568.

NOTE ON THE INTEGRALS $\int_0^\infty \cos x^2\, dx$ AND $\int_0^\infty \sin x^2\, dx$.

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MR WALTON has raised, in relation to these integrals, a question which it is very interesting to discuss. Taking for greater convenience the limits to be $-\infty$, $+\infty$, and writing

$$2u = \int_{-\infty}^{\infty} \cos x^2 dx,\ 2v = \int_{-\infty}^{\infty} \sin x^2 dx,$$

then we have

$$4(u^2 - v^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(x^2 + y^2)\, dx\, dy,$$

$$8uv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin(x^2 + y^2)\, dx\, dy,$$

and writing herein $x = r\cos\theta$, $y = r\sin\theta$, and therefore $dxdy = rdr\, d\theta$, it would thence appear that we have

$$4(u^2 - v^2) = \int_0^\infty \int_0^{2\pi} \cos r^2 . rdr\, d\theta = 2\pi \int_0^\infty \cos r^2 . rdr,$$

$$8uv = \int_0^\infty \int_0^{2\pi} \sin r^2 . rdr\, d\theta = 2\pi \int_0^\infty \sin r^2 . rdr,$$

or, finally

$$4(u^2 - v^2) = \pi \sin\infty,$$

$$8uv = \pi(1 - \cos\infty);$$

that is, either the integrals have their received values $\left\{\text{each } = \frac{\sqrt{(\pi)}}{2\sqrt{(2)}}\right\}$, and then $\sin\infty = 0$, $\cos\infty = 0$; or else the integrals, instead of having their received values, are indeterminate.

The error is in the assumption as to the limits of r, θ; viz. in the original expressions for $4(u^2-v^2)$, $8uv$, we integrate over the area of an indefinitely large square (or rectangle); and the assumption is that we are at liberty, instead of this, to integrate over the area of an indefinitely large circle.

Consider in general in the plane of xy, a closed curve, surrounding the origin, depending on a parameter k, and such that each radius vector continually increases and becomes indefinitely large as k increases and becomes indefinitely large: the curve in question may be referred to as the bounding curve; and the area inside or outside this curve as the inside or outside area. And consider further an integral $\iint z\,dx\,dy$, where z is a given function of x, y, and the integration extends over the inside area. The function z may be such that, for a given form of the bounding curve, the integral, as k becomes indefinitely large, continually approaches to a determinate limiting value (this of course implies that z is indefinitely small for points at an indefinitely large distance from the origin); and we may then say that the integral taken over the infinite inside area has this determinate value; but it is by no means true that the value is independent of the form of the bounding curve; or even that, being determinate for one form of this curve, it is determinate for another form of the curve.

I remark, however, that if z is always of the same sign (say always positive) then the value, assumed to be determinate for a certain form of the bounding curve, is independent of the form of this curve and remains therefore unaltered when we pass to a different form of bounding curve. To fix the ideas, let the first form of bounding curve be a square ($x=\pm k$, $y=\pm k$), and the second form a circle ($x^2+y^2=k^2$). Imagine a square inside a circle which is itself inside another square; then z being always positive, the integral taken over the area of the circle is less than the integral over the area of the larger square, greater than the integral over the area of the smaller square. Let the sides of the two squares continually increase, then for each square the integral has ultimately its limiting value; that is, for the area included between the two squares the value is ultimately $=0$, and consequently for the circle the integral has ultimately the same value that it has for the square. When z is not always of the same sign the proof is inapplicable; and although, for certain forms of z, it may happen that the value of the integral is independent of the form of the bounding curve, this is not in general the case.

We have thus a justification of the well known process for obtaining the value of the integral $\int_0^\infty e^{-x^2}\,dx$, viz. calling this u, or writing

$$2u=\int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

then

$$4u^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy=\int_0^\infty\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$=2\pi\,.\,\tfrac{1}{2},\quad\text{or}\quad u=\tfrac{1}{2}\sqrt{(\pi)},$$

but in consequence of the alternately positive and negative values of $\cos x^2$ and $\sin x^2$, we cannot infer that the like process is applicable to the integrals of these functions.

