

16.

ON THE THREE LAWS OF MOTION IN THE WORLD OF UNIVERSAL ALGEBRA.

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In the preceding *Circular* allusion was made to the three cardinal principles or conspicuous landmarks in Universal Algebra; these may be called, it seems to me (without impropriety), its Laws of Motion, on the ground that as motion is operation in the world of pure space, so operation is motion in the world of pure order, and without claiming any exact analogy between these and Newton's laws, it will be seen that there is an element in each of the former which matches with a similar element in the latter, so that there is no difficulty in pairing off the two sets of laws and determining which in one set is to be regarded as related by affinity with which in the other. They may be termed the law of *concomitance* or *congruity*, the law of *consentaneity* and the law of *mutuality* or *community*.

The law of congruity is that which affirms that the latent roots of a matrix follow the march of any functional operation performed upon the matrix, not involving the action of any foreign matrix; it is the law which asserts that any function of a latent root to a matrix is a latent root to that same function of the matrix; in so far as it regards a matrix *per se*, or with reference solely to its environment, it obviously pairs off with Newton's first law.

The law of *consentaneity*, which is an immediate inference from the rule for combining or multiplying substitutions or matrices, is that which affirms that a given line (or parallel of latitude) can be followed out in the matrices resulting from the continued action of a matrix upon a fixed matrix of the same order, that is, in the series M, mM, m^2M, m^3M, \dots (which may be regarded as so many modified states of the original matrix) without reference to any other of the lines or parallels of latitude in the series, or again any column or parallel of longitude in the correlated series M, Mm, Mm^2, \dots without reference to any other such column or parallel of longitude.

An immediate consequence of this obvious fact (a direct consequence for the rule of multiplication) obtained by dealing at will with either of the systems of parallels referred to, is that a system of simultaneous linear equations in differences may be formed for finding each term in any given line or in any given column at any point in the series, and the integration of these equations leads at once to the conclusion that any term of given latitude and longitude in the i th term of either series is a syzygetic function of the i th powers of the latent roots of m .

If, then, M be made equal to multinomial unity, this at once shows that supposing ω to be the order of m , on substituting m for the *carrier* (or latent variable) in the latent function to m , and multiplying the last term by the proper multinomial unit, the matrix so formed is an absolute null, which proves the proposition concerning the "identical equation" first enunciated by Professor Cayley in his great paper on Matrices in the *Philosophical Transactions* for 1858.

This proposition admits of augmentation, (1), from within, as shown in a former note, by applying to it the limiting law of the nullity of a product (a branch of the 3rd law), which leads to the very important conclusion that the nullity of any factor of the function of a matrix which is an absolute null, or more generally of any product of powers of its linear factors, is exactly equal to the number of distinct linear factors which such factor or product contains, at all events, in the general case where the latent roots are all unequal; and (2), from without, by substituting for m , $m + \epsilon n$ where n is any second matrix whatever and ϵ is an infinitesimal. This leads to the *catena* of identities, to which allusion has been made in the preceding *Circular*. Then, again, the *endogenous* growth of the theorem (that which determines the exact nullity of any factor of the left-hand side of the identical equation) in its turn seems to lead to a remarkable theorem concerning the form of the general term of any power of m into M .

Observe that every such term is expressed as a syzygetic function of powers of the ω latent roots, and contains, therefore, ω constants, so that the total number of syzygetic multipliers is ω^3 ; but the number of variables in m and M together is $2\omega^2$; and, consequently, apart from the ω arbitrary latent roots the number of independent constants in $m^i M$ should be $2\omega^2 - \omega$. The ω^3 syzygetic multipliers ought then to contain only $\omega(2\omega - 1)$ arbitrary constants, and such will be found to be the case by virtue of the following hypothetical theorem: Calling λ any one of the latent roots, the multipliers of λ^i in $m^i M$ will form a square of ω^2 quantities; the theorem in question* is that every minor of the second order in such square is zero, so that the ω^2 terms in the square is given when the bounding angle containing

* I have not had leisure of mind, being much occupied in preparing for my departure, to reduce this theorem to apodictic certainty. I state it therefore with all due reserve.

$2\omega - 1$ terms is given; and the same being true for the multipliers of each latent root (which resolve themselves into ω squares) the number of arbitrary quantities in all is $\omega(2\omega - 1)$ as has to be shown.

The law of *consentaneity* in so far as it relates to the decomposition of the motion of a matrix into a set of parallel motions, has an evident affinity with Newton's second law*.

