4.3$	$7.3.2$	$9.2.1$	11.1	12

Thus we obtain for the V system:

$7.5 \quad 6.5.1 \quad 8.4 \quad 5.4.2.1 \quad 7.4.1 \quad 9.3 \quad 6.3.2.1 \quad 8.3.1$
$6.4.2 \quad 10.2 \quad 5.4.3 \quad 7.3.2 \quad 9.2.1 \quad 11.1 \quad 12$

which are all the ways in which 12 can be broken up into unequal parts*.

The U's corresponding to those given by the arithmetical method of effecting correspondence would be:

$7.5 \quad 1.3^2.5 \quad 1^{12} \quad 1^7.5 \quad 1^5.7 \quad 3.9 \quad 1^3.3^3 \quad 1^9.3 \quad 1^6.3^2$
$1^2.5^2 \quad 3.1^4.5 \quad 1^2.3.7 \quad 1^3.3^3 \quad 11.1 \quad 3^4$

instead of

$11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.3.1^2 \quad 7.1^5 \quad 5^2.1^2 \quad 5.3.1^4$
$5.3^2.1 \quad 5.1^7 \quad 3^4 \quad 3^3.1^3 \quad 3^2.1^6 \quad 3.1^9 \quad 1^{12}$

so that there is absolutely not a single pair the same in the two methods of conjugation.

(45) The object, however, of instituting the graphical correspondence is not to exhibit this variation, however interesting to contemplate, but to find a correspondence between the two systems which shall resolve itself into correspondences between the classes into which each may be subdivided.

Thus we may call U_i that class of U's in which there are i distinct odd numbers, and V_i that class of V's in which there are i sequences with a gap between each two successive ones: the theorem now to be established is that the V corresponding to any U_i is a V_i, so that class corresponds with class, and as a corollary, that the number of ways in which n can be made up by a series of ascending numbers constituting i distinct sequences is the same as the number of ways in which it can be composed with any i distinct odd numbers each occurring any number of times. This part of the investigation which I will presently enter upon is purely graphical. A few remarks and illustrations may usefully precede.

In the example above worked out it will be observed that there are three classes of U's, namely,

$1^{12} \quad 3^4: \quad 11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.1^5 \quad 5^2.1^2$
$3^3.1^3 \quad 3^2.1^6 \quad 3.1^9: \quad 7.3.1^2 \quad 5.3.1^4 \quad 5.3^2.1$

* In Note D, *Interact*, Part 2, I show how this transformation can be accomplished by the continual doubling of a string on itself.

and three classes of V's agreeing with those above in the number of partitions in each, namely,

12 3.4.5: 11.1 9.3 10.2 8.4 7.5 9.2.1
7.3.2 6.5.1 5.4.2.1: 8.3.1 7.4.1 6.4.2.

So again for $n=16$ there will be found to be eleven partitions into odd parts of the third class, which, with their quasi-graphs and corresponding partitions into unequal parts are exhibited below:

$11.3.1^2$	$9.5.1^2$	$9.3^2.1$	$9.3.1^4$	$7.5.1^4$
* * * * * *	* * * * *	* * * * *	* * * * *	* * * *
* *	* * *	* *	* *	* * *
$(*)^2$	$(*)^2$	* *	$(*)^4$	$(*)^4$
		*		
9.6.1	8.5.2.1	8.6.2	10.5.1	9.4.2.1

$7.3.1^6$	$7.3^2.1^3$	$5^2.3.1^3$	$5.3^3.1^2$	$5.3^2.1^5$	$5.3.1^8$
* * * *	* * * *	* * *	* * *	* * *	* * *
* *	* *	* * *	* *	* *	* *
$(*)^6$	* *	* *	* *	* *	$(*)^8$
	$(*)^3$	$(*)^3$	* *	$(*)^5$	
			$(*)^2$		
11.4.1	9.5.2	8.4.3.1	8.5.3	10.4.2	12.3.1

The transformed partitions above written are all of them of the third class (that is consist of three distinct sequences) and comprise all that exist of that class. 16 will correspond to 1^{16} and 1.3.5.7 to itself. All the other partitions of each of the two systems will be of the second class, and will necessarily have a one-to-one graphical correspondence inasmuch as the entire systems have been proved to have such correspondence.

It is worthy of preliminary remark that the association of the first classes of U's and V's given in the previous section will be identical with the association furnished by the graphical method—but whereas in converting V into U by the antecedent process, the two cases of the sequence being of an odd or even order had to be separately considered, the graphical method is uniform in its operation.

Thus 9 8 7 6 a sequence of an even order will be given graphically by

* * * * * * * *
* * * * * * * *

corresponding to 15^2, and 9 8 7 6 5 a sequence of an odd order will be given graphically by

* * *
* * *
* * *
* * *
* * *
* * *
* * *

corresponding to 5^7, whereas it will be observed that $15^2 = (9+6)^{\frac{4}{2}}$ and $5^7 = 5^{\frac{9+5}{2}}$.

It may be noticed that when the major component is an oblate rectangle it gives rise to a sequence of an even order, and when a quadrate or prolate rectangle to one of an odd order.

I subjoin an example of the algorithm by means of which a given V can be transformed into its corresponding U, taking as a first example $V = 10\ 9\ 8\ 5\ 4\ 1$.

The process of finding U is exhibited below:

$$\begin{array}{rrrrl}
3 & 3 & 5 & 5 & (9) \\
2 & 2 & 3 & 3 & (8) \\
 & 4 & 4 & 2 & (7) \\
 & 1 & 3 & 3 & (6) \\
\hline
 & 10 & 8 & 4 & (1) \\
 & 9 & 5 & 1 & (2) \\
\hline
 & 1 & 1 & 1 & (3) \\
 & 4 & 4 & 4 & (4) \\
 & 7 & 7 & 7 & (5)
\end{array}$$

$3^2 . 5^2 . 7^3$ will be the U required.

As a second example let $V = 12\ 10\ 9\ 8\ 5\ 4\ 1$; the algorithm will be as shown below:

$$\begin{array}{rrrrl}
 & & & 1 & (9) \\
 & & & 1 & (8) \\
1 & 0 & 0 & 0 & (7) \\
2 & 1 & 1 & 1 & (6) \\
\hline
12 & 9 & 5 & 1 & (1) \\
10 & 8 & 4 & 0 & (2) \\
\hline
1 & 3 & 3 & 0 & (3) \\
8 & 8 & 6 & 4 & (4) \\
15 & 15 & 11 & 7 & (5)
\end{array}$$

1 7 11 15 15 will be the U required. Lines (1) and (2) are the parts of the given V written alternately in the upper and lower line; lines (3) and (6) are obtained by oblique and direct subtraction performed between (1) and (2); line (4) is obtained from (3) by adding the number of terms in (1) to the last term in (3) which gives the last term in (4) and then adding in successively the other terms in (3) each diminished by one unit; (7) is derived from (6) by diminishing each term in the latter by a unit and taking the continued sum of the terms thus diminished; (8) is found by the usual

rule of "calling"* from its conjugate (7); and finally (5) and (9) are obtained by subtracting a unit from the doubles of the several terms in (4) and (8).

It thus becomes apparent that the passage back from a V to a U is a much more complicated operation than that of making the passage from a U to a V, so much more so that it would seemingly have been labour in vain to have attacked the problem of transformation by beginning from the V end.

(46) I now proceed to the main business, which is to show that any U containing i distinct odd numbers will, by the method described, be graphically converted into a V containing i distinct sequences.

Let G be any regular graph; H what G becomes when the first column of G is removed; a, b, c, d ... the contents of the angles of G, H taken in succession.

Also let i be the number of lines of unequal content in G, j the number of distinct sequences in $a, b, c, d, e, \ldots$.

The two first lines of G, say L, L', and also the two first columns, say K, K', may be equal or unequal†.

If $L=L'$ and $K=K'$, $a-1=b$, $b-1=c$.

If $L=L'$ and $K>K'$, $a-1=b$, $b-1>c$.

If $L>L'$ and $K=K'$, $a-1>b$, $b-1=c$.

If $L>L'$ and $K>K'$, $a-1>b$, $b-1>c$.

Let G', H' represent what G, H become on removing the first bend, that is the first line and the first column, and let i', j' be the values of i, j for G', H', so that j' is the number of sequences in $c, d, e \ldots$.

It is obvious from what precedes that in the four cases considered $j'=j$, $j'=j-1$, $j'=j-1$, $j'=j-2$ respectively. But in these four cases $i'=i$, $i'=i-1$, $i'=i-1$, $i'=i-2$ respectively.

Hence on each supposition $i-j=i'-j'$, and continuing the process by removing each bend in succession, $i-j$ must for any number of bends have the same values as it has for *one* bend; but in that case if h and k are the contents of the line and column of the bend, the reading of the corresponding G, G' will be $h+k-1$, $h-1$, so that for that case j will be 1 or 2 according as h and k are not or are both greater than 1, that is according as i is 1 or 2‡.

* I borrow this term from the vernacular of the American Stock Exchange.

† For brevity I use line and column to signify the extent of (that is, the number of nodes in) either.

‡ The final graph after denudation pushed as far as it will go must be either a single bend, a column, a line or a single node. In the first case $i=2, j=2$, in each of the remaining three cases $i=1, j=1$.

Hence $i-j$ is always equal to zero, consequently a U of the ith class will be transformed by the graphical process into a V of the ith class, as was to be proved.

(47) I have previously noticed [p. 25 above] that the simplest case of $i=j=1$ leads to the formula

$$\frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\frac{q^7}{1-q^7}+\ldots=\frac{q}{1-q}+\frac{q^3}{1-q^2}+\frac{q^6}{1-q^3}+\frac{q^{10}}{1-q^4}+\ldots,$$

which is a sort of pendant to Jacobi's formula

$$\frac{q}{1+q}-\frac{q^3}{1+q^3}+\frac{q^5}{1+q^5}-\frac{q^7}{1+q^7}+\ldots=\frac{q}{1+q}-\frac{q^3}{1+q^2}+\frac{q^6}{1+q^3}-\frac{q^{10}}{1+q^4}+\ldots *.$$

These formulae may be derived from one another or both obtained simultaneously as follows: From addition of the left-hand sides of the two equations there results the double of

$$\frac{q}{1-q^2}+\frac{q^6}{1-q^6}+\frac{q^5}{1-q^{10}}+\frac{q^{14}}{1-q^{14}}+\ldots \text{ or of } \sum_{i=1}^{i=\infty}\left(\frac{q^{4i-3}}{1-q^{8i-6}}+\frac{q^{8i-2}}{1-q^{8i-2}}\right),$$

and from addition of the right-hand sides of the same there results the double of

$$\frac{q}{1-q^2}+\frac{q^5}{1-q^4}+\frac{q^6}{1-q^6}+\frac{q^{14}}{1-q^8}+\ldots \text{ or of } \sum_{i=1}^{i=\infty}\left(\frac{q^{i(2i-1)}}{1-q^{4i-2}}+\frac{q^{i(2i+3)}}{1-q^{4i}}\right).$$

Consequently in order by the operation of addition of the two equations to deduce one from the other we must be able to show that these expressions are identical: observing then that $4i-3$ and $8i-2$ are odd and even respectively for all values of i, but $i(2i-1)$ and $i(2i+3)$ odd or even, according as for i, $2i-1$ or $2i$ be written, it has to be shown that

$$\sum_1^\infty \frac{q^{4i-3}}{1-q^{8i-6}}=\sum_1^\infty\left(\frac{q^{2i-1.\,4i-3}}{1-q^{8i-6}}+\frac{q^{2i-1.\,4i+1}}{1-q^{8i-4}}\right) \qquad (A)$$

and

$$\sum_1^\infty \frac{q^{8i-2}}{1-q^{8i-2}}=\sum_1^\infty\left(\frac{q^{i(8i-2)}}{1-q^{8i-2}}+\frac{q^{i(8i+6)}}{1-q^{8i}}\right). \qquad (B)$$

(A) is equivalent to
$$\sum_1^\infty q^{4i-3}\,\frac{1-q^{i-1.\,8i-6}}{1-q^{8i-6}}=\sum_1^\infty\frac{q^{2i-1.\,4i+1}}{1-q^{8i-4}}$$

or
$$\sum_1^\infty q^{4i+1}\,\frac{1-q^{i(8i+2)}}{1-q^{8i+2}}=\sum_1^\infty\frac{q^{2i-1.\,4i+1}}{1-q^{8i-4}}.$$

Hence if i signify any number from 1 to ∞ and k signify any number from 0 to $i-1$, it has to be shown that $(4i+1)(2k+1)$ contains the same integers and each taken the same number of times as $(2m-1)(4m+1+4n)$, where m is any number from 1 to ∞ and n is any number from 0 to ∞. But the $(4i+1)(2k+1)$ is the same as $(2k+1)\{4(k+l+1)+1\}$ where k and l

* My formula is what Jacobi's becomes when every middle *minus* sign in it is changed into *plus* and every inferior *plus* sign into *minus*.

each extend from 0 to ∞, and the $(2m-1)(4m+4n+1)$ is the same as $(2m+1)\{4(m+n+1)+1\}$ where m and n each extend from 0 to ∞, and the two latter expressions on writing $k=m$, $l=n$ become identical.

Again (B) is equivalent to

$$\sum_1^\infty q^{8i-2}\frac{1-q^{i-1.8i-2}}{1-q^{8i-2}}=\sum_1^\infty \frac{q^{i(8i+6)}}{1-q^{8i}}.$$

Hence we have to show that $(8i-2)(1+j)$ when $i=2, 3, \ldots \infty$ and $j=0, 1, 2, \ldots, (i-2)$, or say $(8i+6)(1+j)$, where $i=1, 2, \ldots \infty$ and $j=0, 1, 2, \ldots (i-1)$ is identical with $l(8l+6+8m)$, where $l=1, 2, \ldots \infty$ and $m=0, 1, 2, \ldots \infty$; the former of these is identical with

$$(1+j)\{8(j+k+1)+6\},$$

where $j=0, 1, \ldots \infty$; $k=0, 1, \ldots \infty$, and the latter is identical with

$$(1+l)\{8(l+m+1)+6\},$$

where $l=0, 1, \ldots \infty$; $m=0, 1, \ldots \infty$, consequently the two expressions are coextensive, which proves (B), and (A) has been already proved. Hence we see that either of the two original equations can be deduced from the other from the fact that their sum leads to an identity.

