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[ADDITION TO MR. WALTON'S PAPER "ON A THEOREM IN MAXIMA AND MINIMA."]

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In what follows I write x, y, z in place of Mr Walton's u, v, w: (so that if $i = \surd(-1)$, as usual, we have

$$f(x+iy) = P + iQ):$$

and I attend exclusively to the case where the second differential coefficients of P, Q do not vanish.

There are not on the surface $z = P$ any proper maxima or minima; but only level points, such as at the top of a pass: say there are not any summits or imits, but only cruxes; and moreover at any crux, the two crucial (or level) directions intersect at right angles. Every node of the curve $Q = 0$ is subjacent to a crux of the surface $z = P$; and moreover the two directions of the curve $Q = 0$ at the node are at right angles to each other; hence, considering the intersection of the surface $z = P$ by the cylinder $Q = 0$, the path $Q = 0$ on the surface has a node at the crux; or say there are at the crux two directions of the path; these cross at right angles, and are consequently separated the one from the other by the crucial directions; that is to say, there is one path ascending, and another path descending, each way from the crux. And the complete statement is; that the elevation of the path is then only a maximum or minimum when the path passes through a crux; and that at any crux there are two paths, one ascending, the other descending, each way from the crux.

The analytical demonstration is exceeding simple; we have

$$\left(\frac{dP}{dy} + i\frac{dQ}{dy}\right) = i\left(\frac{dP}{dx} + i\frac{dQ}{dx}\right);$$

that is,

$$\frac{dP}{dy} = -\frac{dQ}{dx}, \quad \frac{dQ}{dy} = \frac{dP}{dx},$$

and passing thence to the second differential coefficients, we may write

$$\frac{dP}{dx} = \frac{dQ}{dy} = L, \quad \frac{dP}{dy} = -\frac{dQ}{dx} = M,$$

$$\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dx^2} = \frac{d^2Q}{dy^2} = a,$$

$$\frac{d^2Q}{dx\,dy} = \frac{d^2P}{dx^2} = -\frac{d^2P}{dy^2} = b,$$

so that we have

$$\delta P = L\delta x + M\delta y, \qquad \delta Q = -M\delta x + L\delta y,$$

$$\delta^2 P = (b,\ a,\ -b \between \delta x,\ \delta y)^2, \quad \delta^2 Q = (-a,\ b,\ a \between \delta x,\ \delta y)^2.$$

Hence, for the maximum or minimum elevation of the path, we have $0 = \delta P$, where $\delta Q = 0$; that is, $0 = \dfrac{L^2 + M^2}{L}\delta x$, and therefore $L^2 + M^2 = 0$; that is, $L = 0$, $M = 0$; and at any such point $\delta z = 0$, that is, there is a crux of the surface $z = P$; and $\delta Q = 0$, that is, there is a node of the curve $Q = 0$. Moreover the crucial directions for the surface $z = P$ are given by the equation $(b,\ a,\ -b \between \delta x,\ \delta y)^2 = 0$, or these are at right angles to each other; and the nodal directions for the curve $Q = 0$ are given by $(-a,\ b,\ a \between \delta x,\ \delta y)^2 = 0$; or these are likewise at right angles to each other.