To show that it is in fact inapplicable, it will be sufficient to prove that the integrals in question have determinate values; for this being so, the double integrals $\iint \cos(x^2+y^2)\, dx\, dy$ and $\iint \sin(x^2+y^2)\, dx\, dy$, taken over an infinite square (or, if we please, over a rectangle the sides of which are both infinite, the ratio having any value whatever), will have determinate values; whereas, by what precedes, the values taken over an infinite circle are indeterminate. The thing may be seen in a very general sort of way thus: consider the surface $z = \sin(x^2+y^2)$, and let the plane of xy be divided into zones by the concentric circles, radii $\sqrt{(\pi)}$, $\sqrt{(2\pi)}$, $\sqrt{(3\pi)}$, &c...., then in the several zones z is alternately positive and negative, the maximum (positive or negative) value being ± 1; and though the breadths of the successive zones decrease, the areas and values of the integral remain constant for the successive zones; the integral over the circle radius $\sqrt{(n\pi)}$ is thus given as a neutral series having no determinate sum. But if the plane xy is divided in like manner into squares by the lines $x = \pm\sqrt{(n\pi)}$, $y = \pm\sqrt{(n\pi)}$, then in each of the bands included between successive squares, z has a succession of positive and negative values; the breadths continually diminish, and although the areas remain constant, yet, on account of the succession of the positive and negative values of z, there is a continual diminution in the values of the integral for the successive bands respectively, and the value of the integral for the whole square is given as a series which may very well be, and which I assume is in fact, convergent. Observe that I have not above employed this mode of integration (but by considering the single integral have in effect divided the square into indefinitely thin slices, and considered each slice separately); it would be interesting to carry out the analytical division of the square into bands, and show that we actually obtain a convergent series; but I do not pursue this inquiry.

Consider the integral

$$v = \int_0^\infty \sin x^2\, dx,$$

and taking for a moment the superior limit to be $(n+1)\pi$, then the quantity under the integral sign is positive from $x^2 = 0$ to $x^2 = \pi$, negative from $x^2 = \pi$ to $x^2 = 2\pi$, and so on; we may therefore write

$$\int_0^{(n+1)\pi} \sin x^2\, dx = A_0 - A_1 + A_2 \ldots + (-)^n A_n,$$

where

$$A_r, = (-)^r \int_{r\pi}^{(r+1)\pi} \sin x^2\, dx,$$

is positive. Writing herein $x^2 = r\pi + u$, we have

$$A_r = \tfrac{1}{2}\int_0^\pi \frac{\sin u du}{\sqrt{(r\pi + u)}},$$

which, for r large, may be taken to be

$$=\tfrac{1}{2}\int_0^\pi \frac{\sin u du}{\sqrt{(r\pi)}},\ =\frac{1}{\sqrt{(r\pi)}},$$

viz. r being large, we have A_r differing from the above value $\frac{1}{\sqrt{(r\pi)}}$ by a quantity of the order $\frac{1}{r^{\frac{3}{2}}}$.

It is obviously immaterial whether we integrate from $x^2 = 0$ to $(n+1)\pi$ or to $(n+1)\pi+\epsilon$, where ϵ has any value less than π; for by so doing, we alter the value of the integral by a quantity less than A_{n+1}, and which consequently vanishes when n is indefinitely large. And similarly, it is immaterial whether we stop at an odd or an even value of n.

We have therefore

$$v=\int_0^{(n+1)\pi} \sin x^2\, dx = A_0 - A_1 + A_2 \ldots + (-)^n A_n,$$

or, taking n to be odd, this is

$$= A_0 - A_1 + A_2 \ldots - A_n,$$

or, say it is

$$= (A_0 - A_1) + (A_2 - A_3) \ldots + (A_{n-1} - A_n),$$

viz. n here denotes an indefinitely large odd integer.