In like manner subtraction performed between the two allied equations leads to the fissiparous equation

$$\sum_0^\infty\left\{\frac{x^{8i+2}}{1-x^{8i+2}}+\frac{x^{4i+3}}{1-x^{8i+6}}\right\}=\sum_0^\infty\left\{\frac{x^{(i+2)(2i+1)}}{1-x^{4i+2}}+\frac{x^{i+1.2i+3}}{1-x^{4i+4}}\right\},$$

which gives birth to the pair

$$\sum_0^\infty \frac{x^{4i+3}}{1-x^{8i+6}}=\sum_0^\infty\left\{\frac{x^{2i+3.4i+3}}{1-x^{8i+6}}+\frac{x^{2i+1.4i+3}}{1-x^{8i+4}}\right\} \qquad (C)$$

and

$$\sum_0^\infty \frac{x^{8i+2}}{1-x^{8i+2}}=\sum_0^\infty\left\{\frac{x^{2i+2.4i+1}}{1-x^{8i+2}}+\frac{x^{2i+2.4i+5}}{1-x^{8i+8}}\right\}. \qquad (D)$$

(C) is equivalent to

$$\sum_0^\infty \frac{x^{4i+3}(1-x^{i+1.8i+6})}{1-x^{8i+6}}=\sum_0^\infty \frac{x^{2i+1.4i+3}}{1-x^{8i+4}},$$

which is an identity by virtue of the equivalence of $(4i+3)[1+2\{j<(i+1)\}]$ that is $(4j+4k+3)(1+2j)$ to $(2\lambda+1)(4\lambda+3+4\mu)$ where j, k, λ, μ each extend from zero to infinity, and

(D) is equivalent to

$$\sum_0^\infty \frac{x^{8i+2}(1-x^{i(8i+2)})}{1-x^{8i+2}}=\sum_0^\infty \frac{x^{2i+2.4i+5}}{1-x^{8i+8}},$$

which is an identity by virtue of the equivalence of $(8i+2)\{1+(j<i)\}$ that is $\{8(j+k+1)+2\}(1+j)$ to $(2\lambda+2)(4\lambda+5+4\mu)$, each symbol j, k, μ having as before the same range, namely from zero to infinity. Thus then the difference of the two allied equations (as previously their sum) is reduced to an identity which establishes the validity of each of them.

INTERACT, PART 2.

With notes of many a wandering bout,
Of linkèd sweetness long drawn out.
L'Allegro.

(48) D. *Transformation of Partitions by the Cord Rule.*—The figures below are designed to show how it is possible by means of the continuous doubling of a string upon itself to pass from an arrangement of groups of repetitions of r distinct odd integers to the corresponding one with like sum, made up of r distinct sequences. Each of the two figures duplicated by rotation about its upper horizontal boundary of nodes through two right angles will represent an arrangement of repeated odd numbers, the parts being represented by the contents of the *vertical* lines in the figures so duplicated.

Fig. 1.

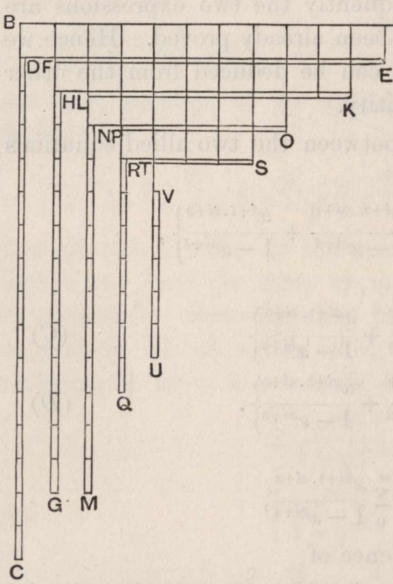
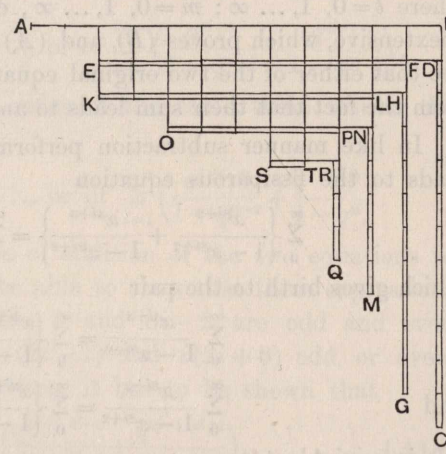


Fig. 2.



The first duplicated figure represents the arrangement 33, 29^2, 23, 21, 9^3, 7, 5^2, 3, 1 whose sum is 183; its correspondent will be the contents of the lengths of * *ABC*, *CDE*, *EFG*, *GHK*, *KLM*, *MNO*, *OPQ*, *QRS*, *STU*, *UV*, namely the arrangement 29, 27, 24 (22, 21,), 18, 14, 12, 10, 6 which is the same number 183 partitioned into (ten parts but) nine sequences: the second duplicated figure represents the arrangement 25, 23, 17, 15, 9^2, 7^3, 5^2, 1^2, whose sum is 130; its correspondent is represented by the lengths of *ABC*, *CDE*, *DEF*, *FGH*, *HKL*, *LMN*, *NOP*, *PQR*, *RST*, *TU*, which is the same number 130 partitioned into the (nine parts but) eight sequences 25, 22 (20, 19,), 15, 12, 10, 6, 1.

* A line containing i units of length represents $(i+1)$ nodes.

(49) E. *On Graphical Dissection.*—It may be not unworthy of notice that there is a sort of potential anticipation of Mr Durfee's dissection of a symmetrical graph, in a method which, whether it is generally known or not I cannot say, but is substantially identical with Dirichlet's for finding approximately $\sum_1^n \left[\frac{n}{i}\right]$ and other such like series (a bracketed quantity being used to signify that quantity's integer part). Constructing the hyperbola $xy = n$, drawing its ordinates to the abscissas 1, 2, 3, ... n, and in each of them planting nodes to mark the distances 1, 2, 3, ... from its foot, there results a *symmetrical graph* included between one branch of the curve, its two asymptotes, and lines parallel to and cutting each of them at the distance n from the original. Its content will be the sum in question. The Durfee-square to it will be limited by the square whose side is $[\sqrt{n}]$, and this added to the original area gives twice over the area in which the number of nodes is $\sum_1^{\sqrt{n}} \left[\frac{n}{i}\right]$, and consequently neglecting magnitudes of the order $\sqrt{n}$,

$$\sum_1^n \left[\frac{n}{i}\right] = 2n \sum_1^{\sqrt{n}} \frac{1}{i} - i^2 = n(\log n + 2C - 1)$$

and as a corollary

$$\sum_1^{\sqrt{n}} \left\{\frac{n}{i} - \left[\frac{n}{i}\right]\right\} = n(C - 2C + 1) = (1 - C)\,n,$$

where C is Euler's number ·57721, so that $1 - C$ for large values of n will be the average value of the fractional part of n divided by an inferior number. Furthermore a similar graph, but with $xy = 2n$ diminished by the portion contained between a branch of the new curve, one of its asymptotes and two parallel ordinates cutting that asymptote at distances n and $2n$ from the origin (which portion obviously contains $(2n - n)$ that is n nodes) will represent $\sum_1^n \left[\frac{2n}{i}\right]$, and consequently the sum $\sum_1^n \left\{\left[\frac{2n}{i}\right] - 2\Sigma\left[\frac{n}{i}\right]\right\}$, that is (see *Berl. Abhand.* 1849, p. 75) the number of times that $\frac{n}{i} - \left[\frac{n}{i}\right]$ equals or exceeds $\frac{1}{2}$, as i progresses from 1 to n (within the same limits of precision as previously) $= 2n(\log 2n + 2C - 1) - n$ less $2n(\log n + 2C - 1)$, that is $= (\log 4 - 1)\,n$, so that the probability of the fractional part of n divided by an inferior number not falling under $\frac{1}{2}$ is $\log 4 - 1$*.

* What precedes I recall as having been orally communicated to me many years ago by the late ever to be regretted Prof. Henry Smith, so untimely snatched away when in the very zenith of his powers, and so to say, in the hour of victory, at the moment when his intellectual eminence was just beginning to be appreciated at its true value, by the outside world. I was under the impression until lately that he was quoting literally from Dirichlet when so communicating with me, but as the geometrical presentation given in the text is not to be found in the

(50) F. *Mr Ely's method of finding the asymptotic value of the number of improper fractions with a very large given numerator which are nearer to the integer below than to the integer above**.

"Let a number n be divided by all the numbers from 1 to n; then a value is required for the number of residues which are equal to or greater than $\frac{1}{2}$. An example will make evident a method by which we may obtain limits to the value sought. If n be 100 the residues $= > \frac{1}{2}$ are

$$(1)\quad \frac{49}{51}\ \frac{48}{52}\ \frac{47}{53}\ \frac{46}{54}\ \frac{45}{55}\ \frac{44}{56}\ \frac{43}{57}\ \frac{42}{58}\ \frac{41}{59}\ \frac{40}{60}\ \frac{39}{61}\ \frac{38}{62}\ \frac{37}{63}\ \frac{36}{64}\ \frac{35}{65}\ \frac{34}{66}$$

$$(2)\quad \frac{32}{34}\ \frac{30}{35}\ \frac{28}{36}\ \frac{26}{37}\ \frac{24}{38}\ \frac{22}{39}\ \frac{20}{40}$$

$$(3)\quad \frac{22}{26}\ \frac{19}{27}\ \frac{16}{28}$$

$$(4)\quad \frac{16}{21}\ \frac{12}{22}$$

$$(5)\quad \frac{15}{17}\ \frac{10}{18}$$

$$(6)\quad \frac{10}{15}$$

$$(a)\quad \frac{4}{6}\ \frac{4}{8}\ \frac{9}{13}$$

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memoir cited from the *Berlin Transactions*, I infer that it originated with himself. In comparing Mertens' memoir, *Crelle*, 1874, with Dirichlet's (1849), upon which it is a decided step in advance, one cannot fail to be struck with surprise that the point to which the closer drawing of the limits to the values of certain transcendental arithmetical functions achieved by the former is owing, should have escaped the notice of so profound and keen an intellect as Dirichlet's, and those who came after him in the following quarter of a century. The point I refer to is the almost self-evident fact that if in the cases under consideration

$$\Sigma\phi\,(Fi\,.\,x)=\psi x \text{ then } \phi x=\Sigma\mu\,(i)\,\psi\,(Fi\,.\,x)$$

where $\mu\,(i)$ means 0, if i contains any repeated prime factors, but otherwise 1 or $\bar{1}$ according as the number of prime factors in i is even or odd. Dirichlet works with a function given implicitly by an equation, Mertens with the same function expressed in a series, wherein exclusively lies the secret of his success.

* It is proper to state that what follows in the text was handed in to me by Mr Ely on the morning after I had proposed to my class to think of some "common sense method" to explain the somewhat startling fact brought to light by Dirichlet, of more than three-fifths of the residues of n in regard to $i=1, 2, 3, \ldots n$ being less than $\frac{i}{2}$. Mr Ely's method shows at once, in a very common sense manner, why the proportion must be considerably greater than the half, inasmuch as whilst the terms in the first few harmonic ranges are approximately $\frac{n}{1\,.\,2}, \frac{n}{2\,.\,3}, \frac{n}{3\,.\,4}$, etc., in number, the number of them which employed as denominators to n give fractional parts greater than $\frac{1}{2}$, instead of being the halves of these are only $\frac{n}{2\,.\,3}, \frac{n}{3\,.\,5}, \frac{n}{4\,.\,7}$, etc. The mean value in both methods to quantities of the order of $\sqrt{n}$ *inclusive*, turns out to be the same, whichever method is employed, but the margin of unascertained error by the use of Mr Ely's method (as compared with Dirichlet's) is reduced in the proportion of $1 : 1+\sqrt{2}$, that is, nearly $2 : 5$.

In which it will be observed that the residues $= > \frac{1}{2}$ occur in batches. Let X be the whole number, and x_i the number in batch i. In batch i the numerators decrease by i and the denominators increase by 1. (Those marked (a) of which the denominators are less than $\sqrt{200}$ are left out of account for the present.) It is evident for the general case we have approximately

$$\frac{\left[\frac{n}{i+1}\right] - ix_i}{\left[\frac{n}{i+1}\right] + x_i} = \frac{1}{2}$$

or accurately

$$x_i = \left[\frac{n}{(i+1)(2i+1)}\right] \text{ or } \left[\frac{n}{(i+1)(2i+1)}\right] + 1 * ."$$

Mr Ely is then able to show that by limiting the calculation of x_i to the values of i which do not exceed $[\sqrt{n}/2]$, so that roughly speaking the character of $\sqrt{2n}$ of the remainders is left undetermined (and no account taken of them in finding the value of X), and giving to x_i its approximate value $\frac{n}{(i+1)(2i+1)}$, and then extending the series $\frac{n}{2.3} + \frac{n}{3.5} + \frac{n}{4.7}$ beyond the $[\sqrt{n}/2]$th term, where it ought to stop, to infinity, the errors arising from each of these three sources† and therefore their combined effect will be of the order $\sqrt{n}$, so that the asymptotic value of X will be

$$\left(\frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \dots\right) n,$$

which is $(2 \log 2 - 1) n$, with an uncertainty of the order $\sqrt{n}$, as was to be shown.

(51) It may be seen that Mr Ely's method consists in distributing the n numbers from n to 1 into what I have elsewhere termed *harmonic ranges* and determining what portions of the several ranges employed as denominators to n give fractional parts, greater or less than $\frac{1}{2}$. It may assist in forming a more vivid idea of this kind of distribution, if the reader takes a definite case, say of $n = 121$, the first (10) harmonic ranges will then comprise

* I find by an exact calculation that if R is the remainder of n in regard to $(i+1)(2i+1)$ and $R = \lambda(i+1) + \mu$, where $\lambda < 2i+1$ and $\mu < i+1$, then for $\lambda = 2\theta - 1$ or 2θ, $x_i = \left[\frac{n}{(i+1)(2i+1)}\right] + 1$ if $\mu = i-1$ or $i-2$ ... or $i-\theta$, and $x_i = \left[\frac{n}{(i+1)(2i+1)}\right]$ for all other values of μ. Hence it follows that out of $(2i^2+3i+1)$ successive values of n, (i^2+i) and (i^2+2i+1) will be the respective numbers of the cases for which the one or the other of these two values of x_i is employed, so that for larger values of i the chances for the two values are nearly the same, but with a slight preponderance in favour of the smaller value. See p. [54].

† The error from the first cause makes the determination of X too small by an unknown amount, that from the third cause too large by a known amount, and that from the second too large or too small (as it may happen) by an unknown amount.

all the numbers from 121 to 12 inclusive, and the remaining 111 harmonic ranges will comprise the remaining 11 numbers from 11 to 1; that is to say 11 of them will contain a single number, and the remaining 100 ranges be vacant of content.

So again if $n = 20$ the first four ranges will contain all the numbers from 20 to 5 inclusive; the 5th, 6th, 9th and 20th range will consist of the sole numbers 4, 3, 2, 1, and the remaining 12 ranges will be vacant. I shall proceed to compare the precision of Mr Ely's result with that of Dirichlet's—for this purpose it will be enough to determine the asymptotic value of the uncertainty and to take no account of quantities of a lower order than $\sqrt{n}$.

Let us then suppose that $\sqrt{(kn)}$ ranges are preserved, and consequently $\sqrt{\left(\frac{n}{k}\right)}$ fractions left out (k being an arbitrary constant which will eventually be determined so as to make the uncertainty a minimum).

The first cause of error necessitates a correction of which the limits are 0 and $\sqrt{\left(\frac{n}{k}\right)}$; the second cause a correction of which the limits are $\sqrt{(kn)}$ and $-\sqrt{(kn)}$; and the third, namely the overreckoning of

$$\frac{n}{(j+1)(2j+1)} + \frac{n}{(j+2)(2j+3)} + \dots$$

where $j = \sqrt{(kn)}$, a correction of which the value is $-\frac{n}{2j}$ or $-\frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}$.

