566.

ON THE TRANSFORMATION OF THE EQUATION OF A SURFACE TO A SET OF CHIEF AXES.

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We have at any point P of a surface a set of chief axes (PX, PY, PZ), viz. these are, say the axis of Z in the direction of the normal, and those of X, Y in the directions of the tangents to the two curves of curvature respectively. It may be required to transform the equation of the surface to the axes in question; to show how to effect this, take (x, y, z) for the original (rectangular) coordinates of the point P, $x+\delta x$, $y+\delta y$, $z+\delta z$ for the like coordinates of any other point on the surface, so that $(\delta x, \delta y, \delta z)$ are the coordinates of the point referred to the origin P; the equation of the surface, writing down only the terms of the first and second orders in the coordinates δx, δy, δz, is

$$A\delta x + B\delta y + C\delta z + \tfrac{1}{2}(a, b, c, f, g, h)(\delta x, \delta y, \delta z)^2 + \&c. = 0,$$

where (A, B, C) are the first derived functions and (a, b, c, f, g, h) the second derived functions of U for the values (x, y, z) which belong to the given point P, if $U=0$ is the equation of the surface in terms of the original coordinates (x, y, z); we have X, Y, Z linear functions of $(\delta x, \delta y, \delta z)$; say

	δx	δy	δz
X	α_1	β_1	γ_1
Y	α_2	β_2	γ_2
Z	α	β	γ

that is, $X = \alpha_1\delta x + \beta_1\delta y + \gamma_1\delta z$, &c. and $\delta x = \alpha_1 X + \alpha_2 Y + \alpha Z$, &c. where the coefficients satisfy the ordinary relations in the case of transformation between two sets of rectangular axes; and the transformed equation is therefore

$$A(\alpha_1 X + \alpha_2 Y + \alpha Z) + B(\beta_1 X + \beta_2 Y + \beta Z) + C(\gamma_1 X + \gamma_2 Y + \gamma Z)$$
$$+(a, b, c, f, g, h)(\alpha_1 X + \alpha_2 Y + \alpha Z,\ \beta_1 X + \beta_2 Y + \beta Z,\ \gamma_1 X + \gamma_2 Y + \gamma Z)^2 = 0,$$

or, as this may be written,

$$\begin{aligned}
&X(A\alpha_1 + B\beta_1 + C\gamma_1) + Y(A\alpha_2 + B\beta_2 + C\gamma_2) + Z(A\alpha + B\beta + C\gamma)\\
&\quad + \tfrac{1}{2}X^2 \ (a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)^2\\
&\quad + \tfrac{1}{2}Y^2 \ (a, \ldots)(\alpha_2, \ \beta_2, \ \gamma_2)^2\\
&\quad + \ XY(a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)(\alpha_2, \ \beta_2, \ \gamma_2)\\
&\quad + \ XZ\,(a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)(\alpha, \ \beta, \ \gamma)\\
&\quad + \ YZ\,(a, \ldots)(\alpha_2, \ \beta_2, \ \gamma_2)(\alpha, \ \beta, \ \gamma)\\
&\quad + \tfrac{1}{2}Z^2 \ (a, \ldots)(\alpha, \ \beta, \ \gamma)^2 \qquad + \&\text{c.} = 0,
\end{aligned}$$

where the &c. refers to terms of the form $(X, \ Y, \ Z)^3$ and higher powers.

But in order that the new axes may be chief axes, we must have

$$A\alpha_1 + B\beta_1 + C\gamma_1 = 0,$$

$$A\alpha_2 + B\beta_2 + C\gamma_2 = 0,$$

$$(a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)(\alpha_2, \ \beta_2, \ \gamma_2) = 0,$$

so that putting for shortness

$$A\alpha + B\beta + C\gamma = \nabla,$$

the equation becomes

$$\begin{aligned}
&\nabla Z + \tfrac{1}{2}X^2 \ (a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)^2 + \tfrac{1}{2}Y^2(a, \ldots)(\alpha_2, \ \beta_2, \ \gamma_2)^2\\
&\quad + \ XZ\,(a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)(\alpha, \ \beta, \ \gamma)\\
&\quad + \ YZ\,(a, \ldots)(\alpha_2, \ \beta_2, \ \gamma_2)(\alpha, \ \beta, \ \gamma)\\
&\quad + \tfrac{1}{2}Z^2 \ (a, \ldots)(\alpha, \ \beta, \ \gamma)^2 \qquad + \&\text{c.} = 0.
\end{aligned}$$

We have

$$A : B : C = \beta_1\gamma_2 - \beta_2\gamma_1 : \gamma_1\alpha_2 - \gamma_2\alpha_1 : \alpha_1\beta_2 - \alpha_2\beta_1,$$

that is,

$$= \alpha \quad : \quad \beta \quad : \quad \gamma \quad ,$$

and thence

$$\alpha, \ \beta, \ \gamma = \frac{A}{\nabla}, \ \frac{B}{\nabla}, \ \frac{C}{\nabla}; \ \nabla = \sqrt{(A^2 + B^2 + C^2)}.$$

I write

$$\frac{1}{\rho_1} = (a, \ldots)(\alpha_1, \ \beta_1, \ \gamma_1)^2,$$

and also for a moment

$$P = \left(a - \frac{1}{\rho_1}, \quad h \quad, \quad g \quad\right)(\alpha_1, \ \beta_1, \ \gamma_1),$$

$$Q = \left(\quad h \quad, \ b - \frac{1}{\rho_1}, \quad f \quad\right)(\alpha_1, \ \beta_1, \ \gamma_1),$$

$$R = \left(\quad g \quad, \quad f \quad, \ c - \frac{1}{\rho_1}\right)(\alpha_1, \ \beta_1, \ \gamma_1).$$