Remains the law of *mutuality*, which is concerned with the effect of the mutual action upon one another of two matrices, and so claims kindred with Newton's third law.

This law branches off into two, one of which may be termed the law of reversibility, the other that of co-occupancy or permeability.

The law of reversibility affirms that the latent function of the product of two matrices is independent of the sense in which either of them operates upon the other, that is, is the same for mn as for nm , just as the kinetic energy developed by the mutual action of two bodies is not affected by their being supposed to change places.

As regards the second branch of the third law, the word co-occupancy refers to the fact that although the space occupied by two similarly shaped figures (say two spheres) is not absolutely determined (in the absence of other data) by the spaces occupied by them each separately (for they may intersect or one of them coincide with or contain the other), a superior as well as an inferior limit to such joint occupation is so determined; the inferior limit being the space occupied by either such figure, that is, the *dominant* of these two given spaces, and the superior limit their arithmetical sum. So the nullity resulting from the action in either sense of two matrices upon one another is not given when their separate nullities are assigned, but has for an inferior limit the dominant of these two nullities and for a superior limit their sum; the nullities of the two component matrices may also be conceived under the figure of two gases or other fluids which are mutually *permeable* and capable of occupying each other's pores.

Although the limits spoken of are independent of the sense in which the two matrices act on one another, it must not however be supposed that the actual resultant nullity is unaffected by that circumstance; thus, for example, if the latent roots of a ternary matrix m are $\lambda, \lambda', \lambda''$, the nullity resulting from $(m - \lambda)(m - \lambda')$ acting sinistrally upon $(m - \lambda'')n$, that is, of $(m - \lambda)(m - \lambda')(m - \lambda'')n$ is 3, but from the same acting dextrally upon the same, that is, of $(m - \lambda'')n(m - \lambda)(m - \lambda')$, need not necessarily exceed 2.

* For another and closer bond of affinity between the two laws see concluding paragraph of this note.

Such then are the three primary Laws of Algebraical Motion; but as Conservation of areas, *Vis viva*, D'Alembert's Principle, the principle of Synchronous Vibrations, of Least action, and various other general laws may be deduced from Newton's three ground laws, so, of course, various subordinate but very general laws may be deduced from the interaction of the above stated three ground laws, namely, the law of Congruity, the law of Consentaneity, and the law of Mutuality.

The deduction of the catena of identical equations connecting two matrices m and n from the second and third laws combined, affords an instance of such derivative general laws. Another instance of the same is the theorem that when the product resulting from the action upon one another of two matrices, is the same in whichever of the two senses the action takes place, the matrices must be functionally related, unless one of them is a scalar, that is, a multiple of multinomial unity, at all events when neither m nor n possesses a pair of equal latent roots.

This very important and almost fundamental law (seemingly so simple and yet so hard to prove) may be obtained as an immediate inference from that identical equation in the catena of such equations connecting the matrices m and n , in which one of the two enters only singly at most in any term. As for example if m and n are of the 3rd order, beside the identical equation $m^3 - 3bm^2 + 3dm - g = 0$ we have* the identity

$$m^2n + mnm + nm^2 - 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0.$$

But if $nm = mn$ then $mnm = m^2n$, $nm^2 = mnm = m^2n$, so that this equation becomes

$$m^2n - 2bmn + dn = m^2c - 2em + h, \quad \text{or} \quad n = \frac{cm^2 - 2em + h}{m^2 - 2bm + d} \dagger,$$

unless $m^2 - 2bm + d$ is vacuous.

The first branch of the third law, namely, the law of *reversibility*, is an almost immediate inference from the rule for the multiplication of matrices, and becomes intuitively evident when the process of multiplication in each of the two senses between m and n is actually set out. The second branch, namely, the law of co-occupancy or permeability, as it is the most far-reaching so it is the most deep seated (the most *caché*) of all the primary laws of

[* See p. 126 above.]