Hence making $(\log 4 - 1)\, n = U$, the superior limit of X is

$$U + \frac{1}{2}\sqrt{\left(\frac{n}{k}\right)} + \sqrt{(kn)},$$

and the inferior limit $U - \frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}\sqrt{(kn)}$. Consequently $X = U + \rho n^{\frac{1}{2}}$ where $\rho < \sqrt{k} + \frac{1}{2}\sqrt{\left(\frac{1}{k}\right)}$, of which the minimum value is found by making $k = \frac{1}{2}$, so that $\rho < \sqrt{2}$ and the *uncertainty* is $\sqrt{2} \,.\, n^{\frac{1}{2}}$. Adopting Mertens' asymptotic value of the *uncertainty* of $\sum_1^n \left[\frac{n}{i}\right]$, namely $\sqrt{n}$, and using Dirichlet's formula, $\sum_n^1 \left[\frac{2n}{i}\right] - 2\sum_n^1 \left[\frac{n}{i}\right]$, X has the same mean value as above, but the uncertainty becomes $(\sqrt{2} + 2)\, n^{\frac{1}{2}}$ which is nearly two and a half times as great as that given by the direct method employed by Mr Ely.

I use the word *uncertainty*, it will be noticed, in a different sense from *error*; the latter is objective, referring to fact, the former subjective, referring to knowledge. Both methods in the case here presented give the same mean value, and therefore the *error* is the same, but the uncertainty is widely different according to the method made use of. Of two formulae referring to the same fact one might very well give the smaller error and the other the smaller uncertainty.

I have shown above that for considerable values of i, the average value of x_i is $\frac{n}{(i+1)(2i+1)}+\frac{1}{2}$; if then it may be assumed (and there seems no reason for suspecting the contrary) that for $i=1, 2, \ldots, \sqrt{2n}$, the mean value of $\frac{n}{i}-\left[\frac{n}{i}\right]$ is $\frac{1}{2}$, U will not only be the mean value of the known limits of X but also the mean value of X itself. The value found for k shows that the most advantageous mode of employing Mr Ely's method is to make the series $\frac{n}{2.3}+\frac{n}{3.5}+\ldots\frac{n}{(i+1)(2i+1)}+\ldots$ stop at one of the terms which is approximately equal to unity.

(52) It is not without interest to consider the exact law for the extent of a harmonic range of a given denomination, say i: this it is easily seen will be always equal to $\left[\frac{n}{i^2+i}\right]$ or $\left[\frac{n}{i^2+i}\right]+1$.

I shall regard i as given and determine the values of n which correspond to the one or the other of the two formulae: this will depend not on the absolute value of n but on its remainder in respect to the modulus i^2+i. To fix the ideas, let $i=4$ so that $i^2+i=20$, and let n take in successively all values from 40 to 59 inclusive.

Then corresponding to n equal to

40	44	48	52	56
41	45	49	53	57
42	46	50	54	58
43	47	51	55	59

the fourth range will be

10, 9	11, 10, 9	12, 11, 10	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11, 10	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11	13, 12	14, 13, 12

that is in half the terms of the period $\left[\frac{n}{i^2+i}\right]$ and in the other half $\left[\frac{n}{i^2+i}\right]+1$ gives the extent of the range.

So in general, if $n=k(i^2+i)+\lambda i+\mu$, where $\lambda=0, 1, 2, \ldots i$, and $\mu\equiv 0, 1, 2, \ldots (i-1)$, when the remainder of n to modulus (i^2+i) is of the form

$\lambda(i^2+i)+\{0, 1, 2, \ldots(\lambda-1)\}$ that is in $\frac{i^2+i}{2}$ cases the extent of the ith harmonic range to n is $\left[\frac{n}{i^2+i}\right]+1$, and when of the form

$$\lambda(i^2+i)+\{\lambda, \lambda+1, \ldots(i-1)\},$$

that is in the remaining $\frac{i^2+i}{2}$ cases it is $\left[\frac{n}{i^2+i}\right]$.

As the sum of the harmonic ranges to n is n itself, and

$$\frac{n}{1.2}+\frac{n}{2.3}+\ldots+\frac{n}{n(n+1)}=n-\frac{n}{n+1},$$

it follows that if we separate all the numbers from 1 to n into two classes, say i's and j's, i being any number for which n is of the form

$$k(i^2+i)+\lambda i+0, 1, 2, \ldots(\lambda-1),$$

and j any other number within the prescribed limits, then

$$\sum_1^n \frac{n}{t}-\sum_1^n\left[\frac{n}{t}\right]=\text{number of } i\text{'s}-\frac{n}{n+1},$$

and consequently the number of the i terms has $(1-C)n$ for its asymptotic value.

(53) In like manner the law previously stated in a footnote, p. [51], for giving the extent of that portion of the ith range for which $\frac{n}{t}$ contains a fractional part not less than $\frac{1}{2}$ may be verified. Thus let $i=3$ then $(i+1)(2i+1)=28$, let $n=56, 57, \ldots 83$. Then for the values of n

28	32	36	40	44	48	52
29	33	37	41	45	49	53
30	34	38	42	46	50	54
31	35	39	43	47	51	55

the portion of the third range having the required character will contain the numbers

8	9	10	11	12	13	14
8	9	10	11	12	14, 13	15, 14
8	9	10	12, 11	12, 11	14, 13	15, 14
8	10, 9	11, 10	12, 11	13, 12	14, 13	15, 14

so that there are $2(1+2+3)$, that is 3.4 forms of n out of 7.4 for which the formula $\left[\frac{n}{4.7}\right]+1$ has to be employed, and so in general if R is the residue of n in respect to $(i+1)(2i+1)$, there are i^2+i cases where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]+1$ and $(i+1)^2$ where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]$ has to be employed.

G. *On Farey Series.*

(54) This note is a natural sequel to and has grown out of the two which precede; it has also a collateral affinity with the subject-matter of the Acts, inasmuch as a graph affords the most simple mode of viewing and stating the fundamental property of an ordinary Farey series, and any series *ejusdem generis*. For instance, let A, B, C be a reticulation in the form of an equilateral triangle, where B is a right angle, and n the number of nodes in the base or height of the triangle; if the hypothenuse be made to revolve in the plane of the triangle about (either end say about) A, the triangle formed by joining A with any two consecutive nodes of greatest proximity to the centre of rotation traversed by the rotating line will be equal in area to the minimum triangle which has any three nodes for its apices, that is its double will be equal to unity. This law of uniform description of areas (say of *equal areas in equal jerks*) is identical with the characteristic law of an ordinary Farey series which deals with terms whose number is the sum-totient τn: but it will also hold good if the triangle be scalene instead of equilateral, which corresponds to Glaisher's extension of a Farey series, to the case where the numerator and denominator of each term has its own separate limit (*Phil. Mag.* 1879), or again, when the rotation takes place about the right angle B as centre, which gives rise to a Farey series of a totally different species, defined by the inequality $ax + by < n$, or again when the hypothenuse is replaced by the quadrant of a circle or ellipse, and in an infinite variety of other cases, as for example when the graph is contained between a branch of an equilateral hyperbola and the asymptotes, which case corresponds to the subject-matter of the theory of Dirichlet (*Berl. Abhand.* 1844) concerning the sum of the number of ways in which all integers up to n can be resolved into the product of two relative primes, which is the same thing as the half of the number of divisors (containing no repeated prime factors) which enter into the several integers up to n, or as the entire number of solutions in relative primes of the inequality $xy =$ or $< n$. The law of equal description of areas ($pq' - p'q = \pm 1$), Mr Glaisher has shown very acutely, is an immediate inference (by an obvious induction) from the well-known fact that between a fraction and its two nearest convergents (namely the one ordinarily so called and that which is obtained by substituting $\delta - 1$ and 1 for the last partial quotient), no other fraction can be interposed whose denominator is not greater than that of the one first named.

From the areal-law obviously follows the equation $\frac{p''}{q''} = \frac{xp' - p}{xq' - q}$ (where $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$ are any three consecutive terms of the series), so that in order to construct explicitly such a series from the two first terms, all we have to do is to give to x at each step the highest value it can assume, consistent with the imposed limit or limits. Thus for example I have found by this method when the limiting inequality is $x+y=$ or <15, the series

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}$$

$$\frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{4}{11}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}\ \frac{1}{1}*$$

and the complements in respect to unity of the several terms which precede $\frac{1}{2}$ taken in reverse order, and again for $xy=$ or <15 the series (which might be called the Dirichlet-Farey series)

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}$$

$$\frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\ \frac{1}{1}.$$

In general if we agree to understand respectively by the *decernent* and the *secernent* to x, the number of divisors without restriction, and the number of divisors restricted to contain no square number, that go into x, and denote the sum-secernent and sum-decernent of n by Sn and Dn respectively, Dirichlet's mode of looking at the question leads immediately to the equation $\sum_1^n S\frac{n}{i^2}=Dn$. Mertens' equation $\left[Sn=\sum_1^n \mu i D\frac{n}{i^2}\right]$ obtained by a longer and somewhat more difficult process is in point of fact merely that equation *reverted.* On pointing out to Mr F. Franklin this elegant passage in Dirichlet's memoir, he remarked to me to the effect that it was an example, which might admit of wide generalization, of a concept resembling that inherent in the subject-matter of the ordinary Farey series; which excellent and keen-witted observation led me to look into the subject from the point of view herein explained. The present theory diverges from the ordinary one in quite another and more natural direction (I imagine) than that pursued by M. Darboux, whose article on the subject of quasi-Farey series (*Bulletin de la Société Mathématique de France,* tome VI.) I have not been able to obtain sight of, and can only conjecture its purport through the reference made to it in a subsequent article which I *have* been able to procure in the same journal by M. Edouard Lucas.

* It is advisable for the purpose of securing generality in reasoning upon Farey series not to omit the initial and final terms $\frac{0}{1}$, $\frac{1}{1}$ which seem generally to have been lost sight of by previous writers on the subject Even then the series is only half complete, for after $\frac{1}{1}$ should follow the reciprocals of the preceding terms until $\frac{1}{0}$ is reached. Thus a complete *ordinary* Farey series beginning with $\frac{0}{1}$ and ending with $\frac{1}{0}$ consists of two symmetrical branches with $\frac{1}{1}$ as their point of junction, each made up of two symmetrical sub-branches meeting respectively in the terms $\frac{1}{2}$ and $\frac{2}{1}$, and such that the sum of a corresponding pair of fractions on the one side of $\frac{1}{1}$ and of their reciprocals on the other side is equal to unity: whereas in the two complete branches the *product* of each corresponding pair is unity.

(55) I prove the persistency of the fundamental property of ordinary Farey series for such series generalized in the manner supposed above, as follows.

Let us use *O. F.* S_i to denote an ordinary Farey series for which the limit is i, and *G. F. S.* a Farey series in which, calling the numerator and denominator of any term x, y, $\phi(x, y) <= i$, $\phi(x, y)$ meaning a rational function which increases when either x or y increases. If in an *O. F.* S_i any two consecutive terms be $\frac{a}{b}$, $\frac{c}{d}$, and in an *O. F.* S_{i+1} $\frac{p}{q}$ intervenes between $\frac{a}{b}$, $\frac{c}{d}$ we know, p being greater than b and d, the two nearest convergents to $\frac{p}{q}$ must be contained in *O. F.* S_i, and consequently must be $\frac{a}{b}$, $\frac{c}{d}$ themselves, so that $p = a + c$, $q = b + d$, and as a corollary if $\frac{a}{b}$, $\frac{c}{d}$ be consecutive terms in any *O. F. S.*, and $\frac{p}{q}$ be any one of the terms which subsequently intervene between $\frac{a}{b}$, $\frac{c}{d}$, we must have $p =$ or $> a + c$, $q =$ or $> b + d$. In order to fix the ideas let us suppose $\phi(x, y)$ to represent $x + y$, so that $x + y <= n$.

For the values 2, 3, 4, 5, 6, 7, 8, 9 ... of n, the *G. F. S.* will be

$$\frac{0}{1}\ \frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{2}\right)\frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{3}\right)\frac{1}{2}\ \frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{4}\right)\frac{1}{3}\ \frac{1}{2}\left(\frac{2}{3}\right)\frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{5}\right)\frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}\ \frac{1}{1};$$

$$\frac{0}{1}\left(\frac{1}{6}\right)\frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\left(\frac{2}{5}\right)\frac{1}{2}\ \frac{2}{3}\left(\frac{3}{4}\right)\frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{7}\right)\frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\left(\frac{3}{5}\right)\frac{2}{3}\ \frac{3}{4}\ \frac{1}{1};$$

$$\frac{0}{1}\left(\frac{1}{8}\right)\frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\left(\frac{2}{7}\right)\frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\left(\frac{4}{5}\right)\frac{1}{1};\ldots$$

where the terms in parenthesis are the new terms which intervene as n increases from any value to the next following integer, and where it will be noticed that if $\frac{p}{q}$ be any such parenthesised fraction lying between $\frac{a}{b}$ and $\frac{c}{d}$, $p = a + c$ and $q = b + d$, just as in the successive form of an *O. F. S.* The theorem to be proved may be made to depend on the following lemma.

If for any given value of n every two consecutive terms in a *G. F. S.* appear as consecutive terms in an *O. F. S.* for the same or any smaller value of n; this will continue to be true for all superior values of n.

The proof is immediate, for let $\frac{a}{b}$, $\frac{c}{d}$ be any two consecutive terms in the *G. F.* S_j which are also consecutive terms in *O. F.* S_i where $i =$ or $< j$.

If a term $\frac{p}{q}$ intervene between $\frac{a}{b}$, $\frac{c}{d}$ in $G.\ F.\ S_{j+1}$, $p =$ or $> a+c$, $q =$ or $> b+d$, by virtue of the remark made. But if $p > a+c$ and $q > b+d$,

$$\phi(a+c, b+d) < \phi(p, q) < j+1,$$

but $\frac{a+c}{b+d}$ is intermediate in value between $\frac{a}{b}$, $\frac{c}{d}$, hence $\frac{a+c}{b+d}$ must have appeared in a $G.\ F.\ S_{j'}$, where $j' < j$, which is contrary to hypothesis.

Hence $\frac{a}{b}$, $\frac{p}{q}$, $\frac{c}{d}$ will have been consecutive terms in some *O. F. S.*, and in like manner any two consecutive terms in *G. F. S.* either remain consecutive in $G.\ F.\ S_{j+1}$ or admit a new term between them which is consecutive to each of them in some *O. F. S.*, so that the supposed relation if it holds good for j is true for all superior values of j; but $\frac{0}{1}$, $\frac{1}{1}$ will in any of the supposed cases be a *G. F. S.*; consequently in all these cases no two terms are consecutive in any *G. F. S.* which are not so in some *O. F. S.*, and therefore the law of equal description of areas will apply to them equally as to the *O. F. S.*, as was to be proved.

The theory may be extended to *G. F. S.*, defined by several concurrent limiting equations. Thus for example Mr Glaisher has proved this for the case of $x <= m$, $y <= n$: I have not had time as yet to consider what are the restrictions to which the limiting functions may be subject, but the theorem is obviously an extremely elastic one, and the above proof suffices for all the special cases which I have enumerated*.

(56) I am indebted to Mr Ely for the following additional examples of Farey series, in the enlarged sense, which may interest some of my readers.

