564.

ON A THEOREM IN ELIMINATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 5, 6.]

I FIND among my papers the following example of a theorem in elimination communicated to me by Prof. Sylvester. Writing

$$\begin{aligned}
\phi &= ax^3 + 3bx^2y + 3cxy^2 + dy^3, \\
\phi_1 &= bx^2 + 2cxy + dy^2, \\
\phi_2 &= cx + dy, \\
\phi_3 &= d; \\
f &= bx^3 + 3cx^2y + 3dxy^2 + ey^3, \\
f_1 &= cx^2 + 2dxy + ey^2, \\
f_2 &= dx + ey, \\
f_3 &= e,
\end{aligned}$$

then we have

$$\Delta_a \,.\, R(f, \phi) = \Delta f \,.\, R(\phi_1, f_1)^2 \, R(\phi_1, f_2)^2,$$

viz. $R(f, \phi)$ is the resultant of the functions (f, ϕ), and similarly $R(\phi_1, f_1)$, $R(\phi_1, f_2)$. Moreover, Δf is the discriminant of f; and $\Delta_a R(f, \phi)$ is the discriminant of $R(f, \phi)$ in regard to a. The equation thus is

$$\Delta_a \left[(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - b^2e - c^3 + 2bcd)^2\right]$$
$$= (b^2e^2 + 4bd^3 + 4c^3e - 3c^2d^2 - 6bcde)^3\,(d^3 - 2cde + be^2)^2;$$

or, what is the same thing, reversing the order of the letters (a, b, c, d, e), it is

$$\Delta_e \left[(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - b^2e - c^3 + 2bcd)\right]$$
$$= (a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd)^3\,(b^3 - 2abc + a^2d)^2,$$

viz. arranging in powers of e, the function is

$$\begin{array}{lll} & e^3 \,. & a^3 \\ +\,3e^2 & . & -\,a^2\,(4bd-3c^2) - 9\,(ac-b^2)^2 \\ +\,3e & . & a\,(4bd-3c^2)^2 + 18\,(ac-b^2)\,(ad^2-2bcd+c^3) \\ +\,1 & . & -\,(4bd-3c^2)^3 - 27\,(ad^2-2bcd+c^3)^2, \end{array}$$

which last coefficient is

$$= -\,d^2\,(27a^2d^2 + 54ac^3 + 64b^3d - 36b^2c^2 - 108abcd),$$

and the discriminant of this cubic function of e is

$$= (a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd)^3\,(b^3 - 2abc + a^2d)^2.$$

The occurrence of the factor

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd$$

is accounted for as the resultant in regard to e of the invariants I, J; we, in fact, have

$$\begin{aligned}(ac-b^2)\,I - aJ &= (ac-b^2)\,(-\,4bd+3c^2) - a\,(-\,ad^2 - c^3 + 2bcd) \\ &= a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd,\end{aligned}$$

and the identity itself may be proved without any particular difficulty.