17.

EQUATIONS IN MATRICES.

[*Johns Hopkins University Circulars*, III. (1884), p. 122.]

I HAVE been lately considering the subject of equations in matrices. Sir William Hamilton in his *Lectures on Quaternions* has treated the case of what I call unilateral equations of the form $x^2+px+q=0$, or $x^2+xp+q=0$, where we may, if we please, regard x, p, q as general matrices of the second order. He has found there are six solutions, which may be obtained by the solution of an ordinary cubic equation. In a paper now in print and which will probably appear in the May number of the *Philosophical Magazine*, I have discussed by my own methods the general *unilateral* equation, say

$$x^\omega+px^{\omega-1}+qx^{\omega-2}+\ldots+l=0,$$

where $x, p, q \ldots l$, are quaternions or matrices of the second order, and have shown, by a method satisfactory if not absolutely rigorous, that the number of solutions is $\omega^3-\omega^2+\omega$, that is to say, the nearest superior integer to the general maximum number of roots (ω^4) divided by the augmented degree $(\omega+1)$.

But after I had done this it occurred to me that there were multitudinous failing cases of which neither Hamilton nor myself had taken account, as for example $x^2+px=0$, besides the solutions $x=0$, $x=-p$, will admit of a solution containing an arbitrary constant, I think; but that is a matter which I shall have to look further into before committing myself to a positive assertion about it. I have only had time to pass in review the more elementary case of a unilateral simple equation, say $px=q$, where p, q are matrices of any order ω.

If p is non-vacuous there is one solution, namely, $x=p^{-1}q$; but suppose p is vacuous: what is the condition that the equation may be soluble?

(1) Suppose $q=0$, p being vacuous has for its identical equation $pP=0$, and consequently we may make $x=\lambda P$ where λ is an arbitrary constant.

(2) Suppose q is finite and that $x=r$ is one solution, then obviously the general solution is $x=r+\lambda P$.

We have now to inquire what is the condition that r may exist. I find from the mere fact of x being indeterminate (and confirm the result by another order of considerations) that the determinant of $q + \lambda p$ must vanish identically; so that for instance when p, q are of the second order and $\begin{matrix} b' & c \\ d\,e & f \end{matrix}$ are the *parameters to the corpus* (p, q), we must have when $d = 0$, which is implied in the vacuity of p, $f = 0$ and $e = 0$. The first of these conditions is known *à priori* immediately from my third law of motion; but not so, without introducing a slight intervening step, the intermediate one (I mean the connective to d and f, namely) $e = 0$.

So in general in order that $px + q = 0$ may be soluble, that is, in order that $p^{-1}q$ where p is simply vacuous may be *Actual* and not Ideal, q must satisfy as many conditions as there are units in the order of p or q, all implied in the fact that the determinant to $p + \lambda q$, where λ is an arbitrary constant, vanishes identically. When these conditions are satisfied $p^{-1}q$ becomes actual but indeterminate. (This, by the way, shows the disadvantage of calling a *vacuous* matrix *indeterminate,* as was done in the infancy of the theory by Cayley and Clifford—for we want this word as you see to signify a combination of the inverse of a vacuous matrix with another which takes the combination out of the ideal sphere and makes it actual.)

So in general in order that $p^{-1}q$ where p is a null of the ith order (that is where all the $(i+1)$th but not all the ith minors of p are zero) shall be an actual (although indeterminate) matrix, it is necessary and sufficient that $p + \lambda q$, where λ is arbitrary, shall be a null of the same (ith) order. What will be the degree of indeterminateness in $p^{-1}q$, that is, how many arbitrary constants are contained in the value of x which satisfies the equation $px = 0$ remains to be considered.

The law as to the conditions is an immediate *corollary* to my third law of motion, for if $px = q$ then $p + \lambda q = p(1 + \lambda x)$; consequently $p + \lambda q$, whatever λ may be, must have at least as high a degree of nullity as p. Q.E.D.