Ex. (1). $x + y =$ or < 20

$$\frac{1}{19} \cdots \frac{1}{9}\ \frac{2}{17}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{3}{17}\ \frac{2}{11}\ \frac{3}{16}$$

$$\frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{4}{13}\ \frac{1}{3}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}$$

$$\frac{5}{12}\ \frac{3}{7}\ \frac{4}{9}\ \frac{5}{11}\ \frac{6}{13}\ \frac{1}{2}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{1}{7}\ \frac{7}{12}\ \frac{3}{5}\ \frac{5}{8}\ \frac{7}{11}\ \frac{2}{3}\ \frac{7}{10}\ \frac{5}{7}$$

$$\frac{8}{11}\ \frac{3}{4}\ \frac{7}{9}\ \frac{4}{5}\ \frac{9}{11}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{8}{9}\ \frac{9}{10}.$$

Ex. (2). $x^2 + y =$ or < 20

$$\frac{1}{19} \cdots \frac{1}{9}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}.$$

* Since the above was in type I have discovered the true principle of Farey series, for which see Note H following the Exodion.

Ex. (3). $y - \sqrt{x} =$ or < 15

$$\frac{1}{16} \ldots \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{3}{16}\ \frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{4}{17}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}$$

$$\frac{5}{17}\ \frac{3}{10}\ \frac{4}{13}\ \frac{5}{17}\ \frac{1}{3}\ \frac{6}{17}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}\ \frac{7}{17}\ \frac{5}{12}\ \frac{3}{7}\ \frac{7}{17}\ \frac{4}{0}\ \frac{5}{11}\ \frac{6}{13}$$

$$\frac{7}{15}\ \frac{8}{17}\ \frac{1}{2}\ \frac{9}{17}\ \frac{8}{15}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{9}{16}\ \frac{4}{7}\ \frac{7}{12}\ \frac{10}{17}\ \frac{3}{5}\ \frac{11}{18}\ \frac{8}{13}\ \frac{5}{8}\ \frac{7}{11}$$

$$\frac{9}{14}\ \frac{11}{17}\ \frac{2}{3}\ \frac{11}{16}\ \frac{9}{13}\ \frac{7}{10}\ \frac{12}{17}\ \frac{5}{7}\ \frac{13}{18}\ \frac{8}{11}\ \frac{11}{15}\ \frac{3}{4}\ \frac{13}{17}\ \frac{10}{13}\ \frac{7}{9}\ \frac{11}{14}$$

$$\frac{4}{5}\ \frac{13}{16}\ \frac{9}{11}\ \frac{14}{17}\ \frac{5}{6}\ \frac{16}{19}\ \frac{11}{13}\ \frac{6}{7}\ \frac{13}{15}\ \frac{7}{8}\ \frac{15}{17}\ \frac{8}{9}\ \frac{17}{19}\ \frac{9}{10}\ \frac{10}{11}\ \frac{11}{12}\ \frac{12}{13}\ \frac{13}{14}$$

$$\frac{14}{15}\ \frac{15}{16}\ \frac{16}{17}\ \frac{17}{18}\ \frac{18}{19}\ \frac{1}{1}.$$

EXODION. *On the Correspondence between certain Arrangements of Complex Numbers.*

At which he wondred much and gan enquere
What stately building durst so high extend
Her lofty towres, unto the starry sphere.

Faerie Queene I. x. 56.

(57) Starting from the expansion in a series of $\Theta_1 x$, multiplying in the usual notation both sides of the equation by

$$(1 - q^2)(1 - q^4)(1 - q^6)\ldots,$$

and intercalating the factors of this product between those of

$$(1 - qz)(1 - q^3 z)\ldots(1 - qz^{-1})(1 - q^3 z^{-1})\ldots$$

taken in alternate order, there results the equation

$$(1 - qz^{-1})(1 - qz)(1 - q^2)(1 - q^3 z^{-1})(1 - q^3 z)(1 - q^4)\ldots = \sum_{i=-\infty}^{i=+\infty} (-)^i q^{i^2} z^i,$$

and writing q^n in place of q and making $z = \mp q^m$, Jacobi (*Crelle*, Vol. XXXII. p. 166) derives the identity

$$(1 \pm q^{n-m})(1 \pm q^{n+m})(1 - q^{2n})(1 \pm q^{3n-m})(1 - q^{3n+m})(1 - q^{4n})\ldots = \sum_{-\infty}^{+\infty} (\pm)^i q^{ni^2 + mi}.$$

From this equation, using the lower sign and making $n = \frac{3}{2}$, $m = \frac{1}{2}$, he observes, may be deduced Euler's expression in a series for

$$(1 - q)(1 - q^2)(1 - q^3)\ldots,$$

and using the *upper* sign and making $n = \frac{1}{2}$, $m = \frac{1}{2}$, another known series "given by Gauss in the first volume of the *Göttingen Commentaries* for the years 1808–11."

It is not without interest, I think, to observe that by making $n=\frac{1}{2}$, $m=\frac{1}{2}+\epsilon$ (where ϵ is an infinitesimal), and using the *lower* sign we may immediately deduce Jacobi's own celebrated postscript (so to say) to Euler's equation, namely,

$$(1-q)^3(1-q^2)^3(1-q^3)^3\ldots=\sum_{-\infty}^{+\infty}(-)^i q^{\frac{i^2+i}{2}+i\epsilon}\div(1-q^{-\epsilon})$$
$$=1-3q+5q^3-7q^6\ldots,$$

the general term being

$$\sum_0^\infty(-)^i\left\{(q^{\frac{i^2+i}{2}+i\epsilon}-q^{\frac{i^2+i}{2}-(i+1)\epsilon})\div\frac{1}{1-q^{-\epsilon}}\right\},$$

which is
$$(-)^i(2i+1)q^{\frac{i^2+i}{2}}.$$

(58) It is obvious, that by the same right and within the same limits of legitimacy as the equation involving q, n, m (or if we please to say so in q, m) has been derived from the equation in (q, z), the equation in q, z may be recovered from the equation in q and m, if this latter can be shown to be true, morphologically interpreted for general values of m. I shall show that regarding m and n as absolutely general symbols, such as $\sqrt{(-1)}$ or $\sqrt{2}$ or ρ or the quaternion units, or any other heterogeneous or homogeneous units we please, the equation in question which I shall write under the equivalent form

$$(1\mp q^a)(1\mp q^b)(1-q^c)(1\mp q^{a+c})(1\mp q^{b+c})(1-q^{2c})\ldots=\sum_{i=-\infty}^{i=+\infty}(\mp)^i q^{\frac{i^2}{2}c+\frac{i}{2}(a-b)}$$

[where $c=a+b$, and a, b are absolutely general symbols or species of units entirely independent of one another] does hold good as a morphological identity*. Thus interpreted, it amounts to a theorem in complex quantities, dealing with arrangements of three sorts of elements which I shall call C's, B's, A's respectively, meaning by a C any non-negative integer (that is zero or any positive integer) multiple of c, by a B such multiple augmented by a single b, and by an A such multiple augmented by a single a.

The C's, the B's and the A's in any such arrangement will be regarded as three separate series, the terms in each of which flow from left to right in descending order, that is the multiples of c which represent totally or with the exception of a single b or a single a, the terms in each such series taken in severalty are to form a continually decreasing series.

* This theorem is less transcendental than Newton's binomial theorem when the same latitude is given to the meaning of the symbols in either case: for $(1+x)^m=1+mx+\frac{m^2-m}{2}x^2+\ldots$ does not admit of *direct* interpretation when m is a general symbol. The passage from numerical proximate equality to absolute identity, prepared but not perfected nor capable of being explained by infinitesimal gradation, brings to mind the analogous transfiguration of sensibility into sensation, or of sensation into consciousness, or of consciousness into thought.

The total number of elements and the number of C's will be called the major and minor parameters respectively—the relation to the modulus 2 (that is the parity or imparity) of either one of them its character: and for brevity, the terms major and minor character will be used to signify the character of the major or minor parameter. The totality of all arrangements whatever of A's, B's, C's in which *no element is repeated,* will constitute the sphere of the investigation, limited only by the absence of what I term the exceptional or isolated arrangements, consisting exclusively of a series of *consecutive* B's ending in b, or of consecutive A's ending in a. Within the prescribed sphere I shall prove that a process may be instituted for transforming any arrangement which shall satisfy the five following conditions:

(1) That it shall be capable of acting on every licit and unexceptional arrangement.

(2) That it shall transform it into another such arrangement.

(3) That operating once upon an arrangement, and then again upon the operate, it brings back the original arrangement.

(4) That it leaves the sum of the elements in the arrangement unaltered.

(5) That it reverses each of its two characters*.

From (3) it will follow that all the arrangements within the prescribed sphere are associated in pairs, and from (1) that the sum of the elements in each such pair is the same. This being so, it is obvious from the fact of the parity of the total number of elements being opposite for any pair of associated arrangements, that in the development in a series of

$$(1-q^a)(1-q^b)(1-q^c)(1-q^{a+c})\ldots,$$

no term will appear in which the index of q is other than the sum of the terms in one of the exceptional (we may now call them unconjugated or unconjugable) arrangements, and from the fact of the parity of the number of the C's being opposite in any pair, the same will be true of the development in a series of

$$(1+q^a)(1+q^b)(1-q^c)(1+q^{a+c})\ldots.$$

As regards the coefficient in this latter series of any term whose index is

* It will presently be seen that all the licit and unexceptional arrangements will be divided into 3 classes and a specific operator be found for each class capable of acting on each arrangement of that class and converting it into another of the same class, and which will satisfy also the 3rd, 4th and 5th of the enumerated conditions. The total operator contemplated in the text may then be regarded as the sum of these specific ones, each of which, within its own sphere, will have to fulfil the five conditions of Catholicity, Homoeogenesis, Mutuality, Inertia and Enantiotropy (the last a word used in the school of Heraclitus to signify "the conversion of the primeval being into its opposite"). See Kant's *Critique of Pure Reason* by Max Müller, Vol. I., p. 18.

the sum of the elements in an unconjugate arrangement it will manifestly be the number of ways in which the same complex number can be thrown under the form of a sum of the arithmetical series

$$a,\ a+c, \ldots,\ a+(i-1)c,$$

which is $$\frac{i^2-i}{2}c+ia,$$

that is $$\frac{i^2}{2}c+\frac{i}{2}(a-b),$$

or of $$b,\ b+c, \ldots,\ b+(i-1)c,$$

which is $$\frac{i^2}{2}c-\frac{i}{2}(a-b).$$

If $$\frac{i^2}{2}c+\frac{i}{2}(a-b)=\frac{j^2}{2}c+\frac{j}{2}(a-b),$$

then $$\frac{i^2+i}{2}a+\frac{i^2-i}{2}b=\frac{j^2+j}{2}a+\frac{j^2-j}{2},$$

which necessitates $i=j$, and if

$$\frac{i^2}{2}c+\frac{i}{2}(a-b)=\frac{j^2}{2}c-\frac{j}{2}(a-b),$$

then $$\frac{i^2+i}{2}a+\frac{i^2-i}{2}b=\frac{j^2-j}{2}a+\frac{j^2+j}{2}b,$$

so that $$i^2+i-(i^2-i)=(j^2-j)-(j^2+j) \quad \text{or} \quad i=-j.$$

Hence the general term is $q^{\frac{i^2}{2}c\pm\frac{i}{2}(a-b)}$, where i is an integer stretching from zero to infinity, and in like manner, and for the same reason, the general term in the former series will be $(-)^i q^{\frac{i^2}{2}c\pm\frac{i}{2}(a-b)}$ with the like interpretation: or which is the same thing, comprising both cases in one and interpreting i to be integer stretching from $-\infty$ to $+\infty$, the general term will be $(\mp)^i q^{\frac{i^2}{2}c+\frac{i}{2}(a-b)}$.

(59) The task before us then is to show the *possibility* of instituting, by *actually* instituting, a law of operation which shall satisfy the five preliminary conditions of catholicity, homoeogenesis, reciprocity, reversal of characters and conservation of sum.

The following notation will be found greatly to conduce to clearness in effecting the needful separation into classes or species. A capital letter with a point above, as $\dot{X}$, will be used to signify the greatest value, and with a point below, as $\underset{\cdot}{X}$, the least value of any term in a series which that letter is used to denote. $X=0$, $X>0$, $X+Y=0$, $X+Y>0$ will signify respectively that there are no X's, that there are X's, that there are no X's and no Y's, that there are either X's or Y's or both in any arrangement under consideration. B's will be separated into $'B$ and B''s, or as we may write it $B = 'BB'$, where $'B$ is the general name for all the B's, which beginning with the highest term $\dot{B}$ form an arithmetical series of which c is the common difference. If there is a gap of more than one c between $\dot{B}$ and the next lowest B, $'B$ is of course the single term $\dot{B}$: B' is any B which is not a $'B$.

So again, A_1 is any A which belongs to a series of A's forming an arithmetical series whose constant difference is c and lowest term a, so that unless $\underset{\cdot}{A} = a$, $A_1 = 0$: any other A will be designated by ${}_1A$. The signs of accent and point may of course be separate or combined: thus for example $\underset{\cdot}{C}$ will mean the smallest C in any given arrangement, $\dot{B}$ will mean the greatest B, $\underset{\cdot}{A}$ will mean the lowest A, ${}_1\underset{\cdot}{A}$ will mean the lowest of the ${}_1A$'s and $\dot{A}_1$ the highest of the A_1's. Every $'B$ is necessarily greater than any B', and every ${}_1A$ than any A_1. If $'\underset{\cdot}{B} - b = 0$, this will indicate that all the B''s will form a consecutive series of terms (that is having a constant difference c) and ending in b, so that here $B' = 0$, that is there are no B's except those that belong to the regular arithmetical progression ending in b. If ${}_1A = 0$, all the A's will form an arithmetical progression ending in a. Thus we see that the arrangements belonging to the 1st terms (those that I have called exceptional) will consist of two species denoted respectively by

$$ {}_1A + B + C = 0 \quad\text{and}\quad ('\underset{\cdot}{B} - b) + A + C = 0. $$

It may sometimes be found convenient to use a point to the left centre of a quantitative letter to signify that the quantity denoted is to be increased, and a point to the right centre to signify that the quantity denoted is to be diminished, by c. Thus $\dot{B}\cdot$ will mean $\dot{B} - c$, and $\cdot\underset{\cdot}{A}_1$ will mean $\underset{\cdot}{A}_1 + c$, the first signifying the greatest B diminished by and the second the smallest A_1 increased by c. When any general letter, say X, is wanting as indicated by the equation $X = 0$, $\underset{\cdot}{X}$ must be understood to mean zero. So for instance if $A = 0$, and consequently ${}_1A = 0$ and $A_1 = 0$, ${}_1\underset{\cdot}{A} = 0$. Again, when there is a gap between the highest B and the one that follows it in any arrangement, the arithmetical progression of $'B$'s reduces as above remarked to a single term and there results $'\dot{B} = '\underset{\cdot}{B}$. It may be noticed also that always $'\dot{B} = \dot{B}$, and $\underset{\cdot}{A}_1 = \underset{\cdot}{A}$.

The arrangements which are comprised under the forms

$$ (\alpha)\quad A,\ A - c,\ A - 2c, \ldots,\ a, $$
$$ (\beta)\quad B,\ B - c,\ B - 2c, \ldots,\ b, $$

may be regarded as belonging to what I shall term the first genus.