If instead of $A_0 - A_1 + A_2 - A_3 +$ &c., the signs had been all positive, then the term A being ultimately as $\frac{1}{\sqrt{(r)}}$, the series would have been divergent, and would have had no definite sum: but with the actual alternate signs, the series is convergent, and the sum has a determinate value. To show this more distinctly, observe that we have

$$A_{r-1} - A_r = (-)^{r-1}\,.\,\tfrac{1}{2}\int_{-\pi}^{\pi} \frac{\sin(r\pi+u)\,du}{\sqrt{(r\pi+u)}},\ = -\tfrac{1}{2}\int_{-\pi}^{\pi} \frac{\sin u dy}{\sqrt{(r\pi+u)}},$$

or, taking the integral from $-\pi$ to 0 and from 0 to π, and in the first integral writing $-u$ in place of u, then

$$A_{r-1} - A_r = \tfrac{1}{2}\int_0^\pi \sin u du \left\{\frac{1}{\sqrt{(r\pi-u)}} - \frac{1}{\sqrt{(r\pi+u)}}\right\},$$

where, r being large, expanding the term in $\{\ \}$ in ascending powers of u, then $A_{r-1} - A_r$ is of the order $\frac{1}{r^{\frac{3}{2}}}$: and the series $(A_0 - A_1) + (A_2 - A_3) \ldots + (A_{n-1} - A_n)$ is therefore convergent, and the sum as n is increased approaches a definite limit. Hence the integral v has a definite value: and similarly, the integral u has a definite value.

The values of u, v being shown to be determinate, I see no ground for doubting that these are the values of the more general integrals

$$\int_0^\infty e^{-ax^2}\cos x^2\,dx,\quad \int_0^\infty e^{-ax^2}\sin x^2\,dx,$$

(a real and positive) when a is supposed to continually diminish and ultimately become $=0$. We have, in fact, (a as above)

$$\int_0^\infty e^{(-a+bi)y}\,y^{n-1}\,dy = \frac{\Gamma(n)\,e^{in\theta}}{(a^2+b^2)^{\frac{1}{2}n}},^*$$

where $\theta = \tan^{-1}\dfrac{b}{a}$, an angle included between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$. Writing herein $n=\frac{1}{2}$, $b=1$, $y=x^2$, then

$$\int_0^\infty e^{(-a+i)x^2}\,dx = \frac{\sqrt{(\pi)}\,e^{\frac{1}{2}i\theta}}{2\,(a^2+1)^{\frac{1}{4}}},$$

where $\theta=\tan^{-1}\dfrac{1}{a}$, an angle included between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$; or, putting herein $a=0$, we have $\theta=\frac{1}{2}\pi$, and therefore

$$\int_0^\infty e^{ix^2}\,dx = \tfrac{1}{2}\sqrt{(\pi)}\,e^{\frac{1}{4}i\pi};$$

that is, equating the real and imaginary parts,

$$u = v = \frac{\sqrt{\pi}}{2\sqrt{2}},$$

which are the received values of the integrals

$$u=\int_0^\infty \cos x^2\,dx,\quad v=\int_0^\infty \sin x^2\,dx.$$

An important instance of the general theory presents itself in the theory of elliptic functions, viz. the integral

$$\iint \frac{dx\,dy}{(\Omega x + \Upsilon y)^2},$$

the ratio $\Omega : \Upsilon$ being imaginary, will, if the bounding curve be symmetrical in regard to the two axes respectively, have a determinate value *dependent on the form of the bounding curve;* if for instance this is a rectangle $x=\pm ak$, $y=\pm bk$, then the value of the integral will depend on the ratio $a : b$ of the infinite sides; and so if the bounding curve be an infinite ellipse, the value of the integral will depend on the ratio and position of the axes. See as to this my papers "On the inverse elliptic

* For brevity I take the integral under this form, but the real and imaginary parts might have been considered separately; and there would have been some advantage in following that course. The like remark applies to a subsequent investigation.

functions," *Camb. Math. Jour.*, t. IV. (1845), pp. 257—277, [24]; and "Mémoire sur les fonctions doublement périodiques," *Liouv.* t. X. (1845), pp. 385—420, [25].