We find

$$P\alpha_1 + Q\beta_1 + R\gamma_1 = (a, \ldots)(\alpha_1, \beta_1, \gamma_1)^2 - \frac{1}{\rho_1}, = 0,$$

$$P\alpha_2 + Q\beta_2 + R\gamma_2 = (a, \ldots)(\alpha_1, \beta_1, \gamma_1)(\alpha_2, \beta_2, \gamma_2) - \frac{1}{\rho_1}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2), = 0,$$

and thence

$$P : Q : R = \beta_1\gamma_2 - \beta_2\gamma_1 : \gamma_1\alpha_2 - \gamma_2\alpha_1 : \alpha_1\beta_2 - \alpha_2\beta_1$$
$$= \alpha : \beta : \gamma ,$$

or say

$$P, Q, R = \theta_1 A, \theta_1 B, \theta_1 C;$$

we have thus the equations

$$\left(a - \frac{1}{\rho_1}, h, g\right)(\alpha_1, \beta_1, \gamma_1) = \theta_1 A,$$

$$\left(h, b - \frac{1}{\rho_1}, f\right)(\alpha_1, \beta_1, \gamma_1) = \theta_1 B,$$

$$\left(g, f, c - \frac{1}{\rho_1}\right)(\alpha_1, \beta_1, \gamma_1) = \theta_1 C,$$

and joining hereto

$$(A, B, C)(\alpha_1, \beta_1, \gamma_1) = 0,$$

we eliminate α_1, β_1, γ_1 and obtain the equation

$$\begin{vmatrix} a - \frac{1}{\rho_1}, & h, & g, & A \\ h, & b - \frac{1}{\rho_1}, & f, & B \\ g, & f, & c - \frac{1}{\rho_1}, & C \\ A, & B, & C, & 0 \end{vmatrix} = 0,$$

and in like manner writing

$$\frac{1}{\rho_2} = (a, \ldots)(\alpha_2, \beta_2, \gamma_2)^2,$$

we have the same equation for ρ_2; wherefore ρ_1, ρ_2 are the roots of the quadric equation

$$\begin{vmatrix} a - \frac{1}{\rho}, & h, & g, & A \\ h, & b - \frac{1}{\rho}, & f, & B \\ g, & f, & c - \frac{1}{\rho}, & C \\ A, & B, & C, & 0 \end{vmatrix} = 0.$$

Moreover, ρ_1, ρ_2 being thus determined, we have, α_1, β_1, γ_1, θ_1 proportional to the determinants formed with the matrix

$$\begin{vmatrix} a-\frac{1}{\rho_1}, & h\ , & g\ , & A \\ h\ , & b-\frac{1}{\rho_1}, & f\ , & B \\ g\ , & f\ , & c-\frac{1}{\rho_1}, & C \end{vmatrix},$$

say, $\alpha_1, \beta_1, \gamma_1, \theta_1 = k\mathfrak{A}_1, k\mathfrak{B}_1, k\mathfrak{C}_1, k\Omega$, where $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{C}_1$, Ω_1 are the determinants in question; and then $1 = k^2(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)$, or we have

$$\theta_1 = \frac{\Omega_1}{\sqrt{(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)}}.$$

But we find at once

$$(a, \ldots)(\alpha_1, \beta_1, \gamma_1)(\alpha, \beta, \gamma) = \theta_1(A\alpha + B\beta + C\gamma) = \theta_1 \nabla,$$

that is,

$$(a, \ldots)(\alpha_1, \beta_1, \gamma_1)(\alpha, \beta, \gamma) = \frac{\nabla\Omega_1}{\sqrt{(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)}},$$

and in the same manner

$$(a, \ldots)(\alpha_2, \beta_2, \gamma_2)(\alpha, \beta, \gamma) = \frac{\nabla\Omega_2}{\sqrt{(\mathfrak{A}_2^2 + \mathfrak{B}_2^2 + \mathfrak{C}_2^2)}}.$$

Hence the transformed equation is

$$\begin{aligned} &\nabla Z + \tfrac{1}{2}\frac{X^2}{\rho_1} + \tfrac{1}{2}\frac{Y^2}{\rho_2} \\ &\quad + XZ\frac{\nabla\Omega_1}{\sqrt{(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)}} + YZ\frac{\nabla\Omega_2}{\sqrt{(\mathfrak{A}_2^2 + \mathfrak{B}_2^2 + \mathfrak{C}_2^2)}} \\ &\quad + \tfrac{1}{2}Z^2\frac{(a, \ldots)(A, B, C)^2}{\nabla^2} + \&c. = 0, \end{aligned}$$

where it will be recollected that $\nabla = \sqrt{(A^2 + B^2 + C^2)}$. The &c. refers as before to the terms $(X, Y, Z)^3$ and higher powers, which are obtained from the corresponding terms in δx, δy, δz, by substituting for these their values $\delta x = \alpha_1 X + \alpha_2 Y + \alpha Z$, &c., where the coefficients have the values above obtained for them. It will be observed, that the radii of curvature are $\nabla\rho_1$, $\nabla\rho_2$, and that the process includes an investigation of the values of these radii of curvature similar to the ordinary one; the novelty is in the terms in XZ, YZ, and Z^2. But regarding X, Y as small quantities of the first order, Z is of the second order, and the terms in XZ, YZ are of the third order, and that in Z^2 of the fourth order.