The second genus, namely that consisting of unexceptional combinations of unrepeated A's, B's, C's, may then be divided into the following three species, the conditions by which they are severally distinguished being attached to each in its proper place.

1st Species. Conditions (γ) $'\underset{.}{B} - b > 0$,

or (γ') $'\underset{.}{B} - b = 0,\ C > 0,\ \underset{.}{C} - c <= '\dot{B} - b$.

2nd Species. (δ) $'\underset{.}{B} - b = 0,\ A + C > 0,\ C = 0$ or $\underset{.}{C} - c > '\dot{B} - b$,

or (δ') $B = 0,\ C > 0,\ A = 0$, or ${}_1\underset{.}{A} - a = > C$.

3rd Species. (ϵ) $B = 0,\ A > 0,\ {}_1A + C > 0,\ C = 0$, or $\underset{.}{C} > {}_1\underset{.}{A} - a$.

Where it is to be understood that the conditions set out in the same line are simultaneous conditions. Thus for example the conditions of an arrangement being of the second species are when all the conditions of the upper or else all the conditions of the lower of the two lines written under that species are fulfilled: the conditions of the upper line (be it noticed) are that $'\underset{.}{B}$ is b, and that there are either some A's or some C's, and that if there are some C's, $\underset{.}{C} - c > '\dot{B} - b$, and of the lower line, that there are no B's and some C's, and that if there are A's, $\underset{.}{A} - a = > \underset{.}{C}$, and so for the interpretation of the conditions of the existence of each of the other two species.

To these (7) systems of conditions $\alpha, \beta, \gamma, \gamma', \delta, \delta', \epsilon$ may be joined the trivial system (ω) $A = 0, B = 0, C = 0$*; the (8) systems thus constituted will easily be seen to be mutually exclusive and between them to comprehend the entire sphere of possibility, leaving no space vacant to be occupied by any other hypothesis. I will now proceed to assign the operators ϕ, ψ, ϑ appropriate to the three species of the second genus.

Office of the Operator ϕ. $\phi = '\phi + \phi'$.

When in Genus 2, Species 1, $C = 0$ or $\underset{.}{C} - c > '\dot{B} - '\underset{.}{B}$, $'\phi$ is to be performed, meaning that for each $'B$, $'B\cdot$ is to be substituted, and the inertia kept constant by forming a new $\underset{.}{C}$ with the sum of the c's thus abstracted. In the contrary case ϕ' is to be performed, meaning that $\underset{.}{C}$ is to be resolved into simple c's and as many of the $'B$'s, commencing with $'\dot{B}$ and taken in regular order to be converted into $'\cdot B$ as are required to maintain the inertia constant, that is c is to be added to each B in succession, until all the c's which together make up $\underset{.}{C}$ are absorbed.

Office of the Operator ψ. $\psi = '\psi + \psi'$.

When in Genus 2, Species 2, $C = 0$ or $\underset{.}{C} > '\dot{B} + \underset{.}{A}$, $'\psi$ is to be performed, meaning that for $'\dot{B}$ and A their sum is to be substituted, producing a C [which, on the second hypothesis, will be a new $\underset{.}{C}$]. In the contrary case ψ' is to be performed, meaning that for $\underset{.}{C}$ is to be substituted $'\cdot\dot{B}$ (which will form a new $'\dot{B}$) and $\underset{.}{C} - '\cdot\dot{B}$ which will form a new $\underset{.}{A}_1$.

* It would be perfectly logical, and indeed is necessary to regard the trivial case as belonging to the cases of exception, and then we might say that there are two genera, each containing three species, those of the first genus solitary, and those of the second, each of them comprising two sub-species, namely the sub-species subject to the action of the left-accented and that subject to the operation of the right-accented operators. The *trivial* species of the first genus consists of a single individual.

Office of the Operator ϑ. $\vartheta = {}'\vartheta + \vartheta'$.

When $C > 0$ and $\underset{.}{C} + \dot{A}_1 < {}_1\underset{.}{A}$, ${}'\vartheta$ is to be performed, meaning that for $\underset{.}{C}$ and $\dot{A}_1$ their sum is to be substituted, producing a new ${}_1\underset{.}{A}$. In the contrary case ϑ' is to be performed, meaning that for ${}_1\underset{.}{A}$, $\cdot\dot{A}_1$ forming a new $\dot{A}_1$ and ${}_1\underset{.}{A} - \dot{A}_1$ forming a new C are to be substituted.

(60) It will be seen that every species of the second genus consists of two contrary sub-species having opposite characters, and it will presently appear that any arrangement belonging to one of these sub-species under the effect of its appropriate operator passes over into the other, which operated upon in its turn by its appropriate operator becomes identical with the original one, so that any two contrary sub-species may be said to be of equal extent: in fact if the sum of the parts is supposed to be given there will be as many arrangements in any sub-species as in its opposite, for each one will be conjugated with some one of the others.

It may not be amiss to call attention here to the fact that the scheme of classification adopted is, in a certain sense, artificial. Thus, for instance, it proceeds upon an arbitrary choice between which shall be regarded as the A and which as the B series, so that by an interchange of these letters a totally different correspondence would be brought about between the arrangements of the second genus, those of the first genus remaining unaltered. Nor is there any reason for supposing that these are the only two correspondences capable of being instituted between the arrangements of the second genus—in particular there is great reason to suspect that a symmetrical mode of procedure might be adopted, remaining unaffected by the interchange between A and B. As a simple example of the effect of interchange, applying the method here given, suppose $A = 0$, $B = 0$, a case belonging to the second species and that sub-species thereof to which ψ' is applicable, and imagine further that the C series is monomial. Then C will be associated according to the scheme here given with b, $C - b$, but in the correlative scheme it would be associated with a, $C - a$.

(61) I need hardly say that so highly organized a scheme, although for the sake of brevity presented in a synthetical form, has not issued from the mind of its composer in a single gush, but is the result of an analytical process of continued residuation or successive heaping of exception upon exception in a manner dictated at each point in its development by the nature of the process and the resistance, so to say, of its subject-matter. The initial step (that applicable to species γ) is akin to the procedure applied by Mr F. Franklin to the pentagonal-number theorem of Euler, of which I shall have more to say presently. It will facilitate the comprehension of the scheme to take as an example the particular case where a and b represent actual and real quantities, say, to fix the ideas, $b = 1$, $a = 2$. Nothing, it will

be noticed, turns upon the fact of this specialization, which is adopted solely for the purpose of greater concision and to afford more ready insight into the *modus operandi.*

To illustrate the classes and laws of transformation consider (with $b=1$, $a=2$*, $c=a+b=3$) all the arrangements, the sum of whose parts is 12, namely 12, 11.1, 10.2, 9.2.1, 8.4, 8.3.1, 7.5, 7.4.1, 7.3.2, 6.5.1, 6.4.2, 5.4.3, 5.4.2.1.

One of these, 7.4.1, belongs to the exceptional genus. The rest will be conjugated and fall into species in the manner shown below, where the first species means where the conditions (γ) or (γ'), the second that where (δ) or (δ'), and the third where the conditions (ϵ) are satisfied. The C's, B's, A's are now numbers whose residues are 0, 1 or 2 in respect to the modulus 3. For greater clearness in each arrangement, numbers belonging to the same series are kept together, the law of descent only applying in this theory to elements belonging to the same series.

Species 1. 10.2 3.7.2; 4.8 3.1.8; 7.5; 3.4.5; 6.4.2 6.3.1.2; 5.7 3.2.7.

Species 2. 9.1.2 9.3; 6.1.5 4.1.5.2;

Species 3. Caret.

Or again let the collection of arrangements be one in which the sum is 18. The partitions of 18 are 18 17.1 16.2 15.3 15.2.1 14.4 14.3.1 13.5 13.4.1 13.3.2 12.6 12.5.1 12.4.2 12.3.2.1 11.7 11.6.1 11.5.2 11.4.3 11.4.2.1 10.8 10.7.1 10.6.2 10.5.3 10.5.2.1 10.4.3.1 9.8.1 9.7.2 9.6.3 9.6.2.1 9.5.4 9.5.3.1 9.4.3.2 8.7.3 8.7.2.1 8.6.4 8.6.3.1 8.5.4.1 8.5.3.2 8.4.3.2.1 7.6.5 7.6.4.1 7.6.3.2 7.5.4.2 7.5.3.2.1 6.5.4.3 6.5.4.2.1. In this case there are no exceptional arrangements.

1st Species. 16.2 3.13.2; 4.14 3.1.14; 13.5 3.10.5; 13.4.1 3.10.4.1; 7.11 3.4.11; 10.8 3.7.8; 12.4.2 12.3.1.2; 10.7.1 6.7.4.1; 6.10.2 6.3.7.2; 10.1.5.2 3.7.1.5.2; 9.4.5 9.3.1.5; 6.7.5 6.3.4.5; 7.1.8.2 3.4.1.8.2; 6.4.8 6.3.1.8; 7.4.5.2 6.4.1.5.2;

2nd Species. 18 17.1; 15.3 15.1.2; 12.6 12.5.1; 6.1.11 4.1.11.2; 9.1.8 4.1.8.5; 9.7.2 9.3.4.2; 9.6.3 9.6.1.2; 11.5.2 3.8.5.2.

3rd Species. Caret.

If the partible number is 11, of which the partitions are 11 10.1 9.2 8.3 8.2.1 7.4 7.3.1 6.5 6.4.1 6.3.2 5.4.2 5.3.2.1, there will be no exceptional arrangements and the pairs of unexceptional ones will be as below.

* No use it will be seen is made of the *accidental* relation $a=b+b$.

1st Species. 10.1 3.7.1; 7.4 6.4.1; 4.5.2 3.1.5.2.

2nd Species. 3.8 1.8.2.

3rd Species. 11 9.2; 6.5 6.3.2.

By interchanging a and b, that is making $a=1$, $b=2$, the correspondence changes into the following:

1st Species. 11, 3.8; 6.3.2, 6.5; 8.2.1, 3.5.2.1; 7.4, 6.4.1.

2nd Species. Caret.

3rd Species. 10.1, 6.4.1; 7.4, 3.7.1.

According to Mr Franklin's process the correspondence takes a form quite distinct from either of the above, namely 11, 10.1; 9.2, 8.2.1; 8.3, 7.3.1; 7.4, 6.4.1; 6.5, 5.4.2; 6.3.2, 5.3.2.1, all these arrangements constituting one single species.

A careful study of the preceding examples will sufficiently explain to the reader the ground of the divisions into species with their appropriate rules of transformation, and might almost supersede the necessity of a formal proof of the operator supplying the conditions of catholicity, homoeogenesis and mutuality; from their very definition they are seen to comply with the other two essential conditions of inertia and enantiotropy.

Signifying by Ω the total operator $\phi+\psi+\vartheta$, it has been already remarked that Ω will in the general case have two values which only come together when $a=b$, or which is the same thing, each of them is 1; a special case of the special case when the complex reduces to simple numbers, namely, it is the case indicated in the well-known equation

$$(1-q)^2(1-q^3)^2(1-q^5)^2\ldots=\frac{1}{(1-q^2)(1-q^4)\ldots}\sum_{i=-\infty}^{i=\infty}q^{i^2}.$$

But besides the two correspondences given by the two values of Ω, if we take the actual (no longer a diagrammatic case) $b=2$, $a=1$, we revert to Euler's theorem concerning the partitions of all pentagonal and non-pentagonal numbers, and can obtain by Dr Franklin's process, given in Art. (12), a totally different distribution into genera and species, namely the first genus instead of containing arrangements of the species

$$1, 4, 7, \ldots 3i-2;\quad 2, 5, 8, \ldots 3i-1$$

will, as previously shown, consist of the very different arrangements (giving the same infinite series of numbers as those for other sums)

$$i, i+1, i+2, \ldots 2i-1;\quad i+1, i+2, i+3\ldots;\ 2i.$$

The character of each arrangement in the new solution depends in part on the relation to the modulus 2 of the whole number of parts and of the number of parts which are divisible by 3, so that we may divide the conjugate arrangements into four groups* designated respectively by *Oo, Oe*; *Eo, Ee,* using the capital letters to signify the oddness or evenness of the whole set of parts, and the small letters the same for the parts divisible by 3. There will thus be a cross classification of the arrangements of the second genus into groups over and above that into species, each species in fact consisting of four groups, which may be denoted as above, and of which *Oo* and *Ee* are one associative couple, and *Oe, Eo* the other†.

(62) The following elegant investigation has been handed in to me by Arthur S. Hathaway, fellow and one of my hearers at the Johns Hopkins University, to which, although it does not exactly strike at the object of the constructive theory here expounded, I gladly give hospitality in these pages.

"The theorem to be proved is as follows:

$$\begin{aligned} &1 + \epsilon x^a . 1 + \epsilon x^{a+h} . 1 + \epsilon x^{a+2h} \ldots \\ \times\, &1 + \epsilon x^b . 1 + \epsilon x^{b+h} . 1 + \epsilon x^{b+2h} \ldots \\ \times\, &1 - x^h . 1 - x^{2h} . 1 - x^{3h} \ldots = \sum_{\delta=-\infty}^{\delta=+\infty} \epsilon^\delta . x^{\frac{a+b}{2}\delta^2 + \frac{a-b}{2}\delta}, \end{aligned}$$

where $\epsilon^2 = 1$ and $h = a + b$, a and b being any quantities whatever.

"The general term contains, say i exponents of x selected from the first line, j from the second line, and k from the third line, namely

$$\begin{gathered} a + \alpha_0 h, \ldots a + \alpha_{i-1} h, \\ b + \beta_0 h, \ldots b + \beta_{j-1} h, \\ \gamma_1 h, \ldots \gamma_k h, \end{gathered}$$

where $\alpha_0 \ldots \alpha_{i-1}$, $\beta_0 \ldots \beta_{j-1}$, $\gamma_1 \ldots \gamma_k$ are respectively sets of i, j, k unequal integers arranged in ascending order, none representing a less integer than its subscript. This term is (remembering that $h = a + b$)

$$\epsilon^{i+j} (-)^k x^{ma+nb},$$

where

$$m = [(\alpha_0 + 1) + \ldots (\alpha_{i-1} + 1)] + [\beta_0 + \ldots \beta_{j-1}] + [\gamma_1 + \ldots \gamma_k] \qquad (1)$$

$$n = [\alpha_0 + \ldots \alpha_{i-1}] + [(\beta_0 + 1) + \ldots (\beta_{j-1} + 1)] + [\gamma_1 + \ldots \gamma_k] \qquad (2)$$

* It will be seen later on that there is a division into sixteen groups analogous to the division into four groups first noticed by Prof. Cayley arising under the Franklin process.

† The *Oe* and *Eo* conjugation has a very striking analogue in nature (as I am informed) in the existence of dissimilar hermaphrodite characters in two sorts of the wild English *primrose* and the American flower *Spring-beauty* or *Quaker-lady*—it being the law of nature that only those of different sorts can fertilize one another. Possibly the double symbolic character of *Oo* and *Ee* will justify or suggest the inquiry whether there may not be a latent duality in the unisexual specimens of such flowers as those just mentioned, where male and female are found codomiciled with the bisexual florets. There is also, it seems, a trace of analogy to the sparsely distributed unconjugate individuals of my first genus in Darwin's "complemental males."

