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ON THE EQUATION TO THE SECULAR INEQUALITIES IN THE PLANETARY THEORY.

[*Philosophical Magazine*, XVI. (1883), pp. 267—269.]

A VERY long time ago I gave, in this *Magazine**, a proof of the reality of the roots in the above equation, in which I employed a certain property of the square of a symmetrical matrix which was left without demonstration. I will now state a more general theorem concerning the *product* of *any* two matrices of which that theorem is a particular case. In what follows it is of course to be understood that the product of two matrices means the matrix corresponding to the *combination* of two *substitutions* which those matrices represent.

It will be convenient to introduce here a notion (which plays a conspicuous part in my new theory of multiple algebra), namely that of the *latent roots* of a matrix—latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf. If from each term in the diagonal of a given matrix, λ be subtracted, the determinant to the matrix so modified will be a rational integer function of λ; the roots of that function are the latent roots of the matrix; and there results the important theorem that the latent roots of any function of a matrix are respectively the same functions of the latent roots of the matrix itself: for example, the latent roots of the square of a matrix are the squares of its latent roots.

The latent roots of the product of two matrices, it may be added, are the same in whichever order the factors be taken. If, now, m and n be any two matrices, and $M=mn$ or nm, I am able to show that the sum of the products of the latent roots of M taken i together in every possible way is equal to the sum of the products obtained by multiplying every minor determinant of the ith order in one of the two matrices m, n by its *altruistic opposite* in the other: the reflected image of any such determinant, in respect to the principal diagonal of the matrix to which it belongs, is its *proper* opposite, and the corresponding determinant to this in the other matrix is its *altruistic opposite.*

[* Vol. I. of this Reprint, p. 378.]

The proof of this theorem will be given in my large forthcoming memoir on Multiple Algebra designed for the *American Journal of Mathematics.*

Suppose, now, that m and n are transverse to one another, that is, that the lines in the one are identical with the columns in the other, and *vice versâ,* then any determinant in m becomes identical with its altruistic opposite in n; and furthermore, if m be a symmetrical matrix, it is its own transverse. Consequently we have the theorem (the one referred to at the outset of this paper) that the sum of the i-ary products of the latent roots of the square of a symmetrical matrix (that is, of the squares of the roots of the matrix itself) is equal to the sum of the squares of ali the minor determinants of the order i in the matrix; whence it follows, from Descartes's theorem, that when all the terms of a symmetrical matrix are real, none of its latent roots can be *pure* imaginaries, and, as an easy inference, cannot be *any kind* of imaginaries; or, in other words, all the latent roots of a symmetrical matrix are real, which is Laplace's theorem.

I may take this opportunity of stating the important theorem that if $\lambda_1, \lambda_2, \ldots \lambda_i$ are the latent roots of any matrix m, then

$$\phi m = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_i)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_i)}\phi\lambda.$$

This theorem of course presupposes the rule first stated by Prof. Cayley (*Phil. Trans.* 1857) for the addition of matrices.

When any of the latent roots are equal, the formula must be replaced by another obtained from it by the usual method of infinitesimal variation. If $\phi m = m^{\frac{1}{\omega}}$, it gives the expression for the ωth root of the matrix; and we see that the number of such roots is ω^i, where i is the order of the matrix. When, however, the matrix is *unitary,* that is, all its terms except the diagonal ones are *zeros,* or *zeroidal,* that is, when all its terms are *zeros,* this conclusion is no longer applicable, and a certain definite number of arbitrary quantities enter into the general expressions for the roots.

The case of the extraction of any root of a unitary matrix of the second order was first considered and successfully treated by the late Mr Babbage; it reappears in M. Serret's *Cours d'Algèbre supérieure.* This problem is of course the same as that of finding a function $\frac{ax+b}{cx+d}$ of any given order of periodicity. My memoir will give the solution of the corresponding problem for a matrix of any order. Of the many unexpected results which I have obtained by my new method, not the least striking is the *rapprochement* which it establishes between the theory of Matrices and that of Invariants. The theory of invariance relative to associated Matrices includes and transcends that relative to algebraical functions.