† Whence it follows that n must be a function of m convertible into an integral polynomial form, unless the numerator and denominator of the fraction to which n is equated vanish simultaneously, which is what happens when m is scalar. If the numerator exactly contains the denominator n becomes a scalar. Seeing that a constant c is a specialized case of a function of a variable x although the converse is not true, we may say that whenever $nm = mn$, one at least of the two matrices m and n is a function of the other, and that each is a function of the other unless that other is a scalar. Compare Clifford's "Fragment on Matrices" in the posthumous edition of his collected works.

motion. I found my proof of it upon the fact that the value of any minor determinant, say of the i th order, in either product of m and n (two matrices of the order ω) may be expressed as the quantitative product of a certain couple of rectangular matrices (in Cauchy's sense of the term), of which one is formed by i columns and the other by i lines in the two given matrices respectively. Such rectangle as shown by Cauchy (and as may be intuitively demonstrated by the simplest of my umbral theorems on compound determinants) is the sum of the

$$\frac{\pi(\omega)}{\pi(\omega-i)\pi i}$$

complete determinants of the one rectangle multiplied respectively by the corresponding complete determinants of the other rectangle.

This shows at once the truth of the proposition in so far as relates to the lower limit, that is, that if $mn = p$, and m, n have the nullities ϵ, ζ , and p the nullity θ , then θ must be at least as great as ϵ and at least as great as ζ . As regards the superior limit the proof is also founded on the theorem in determinants already cited, and the form of it is as follows. If ϵ be any number r , it may be shown that ζ must be at least as great as $\theta - r$; hence giving r all values successively from 0 to $\zeta - 1$, it follows that $\epsilon + \zeta$ cannot be less than θ , that is, that θ cannot be greater than $\epsilon + \zeta$.

The proof of the first law, that of concomitance or congruity, I ought to have stated antecedently, is a deduction from the theory of resultants and the well-known fact that the determinant of a product of matrices is the product of their determinants. Thus each of the three laws of motion is deduced independently of the two others.

As another example of a derivative law of motion, I may quote the very notable one which results from the interaction of the first and second fundamental laws upon one another, and which gives the general expression for any function whatever of a matrix in the form of a rational polynomial function of the same and of its latent roots, to wit, the magnificent theorem that whatever the form of the functional symbol ϕ , and whether it be a single or many valued function, if $\lambda_1, \lambda_2, \dots, \lambda_\omega$ be the latent roots of m ,

$$\phi m = \sum \phi \lambda_1 \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_\omega)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_\omega)}.$$

As for example if $\phi m = m^{\frac{p}{q}}, m^{\frac{p}{q}}$ will have q^ω roots which are completely determined by the above formula.

The first law, as already stated, regards a single body or matrix, uninfluenced by the action of any external force. The second law regards the effect upon a single matrix, subject to external impulses, taking their rise in an external source; whilst the third law has regard to the mutual

action or joint effect of two bodies or matrices simultaneously operating upon one another.

Note. Making [in p. 149] $m^3 - 3bm^2 + 3dm - g = F(m)$, we found

$$(F'm)n = cm^2 - 2em + g.$$

When two of the latent roots of m are equal, it is easy to prove that $F'm$ is vacuous, and conversely, that when $F'm$ is vacuous, two of the latent roots of m are equal; but when $F'm$ is vacuous it is no longer permissible to drive it out of the equation, and accordingly the true statement of the theorem in question is that when m, n are two matrices of (any) the same order, such that $mn = nm$, n must *in general* be a function of m , but that this ceases to be true, when and only when m has two equal roots. The theorem requires further investigation in order to make out what happens when, or how it can happen that, two of the latent roots of one and only one of the two convertible matrices are equal; for supposing this to happen it would seem to lead to the conclusion that n may be a function of m , but m not a function of n ; which, however, is not quite so paradoxical as it looks, inasmuch as in ordinary algebra a constant may be regarded as a specialized function of a variable, whilst a variable in no sense can be regarded as a function of a constant. The following example of two matrices not functions of one another, but forming commutable products, has recently occurred to me in practice, and led to the discovery of the oversight I had committed in stating the theorem in question in too absolute terms.

$$0 \ \rho \ \rho^2 \quad 0 \ 1 \ 1$$

If $x = 1 \ 0 \ 1$, $y = \rho \ 0 \ \rho^2$ where $\rho^2 + \rho + 1 = 0$, it will be found that $xy = yx$,

$$\rho^2 \ \rho \ 0 \quad \rho \ \rho^2 \ 0$$

but that neither x nor y is a function of the other; this may easily be deduced from the fact that $x^2 - \rho^2 x - 2\rho = 0$, so that if y were any function of x , it would be reducible to the form of a linear function thereof, and consequently (on account of the zeros in the two matrices) y must be a multiple of x , which is absurd.

In like manner it will be found that $y^2 - \rho^2 y - 2\rho = 0$, and that consequently x cannot be a function of y .