In addition to these we obtain by subtraction

$$m-n=i-j\equiv i+j \bmod 2. \tag{3}$$

Whence (since $\epsilon^2=1$) $\epsilon^{i+j}=\epsilon^{m-n}$.

"Thus all the above general terms having the same m and the same n divide themselves into positive and negative groups (corresponding to even and odd values of k), a term from one group cancelling a term from the other group. I propose to prove that the number of terms in each of these groups are equal, except when a certain relation exists between m and n, namely

$$m-\frac{(m-n)(m-n+1)}{2}=0, \text{ (or } m=0 \text{ if } m=n),$$

corresponding to which there is but one general term having the same m and the same n which falls into the positive group ($k=0$). This establishes the theorem in question, as we see by putting $m-n=\delta$.

"It is sufficient to consider (1) in connection with (3). In the first place the first two partitions in (1) may be converted by a (1 : 1) correspondence into an indefinite partition (bearing in mind (3)) with a decrease ($m-n>0$) in the sum or content of the integers by $\frac{1}{2}(m-n)(m-n+1)$, as follows: extend α_0+1 in a horizontal line of dots, and *under* the first dot extend β_0 in a vertical line of dots, thus forming an *elbow*; in a similar manner form elbows out of α_1+1, β_1 &c. until one of the partitions is exhausted; this will be according to (3), the first or the second, according as $m<$ or $>n$, leaving in the inexhausted partition $m-n$ integers; place these elbows successively one without the other, and place on top ($m-n>0$) horizontal lines of dots corresponding to the successive unmatched integers decreased respectively by $0, 1, \ldots (n-m-1)$ or $1, 2, \ldots (m-n)$, according as $m<$ or $>n$; in either case the total decrease is $\frac{1}{2}(m-n)(m-n+1)$. In other words, the above tri-partition of m has a (1 : 1) correspondence with a bi-partition of

$$m-\frac{(m-n)(m-n+1)}{2}, \text{ (or } m \text{ if } m=n),$$

consisting of an indefinite partition on one side and a partition of unrepeated integers on the other ($\gamma_1, \ldots \gamma_k$). Such a bi-partition (on removing the line of demarcation) is an indefinite partition; and, conversely, every indefinite partition involving θ different integers gives rise as follows to $(1+1)^\theta$ such bi-partitions, the number of those involving even and odd values of k being respectively the positive and negative parts of the expansion of $(1-1)^\theta$, which are equal: namely, first, the indefinite partition itself ($k=0$); second, the θ bi-partitions obtained by placing each of the θ integers successively on the k side ($k=1$); third, the $\frac{1}{2}\theta(\theta-1)$ bi-partitions obtained by placing the $\frac{1}{2}\theta(\theta-1)$ pairs of the θ integers successively on the k side ($k=2$), and so on.

The only exception to this equality of the number of partitions for even and odd values of k is when the partible number,

$$m - \frac{(m-n)(m-n+1)}{2} \text{ or } m,$$

is zero, for which case there is but one bi-partition $[0]+[0]\,(k=0)$. Q.E.D. The tri-partition of m corresponding to the celibate case reduces to the natural sequence above subtracted whose content is

$$\frac{(m-n)(m-n+1)}{2} \text{ (or 0)},$$

which is the second or the first partition (according as $m <$ or $> n$), the others being wanting."

(63) The same infinitesimal method which applied to the expansion of $\Theta_1 x$ gives rise as was shown to the expression for the cubes of the successive rational binomial functions may be applied to the development of

$$(1+ax)(1+ax^2)(1+ax^3)\ldots$$

given in Art. (35), but will not lead to any new result. Making $a = -x^{-1-\epsilon}$, where ϵ is infinitesimal, we obtain from the general theorem

$$(1-x^\epsilon)(1-x)(1-x^2)(1-x^3)\ldots$$
$$= 1 - \frac{1-x^\epsilon}{1-x}x + \frac{1-x^\epsilon . 1-x}{1-x . 1-x^2}x^5 - \frac{1-x^\epsilon . 1-x . 1-x^2}{1-x . 1-x^2 . 1-x^3}x^{12}\ldots$$
$$- x^\epsilon + \frac{1-x^\epsilon}{1-x}x^3 - \frac{1-x^\epsilon . 1-x}{1-x . 1-x^2}x^9\ldots,$$

or $$(1-x)(1-x^2)(1-x^3)\ldots = 1 - \frac{x-x^3}{1-x} + \frac{x^5-x^9}{1-x^2}\ldots$$
$$= 1 - x(1+x) + x^5(1+x^2)\ldots,$$

the same equation as results from writing $a = -1$.

To arrive at any new result it would be necessary to have recourse to processes of differentiation; the above calculation serves, however, as a verification if any were needed of the accuracy of the theorem to which it refers.

(64) Since sending what precedes to press I have thought it would be desirable in the interest of sound logic to set out the marks or conditions of the several species of the arrangements of unrepeated A, B, C's, somewhat more fully and explicitly than before. And first, I may observe that since it has been convenient to understand that when there are no X terms $\underset{\cdot}{X}$ shall signify zero, the quantitative equation $\underset{\cdot}{X}=0$ dispenses with the necessity of

using the symbolical one $X=0$, and in like manner $\underset{.}{X}>0$ supersedes the symbolical inequality $X>0$, and, of course, the same remark extends to the equality or inequality $X+Y=$ or >0.

We have then for what I shall term the first, second and third species of genus 1, the conditions

$$\underset{.}{C}+\underset{.}{B}+\underset{.}{A}=0, \quad \underset{.}{C}+{}'\underset{.}{B}+\underset{.}{A}=b, \quad \underset{.}{C}+\underset{.}{B}+\underset{.}{A}_1=0$$

respectively—the first, the trivial case of vacuous content; the second, of only a complete natural B progression, that is, one ending with b (the minimum value of B), and the third, the same for A similarly ending with the minimum a. In what follows the conditions in each separate *line* are to be understood to be not disjunctive but simultaneous or accumulative; they of course refer to the species of the second genus.

Marks of species (1) (α) $\underset{.}{B}-b>0$,
or (β) $\underset{.}{B}-b=0$, ${}'\dot{B}-{}'\underset{.}{B}=\,>\underset{.}{C}-c$, $\underset{.}{C}>0$.
" " (2) (α) $\underset{.}{B}-b=0$, $\underset{.}{C}-c>{}'\dot{B}-{}'\underset{.}{B}$,
or (β) $\underset{.}{B}-b=0$, $\underset{.}{C}=0$ [$\underset{.}{A}>0$],
or (γ) $\underset{.}{B}=0$, $\underset{.}{A}-a=\,>\underset{.}{C}$, $\underset{.}{C}>0$,
or (δ) $\underset{.}{B}=0$, $\underset{.}{A}=0$ [$\underset{.}{C}>0$].
" " (3) (α) $\underset{.}{B}=0$, $\underset{.}{C}>\underset{.}{A}-a$, $\underset{.}{A}>0$,
or (β) $\underset{.}{B}=0$, $\underset{.}{C}=0$ [${}_1\underset{.}{A}-a>0$].

The three inequalities included in brackets are only required in order to exclude arrangements belonging to the first genus. Leaving these out of account for the moment, merely for the sake of greater concision of statement, it is easy to see by mere inspection of the above table that the three species are mutually exclusive and share between them the total sphere of possibility, for (1) α exhausts the hypothesis of there being other B's besides those forming a complete natural progression, (1) β and (2) α of the B's forming such progression when there are existent C's, and (2) β when there are not. Also ((2) γ, (2) δ), (3) α exhaust between them the hypothesis of there being no B's when there are some existent C's, and (3) β of neither B's nor C's appearing in an arrangement.

Thus all unexceptional arrangements must bear the marks occurring in one or the other of the first four lines of the table, and all those where no B's occur, either of the last line when there are neither B's nor C's, and of the three preceding ones when there are no B's but some C's, and the total sum of these hypotheses plus the hypothesis of the first genus together make up necessity, as was to be shown.

The convention $\underset{.}{X}=0$ when an arrangement contains no X with the consequent reduction of the conditions to a purely quantitative form has lent itself very advantageously to the above bird's-eye view of the completeness of the scheme (as covering the whole ground of possibility); it also will be found to simplify the expression of the proof. I did not employ it until the necessity for so doing forced itself upon my notice, for a very obvious reason, namely that X is a B (or an A), which is defined to be congruous to b (or a) [mod c], which zero is not: there is thus an apparent paralogism in admitting that any X of these two *where there is a* B (or when there is an A) is congruent to b (or to a), but that *when there is no* B (or no A) then the conventional least B (or A) is zero. It will be seen, however, *ex post facto*, that no inconvenience in working the scheme results from this extended definition which constitutes an important gain to the perfect evolution of the method. It is usually in the form of some apparent contradiction or paradox that a scientific advance makes its first appearance.

(65) Aided by this clearer and fuller expression of the definitions of the genera and species, I will now set out a logical proof that the respective operators fulfil the three additional necessary conditions. I may observe preliminarily that the Greek letterings α, β; $\alpha, \beta, \gamma, \delta$; α, β, do not express sub-species, for one distinguishing mark of species (or sub-species) may be taken to be that conjugation cannot take place except between individuals of the same species or sub-species, but it will be presently seen that individuals belonging to the differently lettered divisions of the above species are susceptible of mutual conjugation—and are therefore in conformity with biological precedent to be regarded as mere varieties. Besides these varieties of each of the species there is another entirely different principle of cross classification applicable to each of them, namely in general an arrangement must belong to one of sixteen groups designated by combining together one out of each of the four pairs of opposite symbols X, C; x, c; O, E; o, e, where the large O, E refer to the oddness or evenness of the major, and the small o, e to the same for the minor parameter; and in like manner the large X and large C to the result of the operation appropriate to any arrangement, being to extend or contract the major, and x, c to extend or contract the minor parameter. There are thus eight pairs of groups, and conjugation can only take place between individuals belonging to the same pair.

The pairs are as follows:

$$\begin{pmatrix} XxOo \\ CcEe \end{pmatrix}, \begin{pmatrix} XxOe \\ CcEo \end{pmatrix}, \begin{pmatrix} XxEo \\ CcOe \end{pmatrix}, \begin{pmatrix} XxEe \\ CcOo \end{pmatrix};$$

and

$$\begin{pmatrix} XcOo \\ CxEe \end{pmatrix}, \begin{pmatrix} XcOe \\ CxEe \end{pmatrix}, \begin{pmatrix} XcEo \\ CxOe \end{pmatrix}, \begin{pmatrix} XcEe \\ CxOo \end{pmatrix}.$$

Species (1) and species (3) it will be seen may each be separately divided into four sub-species denoted by the upper four, and species (2) into the four sub-species denoted by the lower four pairs of combined characters, so that there will be in all twelve (and not as might at first be supposed twenty-four) sub-species of conjugable arrangements. The different sub-species of the same species do not admit of cross-conjugation ; it is the property which they have in common of being subject to the same law of transformation when passage is made from an individual to its conjugate, which binds them together into a single species. In the arrangements peculiar to Euler's problem, we see that there was no division of the second genus at the outset, but that a separation would be made of it into two pairs of groups with conjugation possible only between individuals belonging to the same pair, and consequently there may be said in this case to be two species of the second genus, analogous, however, not to the species but the sub-species in the more general theory. The final separation of a pair of groups into its component elements has nothing to do with the concept of species, sub-species or variety, but may be regarded as similar to the separation of the sexes.

In what follows, a bracket enclosing a letter will be used to denote that it belongs to an arrangement after it has been operated upon by its appropriate operator, or what may be called its operate.

Species (1). When $\underset{.}{B}-b>0$, if $\underset{.}{C}-c>{}'\dot{B}-{}'\underset{.}{B}$ or $\underset{.}{C}=0$, $'\phi$ may be performed, giving $[\underset{.}{C}]={}'\underset{.}{B}-{}'B+C<\underset{.}{C}$ so that the law of descending magnitude is maintained ; we have then $[\dot{B}]-['\underset{.}{B}]=$ or $>\dot{B}-{}'\underset{.}{B}=>[\underset{.}{C}]-c$; hence ϕ' has to be performed and will obviously restore the original arrangement. Again if in the original arrangement $\dot{B}-{}'\underset{.}{B}=>\underset{.}{C}-c$ and $\underset{.}{C}>0$, ϕ' has to be applied; a resolution of $\underset{.}{C}$ can take place into c's and the $\underset{.}{C}/c$ first $'B$'s, and will each be increased by c and $[\dot{B}]-{}'[\underset{.}{B}]=\underset{.}{C}-c$, so that either $[\underset{.}{C}]=0$ or $[\underset{.}{C}]-c<\underset{.}{C}-c<[\dot{B}]-{}'[\underset{.}{B}]$, and $'\phi$ being applicable to the new arrangement will convert it back to the original one.

First Species (β). When $\underset{.}{B}-b=0$ and $\dot{B}-{}'\underset{.}{B}=>\underset{.}{C}-c$ and $\underset{.}{C}>0$, ϕ' can be performed, and the new arrangement as before may be operated upon by ϕ' and so brought back to its original value. If $\underset{.}{C}=0$ or $\underset{.}{C}-c>\dot{B}-{}'\underset{.}{B}$, $'\phi$ could not be performed, for then $\underset{.}{B}=b$ and has no c to part with to help make up $[\underset{.}{C}]$.

These two hypotheses belong to Species (2), which we will now proceed to consider throughout its full extent. When $\underset{.}{B}-b=0$, then $'\underset{.}{B}=b$, and I shall first suppose [(α) and (β)] that $\underset{.}{C}=0$ or $\underset{.}{C}-c>B-b$. When $\underset{.}{C}=0$ or $\dot{B}+\underset{.}{A}>\underset{.}{C}$, then $'\psi$ will be applicable, making $[\underset{.}{C}]=\dot{B}+\underset{.}{A}$; if now $[\underset{.}{B}]>0$ and $[\underset{.}{A}]>0$, $[\underset{.}{B}]+[\underset{.}{A}]=>(\dot{B}-c)+(\underset{.}{A}+c)=>\dot{B}+\underset{.}{A}=>[\underset{.}{C}]$, and

$$[\underset{.}{C}]-c=\dot{B}+\underset{.}{A}-c=[\underset{.}{B}]+\underset{.}{A}>[\underset{.}{B}]-b.$$

Hence we are still within Species 2 and have fallen upon the case to which the reversing operator ψ' has to be applied. If $[\underset{.}{B}]=0$, $[\underset{.}{A}]=0$ we must have $B[\underset{.}{C}]>0$, inasmuch as the original content (or inertia) is originally greater than zero and is kept constant, and this is a case which still belongs to Species 2 and falls under the operation of ψ'.

If $[\underset{\cdot}{B}]=0$ so that $\dot{B}=\underset{\cdot}{B}=b$ and $[\underset{\cdot}{A}]>0$, then

$$[\underset{\cdot}{A}]-a=>\underset{\cdot}{A}+c-a=>\underset{\cdot}{A}+\dot{B}=>\underset{\cdot}{C},$$

which also falls within the second species and is amenable to the reversing operator ψ'.

Finally, if $[\underset{\cdot}{B}]>0$, that is $\underset{\cdot}{B}-b=0$ and $[\underset{\cdot}{A}]=0$,

$$[\underset{\cdot}{C}]-c=\dot{B}+\underset{\cdot}{A}-c=>[\dot{B}]-b,$$

that is $=>[\dot{B}]-{}'\underset{\cdot}{B}$, and we are still within Species (2) and in the case amenable to the reversing operator ψ'.

If now on the other hand we begin with an arrangement of the second species in the case amenable to ψ' we must suppose either $\underset{\cdot}{B}=0$ or $\underset{\cdot}{A}=0$, or else $\underset{\cdot}{C}>0$ and $\underset{\cdot}{C}<=\dot{B}+A$.

Take first this last supposition. The operation of ψ' gives $[\underset{\cdot}{C}]=>\underset{\cdot}{C}+c$,

$$[\dot{B}]=\dot{B}+c \text{ and } [\underset{\cdot}{A}]=\dot{C}-c-\dot{B}>\dot{B}-b-\dot{B}>-b=>c-b=>a.$$

And $\quad [\dot{B}]+[\underset{\cdot}{A}]=\dot{B}+\underset{\cdot}{C}-\dot{B}=\underset{\cdot}{C}<[\underset{\cdot}{C}]$,

$$[\underset{\cdot}{C}]-c=>(\underset{\cdot}{C}-c)+c=>B-b+c=>[B]-[\underset{\cdot}{B}].$$

Hence the operate is licit, belongs to the second species and is amenable to the reversing operator ${}'\psi$.

If $\underset{\cdot}{B}=0$ and $\underset{\cdot}{A}=0$, $[\dot{B}]=[\underset{\cdot}{B}]=b$ and $[\underset{\cdot}{A}]=\underset{\cdot}{C}-b$ and $[\underset{\cdot}{C}]=0$ or $>\underset{\cdot}{C}$.

If $[\underset{\cdot}{C}]=0$ since $[\underset{\cdot}{A}]>0$, the operate is included in variety (β) of the second species and amenable to the reversing operator ${}'\psi$, and if

$$[\underset{\cdot}{C}]>\underset{\cdot}{C}\,[\underset{\cdot}{C}-c]>\underset{\cdot}{C}-c>0,$$

that is $>[\dot{B}]-\underset{\cdot}{B}$ which belongs to variety (α) of the second species; and since $[\underset{\cdot}{C}]>\underset{\cdot}{C}>[\dot{B}]+[\underset{\cdot}{A}]$ is amenable to the reversing operator ${}'\psi$.

If $\underset{\cdot}{B}>0$ and $\underset{\cdot}{A}=0$, then $\underset{\cdot}{C}>0$ [otherwise it would be an arrangement in Genus 1, Species 2] $[\underset{\cdot}{C}]=0$ or $>\underset{\cdot}{C}$, $[\dot{B}]=\dot{B}+c$,

$$[\underset{\cdot}{A}]=\underset{\cdot}{C}-[\dot{B}]>(c+\dot{B}-b)-(c+\dot{B})=>a,$$

and either $[\underset{\cdot}{C}]=0$ and $[\underset{\cdot}{A}]>0$ or

$$[\underset{\cdot}{C}]-c>(\underset{\cdot}{C}-c)+c>\dot{B}+c-b>[\dot{B}]-{}'\underset{\cdot}{B}$$

and $[\underset{\cdot}{A}]+[\dot{B}]=\underset{\cdot}{C}>[\underset{\cdot}{C}]$. Hence in either hypothesis the operate is still in Species (2) and amenable to the reversing operator ${}'\psi$.

Lastly, if $\underset{\cdot}{B}=0$, $\underset{\cdot}{A}-a=>\underset{\cdot}{C}$ and $\underset{\cdot}{C}>0$, the arrangement is amenable to the operator ψ', which will make $[\underset{\cdot}{B}]=b$, $[\underset{\cdot}{A}]=\underset{\cdot}{C}-b<\underset{\cdot}{C}+a<\underset{\cdot}{A}$. We have then $[\underset{\cdot}{B}]-b=0$ and $[\underset{\cdot}{C}]=0$, and consequently also $\underset{\cdot}{A}>0$ or

$$[\underset{\cdot}{C}]-c>\underset{\cdot}{C}-c>0,$$

that is $>[\dot{B}]-{}'[B]$, and the result is still contained within Species (2) and is amenable to the reversing operator ${}'\psi$.

(66) The following are examples of paired arrangements belonging to the first species, adapted to the case of $a=2$, $b=1$. The C and B terms are

expressed; the A line is the same for each of any pair of this species, and may be filled in at will.

$$\phi'\left\{\begin{matrix} X.9. \\ 16.13.10.Y \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 19.16.13.Y \end{matrix}\right\}$$

where X, Y represent any licit series of C's and B's respectively.

$${}'\phi\left\{\begin{matrix} X.9 \\ 16.13.7.Y \end{matrix}\right\} = \left\{\begin{matrix} X.9.6. \\ 13.10.7.Y \end{matrix}\right\} \quad \phi'\left\{\begin{matrix} X.9 \\ 16.13.10.4 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 19.16.13.4 \end{matrix}\right\}$$

$$\phi'\left\{\begin{matrix} X.9 \\ 7.4.1 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 10.7.4 \end{matrix}\right\} \quad {}'\phi\left\{10.7.4\right\} = \left\{\begin{matrix} 9. \\ 7.4.1 \end{matrix}\right\}$$

$$\phi'\left\{\begin{matrix} 3. \\ 13.7.4.1 \end{matrix}\right\} = \left\{16.7.4.1\right\}.$$

The following are examples of paired arrangements of the second species with $a=2$ and $b=1$ as usual.

$${}'\psi\left\{\begin{matrix} X.12. \\ 7.4.1. \\ Y.2 \end{matrix}\right\} = \left\{\begin{matrix} X.12.9. \\ 4.1 \\ Y \end{matrix}\right\} \quad \psi'\left\{\begin{matrix} X.12. \\ 7.4.1. \\ Y.5 \end{matrix}\right\} = \left\{\begin{matrix} X \\ 10.7.4.1. \\ Y.5.2 \end{matrix}\right\}$$

$${}'\psi\left\{\begin{matrix} 7.4.1. \\ Y.5. \end{matrix}\right\} = \left\{\begin{matrix} 12. \\ 4.1. \\ Y \end{matrix}\right\} \quad \psi'\left\{\begin{matrix} X.15 \\ 7.4.1 \\ Y.8 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 10.7.4.1 \\ Y.8.5 \end{matrix}\right\}$$

$$\psi'\left\{\begin{matrix} X.9. \\ \ldots. \\ \ldots. \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 1. \\ 8 \end{matrix}\right\} \quad \psi'\left\{\begin{matrix} 6. \\ 1. \\ 8 \end{matrix}\right\} = \left\{\begin{matrix} \ldots \\ 4.1. \\ 8.2 \end{matrix}\right\}$$

$$\psi'\left\{\begin{matrix} X.9. \\ \ldots \\ Y.11 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 1. \\ Y.11.8 \end{matrix}\right\}.$$

We come now to the third species. Here, I think, the reader will find it a great relief to the strain upon his attention if I invite him before attacking the demonstration to consider the annexed diagrammatic cases accommodated to the supposition $a=2$, $b=1$. The B's it will be remembered in this species do not exist, and the action neither of ${}'\vartheta$ nor ϑ' introduces any B into the transformed arrangement. In the examples given below the C and A terms occupy the higher and lower lines respectively—the comma is used in the latter to mark off the ${}_1A$'s from the A_1's.

$${}'\vartheta\left\{\begin{matrix} 9.6. \\ 14.11.8.5, \end{matrix}\right\} = \begin{matrix} 9.6.3. \\ 14.11.8,\ 2 \end{matrix} \quad \vartheta'\left\{\begin{matrix} 6.3. \\ 14.11.8,\ 2 \end{matrix}\right\} = \begin{matrix} 6. \\ 14.11.8.5, \end{matrix}$$

$${}'\vartheta\left(17.8.5\right) = \begin{matrix} 3. \\ 17.8,\ 2 \end{matrix} \quad {}'\vartheta\left(17.8.5,\right) = \begin{matrix} 3. \\ 17.8,\ 2 \end{matrix}$$

$${}'\vartheta\left(17,\ 8.5.2\right) = \begin{matrix} 6. \\ ,\ 11.8.5.2 \end{matrix} \quad {}'\vartheta\left(17.14,\ 8.5.2\right) = \begin{matrix} 3. \\ 17,\ 11.8.5.2 \end{matrix}$$

$${}'\vartheta\, 11, = \begin{matrix} 9. \\ ,\ 2 \end{matrix} \quad \vartheta'\left\{\begin{matrix} 12.9.3. \\ ,\ 11.8.5.2 \end{matrix}\right\} = \begin{matrix} 12.9. \\ 14,\ 8.5.2 \end{matrix}$$

$$\vartheta'\left\{\begin{matrix} 9.6.3. \\ ,\ 11.8.5.2 \end{matrix}\right\} = \begin{matrix} 9.6 \\ 14,\ 8.5.2 \end{matrix}.$$

The left-hand accent is used here as elsewhere to signify that phase of the operator which brings about an increase and the right-hand one a decrease in the number of C's. It will readily be seen that the action of the operator in each of the above examples prepares the arrangement for the action of the contrary one which will restore it to its original value. It is worthy of notice that in any two associated arrangements above, an a (here 2) *may* appear in each and *must* appear in one of them. I will now proceed to the general demonstration.

(67) Let us first suppose $\underset{\cdot}{A}_1 = 0$, then ${}_1\underset{\cdot}{A} > 0$, otherwise we shall be dealing with the antecedent species and ${}'\vartheta$ will be applicable, making $[\underset{\cdot}{A}] = [\dot{A}_1] = a\, [\underset{\cdot}{C}] = \underset{\cdot}{A} - a < \underset{\cdot}{C}$ and $> (\underset{\cdot}{A} - a)$. Thus the generated arrangement is licit and belongs still to the third species; but now $[\underset{\cdot}{C}] + [\dot{A}_1] = \underset{\cdot}{A}$ and $[{}_1\underset{\cdot}{A}] = 0 > \underset{\cdot}{A}$. Hence the reversing operator ϑ' is applicable to the new arrangement; the remaining cases to consider (in which $\underset{\cdot}{A} = a$ for the arrangement as well before as after being operated upon) may be separated into those where $\underset{\cdot}{C} > 0$, and at the same time either $\underset{\cdot}{C} + \dot{A}_1 < {}_1\dot{A}$ or ${}_1\dot{A} = 0$, which are amenable to the operator ϑ' and the complementary cases which are amenable to ${}'\vartheta$.

In the cases first considered $[\dot{A}_1] = \dot{A}_1 - c$, $[{}_1\dot{A}] = \underset{\cdot}{C} = \underset{\cdot}{A}_1 \vartheta\, [\underset{\cdot}{C}] + 0$ or $> \underset{\cdot}{C}$ (and *à fortiori* > 0), consequently the new arrangement is licit and still belongs to the third species, and since either $[\underset{\cdot}{C}] = 0$ or else

$$[\underset{\cdot}{C}] + [\dot{A}_1] > \underset{\cdot}{C} + \dot{A}_1 - \underset{\cdot}{C} = > [{}_1\dot{A}]$$

and $[{}_1\dot{A}] > 0$, it is one of the complementary cases and is subject to the reversing operator ${}'\vartheta$.

Again, any arrangement for which $\underset{\cdot}{A} = a$ belonging to the complementary cases is defined by the conditions ${}_1\dot{A} > 0$ and $\underset{\cdot}{C} + \dot{A}_1 = > {}_1\underset{\cdot}{A}$ and is by hypothesis to be subjected to the operator ${}'\vartheta$ which will make $[\dot{A}_1] = \dot{A}_1 + c$, $[{}_1\underset{\cdot}{A}] = 0$ or $> {}_1\underset{\cdot}{A}\, [\underset{\cdot}{C}] = {}_1\underset{\cdot}{A} - \dot{A}_1 - c$, and since $\underset{\cdot}{C} = > {}_1\underset{\cdot}{A} - \dot{A}_1, [\underset{\cdot}{C}] < \underset{\cdot}{C}$, so that the operation leads to a licit new arrangement.

Also $[\underset{\cdot}{C}] + [\dot{A}_1] = {}_1\underset{\cdot}{A}$, and consequently either $[{}_1\underset{\cdot}{A}] = 0$ or $[\underset{\cdot}{C} + \dot{A}_1] < [{}_1\underset{\cdot}{A}]$, which is a condition belonging to the first considered class of cases, subject to the reversing operator ϑ', and thus for the third as for both the antecedent species of the second genus, it has been proved that each designated operator prior to any arrangement being performed does not take away its licit character nor carry it out of the species to which it belongs, and on being repeated brings it back to its original form, and that the effect of any single operation is to maintain the content (or inertia) of the arrangement constant but to reverse each of its characters. This is the thing that was to be proved and brings my wearisome but indispensable task to an end.

(68) Another and perhaps somewhat clearer image of the classification of the numbers of the second Genus may be presented as follows: The combinations of the characters $XCOExcoe$ give rise to eight pairs of groups, say eight classes. Of these classes four belong to Species 2, and may be represented by four indefinite vertical parallelograms, set side to side, and subdivided each of them into four, (say) black, white, grey and tawny stripes, corresponding to the four varieties of the second species. The other four classes may be similarly represented by four such parallelograms as before, but separated by a transverse horizontal line into eight sub-classes, four corresponding to the first species and four to the second. The upper parallelograms may then be each divided into blue and green, the lower into yellow and red stripes to represent the respective couples of varieties of the first and third species. There will thus be in all thirty-two stripes, namely four blue, green, yellow and red, and four black, white, grey and tawny, each of which is bifid, representing two groups of opposite sexual characters, which may be fittingly represented by the upper and under sides of the sixteen unlimited single-coloured stripes of the first and the eight unlimited double-coloured stripes of the second set of parallelograms.

The above logical scheme is not intended to convey any notion of the relative frequency of the three species. The general case is that of the first species. The second is conditioned by $'\underset{.}{B} = b$ or $\underset{.}{B} = 0$, and the third by $\underset{.}{B} = 0$. When $'\underset{.}{B} = b$ it is about an even chance whether the arrangement is of the second or first species, and when $\underset{.}{B} = 0$ of the second or third. Either equality is a particularization of the B series, the latter signifying that there are no B's in the arrangement, the former that there are B's descending in rational progression down to b: this supposition is apparently infinitely more general than the former, because there is no limit to the number of terms in the progression, and the case of a natural progression of B's of the kind mentioned with any given number of terms as regards the probability of its occurring in an arrangement seems to be on a par with the case of the B's being all wanting. Hence the first species is infinitely more frequent than the second, and the second than the third. According to Prof. Max Müller's theory of the relation of thought to language (if I interpret it rightly) I ought to have thought out my divisions and schemes of operation in language, but I certainly had formed in my mind a dim abstract of them before I had found the language that was competent to give them expression.

In conclusion, I may remark that whilst the experience of the past indicated the probability that there did exist (if one could find it) a method of distributing the arrangements of the second genus into pairs, in such a way that in each pair the total or partial character should be reversed in passing from the one to the other, there was nothing to induce a reasonable degree of assurance that both those characters should be found simultaneously reversed in one and the same distribution; for aught that could have been foreseen to the contrary, it might very well have happened that one mode of distribution might have been needed to prove Jacobi's theorem for the case of only negative signs appearing in the factors on the left-hand side of the equation, and a different one for the other case where only every third factor contains such sign—indeed upon the principle of *divide et impera* or doing one thing at a time (as invaluable a maxim to the algebraist as to the politician) I had completed the proof for the former case without thinking of the latter, and only when on the point of attacking it was agreeably surprised to find that there was nothing left to be done, for that the proof found for the one extended to the other—in familiar phrase, I had hit two birds with one stone. We may now ask whether this was a happily found chance solution or was predestined by the nature of things, and that *simple* necessarily implies *double* enantiotropy of conjugation. Probably I think not, and if so, a question arises as to the number of solutions for each of the two sorts of enantiotropy and whether the number of each kind of simply-enantiotropic conjugations is the same.

Viewed merely as a question of direct multiplication, I think it must be allowed that what I have here called Jacobi's theorem (including Euler's marvellous one, as the ocean a drop of water) is the most surprising revelation that has been made in elementary algebra since the discovery of the general binomial theorem, and that the space devoted to its independent, and so to say, materialistic proof in these pages, although considerable, is not out of proportion to its intrinsic importance.

H. *Intuitional Exegesis of Generalized Farey Series**.

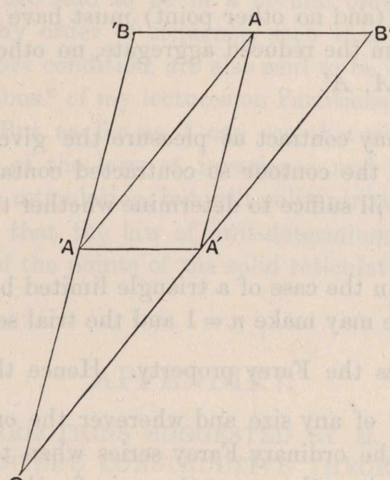
(69) The demands of the press will only admit of a rapid sketch of what appears to me to be the true underlying principles of the theory initiated by Farey, honoured by the notice of Cauchy, and to a certain extent generalized by Mr Glaisher, whose inductive method in the cases treated by him finds its full development in the method of continuous change of boundary, explained in the course of what follows. Let us start from the conception of an infinite cross-grating formed by two orthogonal systems of parallel lines in a plane, the distance between any two parallels being made equal to unity. The intersections of any two lines of the grating may, as heretofore, be termed nodes. A triangle which has nodes at its apices and at no other point on or within its periphery, may be termed an elementary triangle, and the double of the area of any such triangle will be unity. If any finite aggregate of nodes be given it must be possible to pick out a certain number of them which may be formed together by right lines so as to form a sort of ring-fence, within which all the rest are included: the area thus formed, if it

* Continued from note G, Interact, Part 2.

admits of being mapped out into elementary triangles, may be termed a *complete* nodal aggregate. Any other contour consisting of lines of any form (curved or straight) drawn outside of this ring-fence in such a manner that no nodes occur between the two, may be termed a regular contour.

If any node O be taken as origin and any nodal lines through O as axes of coordinates, and if $'A$, A' are the nearest nodes to O in the radial lines on which they lie, and if no nodes of the given aggregate are passed over as an indefinite line rotating round O, passes from one of these radial lines to the other, $'AOA$ is an elementary triangle, and if $'p, 'q$; p, q be the coordinates of $'A, A$ respectively, $'pq - p'q = \epsilon$ where ϵ is $+1$ or -1 but is fixed in sign when the direction of the rotation is given.

When the aggregate is *complete*, if the values of the coordinates of the successive points passed over by the rotating line be called $\ldots ''p, ''q$; $'p, 'q$; p, q ; p', q', p'', q'' ; ..., we shall have a Farey series formed by the successive couples p, q, that is $p''\,'q - p'q'' = \epsilon$; $p'q - pq' = \epsilon$; $pq' - p'q = \epsilon \ldots$. Thus we see that the Farey property is invariantive in the sense of being independent of the position of the origin.



Next I say, that if any contour to a given aggregate is regular, every contour similar thereto in respect to any node of the aggregate regarded as the centre of similitude is also regular, provided the boundary is simple; meaning that there are no interior limiting lines giving rise to holes or perforations in aggregate, and no loops formed by the boundary cutting itself.

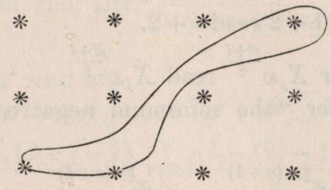
In the above figure $'BOB'$ is any triangle whose sides are bisected in $'A$, A, A'. Suppose O to be the origin, $'A$, A' two nodes of greatest proximity to O successively passed over by the rotating line for a given contour. As this contour expands uniformly in all directions through O, the line $'AA'$ remains parallel to itself. Since $'AOA'$ is an elementary triangle so also must the similar triangles $'AAA'$, $A'AB'$, $'AA'B$ be all elementary, consequently A will be the first new node intervening between $'A$, A' brought into the enlarged aggregate as $'AA'$ moves continuously parallel to itself, and $'AOA$, AOA' will be elementary triangles; it may be noticed in order to bring this method into relation with that indicated by Mr Glaisher, that the coordinates of this new node A are the sums of the coordinates of its neighbours $'A$, A'. If the contour were not supposed to be simple, this condition could not be drawn; for if there were a hole round the middle point of $'AA'$ the node A would be missing in the enlarged aggregate, and if the first node to intervene as the contour went on enlarging be called (A), $'AO(A)$ or $(A)OA'$ or each of them would be a multiple of the elementary triangle, so that the constancy of the value of the successive determinants would no longer hold. In like manner it will be seen that on the same supposition as above made, if in consequence of the contour contracting about O as the centre of similitude, two points $'A$, A' which originally are non-contiguous, at any moment become contiguous, at the moment previous to this taking place A (and no other point) must have intervened, and after A has disappeared from the reduced aggregate, no other point can make its appearance between $'A$, A'.

(70) Hence we may contract at pleasure the given contour about any node as origin, and if the contour so contracted contains at least one node besides the origin, it will suffice to determine whether the given contour is or is not regular.

Thus for example in the case of a triangle limited by the axes and by the right line $x+y=n$, we may make $n=1$ and the trial series will then become $\frac{0}{1}\ \frac{1}{1}\ \frac{1}{0}$ which possesses the Farey property. Hence this will hold good for a triangular boundary of any size and wherever the origin is situated: this includes the case of the ordinary Farey series when the origin is taken at either extremity of the hypothenuse. So again for the area contained within the axes and the hyperbola $xy=n$, we may take $xy=1$ and the trial series is the same as before.

(71) It is easy to form *unperforated* areas of any magnitude which shall not satisfy the Farey law: for example we may as in the annexed figure draw a curve passing through the origin, the point $(0, 1)$, and the point $(2, 3)$, $\frac{0}{1}$, $\frac{2}{3}$ does not satisfy the Farey law, and consequently no similar contour obtained by treating any one of the three nodes which it contains as a centre of similitude will be a "complete contour," and the successive values of (p, q)

obtained by the rotation of a line round the origin in such contour will not constitute a Farey series.



The theory will, I believe, admit of being extended to solid reticulations, formed by the intersections of three systems of equidistant parallel planes, determinants of the third order between the three coordinates of successive points, replacing the $pq' - p'q$ of the plane theory. The chief difference will consist in the introduction of a new element in the multiplicity of the "normal orders" in which a given set (of points in a plane or) of radii *in solido* may be taken. (Points in a plane arranged in any order of sequence, such that the successive determinants formed by their trilinear coordinates are of uniform sign, are said to be in a normal order. Rays of a conical pencil arranged in any order of sequence, such that their intersections by a plane satisfy the above condition, are also said to be in a normal order: see privately printed syllabus* of my lectures on Partitions, 1859, or M. Halphen's theory of *Aspects*.) But as far as I can see this will in no way militate against the existence of the laws of invariance and similitude established for the case of a plane reticulation, but will only introduce a further principle of invariance, namely that the law of unit-determinants if satisfied by one normal arrangement of the points of the solid reticulation will be satisfied by every other.

APPENDIX†.

LIST OF CORRECTIONS SUGGESTED BY M. JENKINS TO PROFESSOR SYLVESTER'S CONSTRUCTIVE THEORY OF PARTITIONS.

Page 5, 5 lines from end, $2n-(i+3)$ should be $n-(i+3)$.

„ 6, between 2nd and 3rd rows of sinister table insert 13 . 2 . 0.

„ „ „ 7th and 8th „ „ „ „ 11 . 2 . 2.

„ „ in 6th row of dexter table, for 8 . 4 . 3 (2) write 8 . 4 . 3 (1).

„ 11, line 8 from the end, interchange protraction and contraction so as to read "*contraction* could not now be applied to A' and B' nor *protraction* to C'."

„ 13, line 25. If $f(x) = (1-x)(1-x^3)(1-x^3)(1-x^7)(1-x^9)$, for the second x^3 read x^5.

[* Vol. II. of this Reprint, p. 119.]

[† These corrections have been included in those made in the text preceding.]

Page 13, line 29, for "latter" read "former."

„ 15, line 11 from end, for l^τ read l^λ.

„ 20, line 4, for $1+2$ read $i+2$.

„ „ line 5, for $1+2$ read $i+2$.

„ 22, line 11, for $X_j x^{\frac{i^2+i}{2}}$ read $X_j x^{\frac{j^2+j}{2}}$.

„ „ line 20, for "the minimum negative residue of $i-1$" read $i+1$.

„ 25, line 7, for $\dfrac{x^{\frac{1}{2}n(n+1)}}{1-x^n}$ read $\dfrac{x^{\frac{1}{2}r(r+1)}}{1-x^r}$.

„ „ line 4 from the end, for "to the 5th now" read "to the 5th row now."

„ 27, line 15, for 15, 7, 3 read 13, 11, 3.

„ „ line 19, for $(1+ax)(1-ax^3)(1-ax^j)\ldots$ read $(1+ax)(1+ax^3)\ldots(1+ax^{2j-1})$.

„ „ line 22, for $\dfrac{x}{1-x}\alpha$ read $\dfrac{x}{1-x^2}\alpha$.

„ „ line 30, for "angle whose *nodes* contain i nodes" read whose *sides*.

„ 28, line 5, for "with $j-i$ or fewer parts" read $j-1$.

„ „ line 12, for $1+\dfrac{1-x^{\omega+1}}{1-x^2}x^\omega+\dfrac{1-x^{\omega+1}\,.\,1-x^{\omega+3}}{1-x\,.\,1-x^4}x^{\omega+1}$ etc.

read $x^\omega+\dfrac{1-x^{\omega-1}}{1-x^2}x^{\omega+1}+\dfrac{1-x^{\omega-1}\,.\,1-x^{\omega-3}}{1-x^2\,.\,1-x^4}x^{\omega+4}+$ etc.

If in the expression in line 9, namely in

$$\frac{1-x^{2i-2j+2}\,.\,1-x^{2i-2j+4}\ldots 1-x^{2i-2}}{1-x^2\,.\,1-x^4\ldots 1-x^{2j-2}}\,x^{j^2-2j+2i},$$

we put $j=3$ we obtain

$$\frac{1-x^{2i-4}\,.\,1-x^{2i-2}}{1-x^2\,.\,1-x^4}\,.\,x^{9-6+2i}=\frac{1-x^{2i-2}\,.\,1-x^{2i-4}}{1-x^2\,.\,1-x^4}\,.\,x^{2i+3}$$

$$=\frac{1-x^{\omega-1}\,.\,1-x^{\omega-3}}{1-x^2\,.\,1-x^4}\,.\,x^{\omega+4},$$

since $\omega=2i-1$, and similarly for other terms when we put $j=2$ and $j=1$.

The correction which I offer seems to me to be right, and the expression in the paper to give a wrong result in the case when n happens to be equal to $\omega+2$; for then the number of parts being supposed to be exactly i, the first bend contains $2i-1$ or ω nodes, and there is then no way of placing the remaining 2 nodes so as to make the partition a conjugate partition—supposing I have not misunderstood the article.

Page 29, line 8, for 19, 7, 6, 6 read 10, 7, 6, 6.

„ „ figure, either insert a node at junction of 5th column and 7th row or remove a node from junction of 7th column and 5th row.

„ „ lines 7 and 8 from the bottom, if we remove a node from the figure no change is required in these two lines; but if we insert a node in the figure, then 11 11 11 7 3 3 should be 11 11 11 7 5 3 and 5 5 5 3 1 1 should be 5 5 5 3 2 1.

Page 31, line 15 from end, after $\dfrac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta}$ insert "or of $x^n a^j$."

„ 34, line 7, for a^j read a^θ.

„ „ line 8, for $(x^\theta + ax^{1\theta})\}$ read $(x^\theta + x^{2\theta})$.

„ 36, line 8, for $\dfrac{l_1(2-j-1)}{2}$ read $\dfrac{l_1-(2j-1)}{2}$.

„ 37, line 4, for x^n read $x^{\frac{n}{2}}$.

„ „ line 7, for x^{2i+1} read x^{2i+2}.

„ 40, line 6, $a_i - i$ is, I believe, the right final term; but it appears as if it were the first of a pair instead of the last of a pair, $a_i - i$ being a quantity which may vanish.

If the pair of expressions which in the text precede $a_i - i$, if definitely expressed and not left to be understood, should be

$$[a_{i-1} + \alpha_{i-1} - (2i-3)],\quad [a_{i-1} + \alpha_{i-2} - (2i-2)],$$

and not as in the text

$$[a_{i-1} + \alpha_{i-1} - (2i-1)],\quad [a_{i-1} + \alpha_i - 2i],$$

the factor which should precede $a_i - i$ is $[a_i + \alpha_i - (2i-1)]$.

I do not quite follow lines 9—13 of p. 40, possibly from the oversight in the subscripts I do not see what is intended. But it seems to me the following proof would be right:

The expressions of the same form succeeding $a_1 + \alpha_1 - 1$ and $a_1 + \alpha_2 - 2$ must be continued so long as they are positive, and must be rejected when they become negative.

Now from the fact of i being the content of the side of the square belonging to the transverse graph $a_i =$ or $> i$, $\alpha_i =$ or $> i$, therefore $a_i + \alpha_i - (2i-1)$ is positive and is therefore one of the terms of the series. Also $a_{i+1} =$ or $< i$ and $\alpha_{i+1} =$ or $< i$, therefore $a_{i+1} + \alpha_{i+1} - (2i+1)$ is negative and must consequently be rejected.

The intermediate expression is $a_i + \alpha_{i+1} - 2i$; and for this we may in all cases put $a_i - i$ as the last term of the series for the following reason:

If the extreme inside bend have more than one node in the row, then $\alpha_{i+1} = i$ and $a_i + \alpha_{i+1} - 2i$ is $= a_i - i$, which is not negative since $a_i =$ or $> i$. If the extreme inside bend degenerate, so that it consists only of a vertical line or of a single point, then $a_i = i$; and since $\alpha_{i+1} < i$ in this case, therefore $a_i + \alpha_{i+1} - 2i$ is negative and inadmissible as a term in the series; but since $a_i - i = 0$ there is no harm in putting it as the final term in the series.

Page 601, *Vol. III. of this Reprint*, line 6 from the end, for 3100 read 3110.