A like theory applies to series, viz. as remarked by Cauchy, although the series $A_0 + A_1 + A_2 + \ldots$ and $B_0 + B_1 + B_2 + \&c.$ are respectively convergent, then arranging the product in the form

$$\begin{aligned} &A_0B_0 + A_0B_1 + A_0B_2 + \ldots \\ +\ &A_1B_0 + A_1B_1 + A_1B_2 + \ldots \\ +\ &A_2B_0 + A_2B_1 + A_2B_2 + \ldots \\ +\ &\ldots, \end{aligned}$$

say the general term is $C_{m,n}$, then if we sum this double series according to an assumed relation between the suffixes m, n (if, for instance, we include all those terms for which $m^2 + n^2 \overline{\lessgtr} k^2$, making k to increase continually) it by no means follows that we approach a limit which is equal to the product of the sums of the original two series, or even that we approach a determinate limit.

Mr Walton, agreeing with the rest of the foregoing Note, wrote that he was unable to satisfy himself that the value of $\int_0^\infty e^{ix^2} dx$ is correctly deduced from that of $\int_0^\infty e^{(-a+bi)y} y^{n-1} dy$. Writing $n = \frac{1}{2}$, the question in fact is whether the formula

$$\int_0^\infty e^{(-a+bi)y} y^{-\frac{1}{2}} dy = \frac{\sqrt{(\pi)}\, e^{\frac{1}{2}i\theta}}{(a^2+b^2)^{\frac{1}{4}}} \left(\theta = \tan^{-1}\frac{b}{a},\ \text{angle between } \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi\right),$$

which is true when a is an indefinitely small positive quantity, is true when $a = 0$; that is, taking b positive, whether we have

$$\int_0^\infty e^{iby} y^{-\frac{1}{2}} dy = \frac{\sqrt{(\pi)}\, e^{\frac{1}{4}i\pi}}{\sqrt{(b)}}.$$

Write in general

$$u = \int_0^\infty e^{(-a+bi)y} y^{-\frac{1}{2}} dy,$$

then, differentiating with respect to b, we have

$$\frac{du}{db} = \int_0^\infty iy^{\frac{1}{2}} e^{(-a+bi)y} dy,$$

or, integrating by parts,

$$\frac{du}{db} = \frac{i}{-a+bi} y^{\frac{1}{2}} e^{(-a+bi)y} - \frac{i}{2(-a+bi)} \int_0^\infty y^{-\frac{1}{2}} e^{(-a+bi)y} dy,$$

where the first term is to be taken between the limits ∞, 0; viz. this is

$$\frac{du}{db}=\left[\frac{i}{-a+bi}\,y^{\frac{1}{2}}e^{(-a+bi)y}\right]_0^\infty-\frac{i}{2(-a+bi)}\,u.$$

When a is not $=0$, the first term vanishes at each limit, and we have

$$\frac{du}{db}=\frac{-i}{2(-a+bi)}\,u.$$

The doubt was in effect whether this last equation holds good for the limiting value $a=0$. When a is $=0$, then in the original equation for $\frac{du}{db}$ the first term is indeterminate, and if the equation were true, it would follow that $\frac{du}{db}$ was indeterminate; the original equation for $\frac{du}{db}$ *is not true*, but we *truly* have

$$\frac{du}{db}=-\frac{1}{2b}\,u,$$

the same result as would be obtained from the general equation, rejecting the first term and writing $a=0$.

To explain this observe that for $a=0$, we have

$$u=\int_0^\infty y^{-\frac{1}{2}}\,e^{iby}\,dy,$$

which for a moment I write

$$u=\int_0^k y^{-\frac{1}{2}}\,e^{iby}\,dy,$$

where, as before, b is taken to be positive. Writing herein $by=x$, we have

$$u=\frac{1}{\sqrt{(b)}}\int_0^{bk} x^{-\frac{1}{2}}\,e^{ix}\,dx,$$

and assuming only that the integral $\int_0^M x^{-\frac{1}{2}}\,e^{ix}\,dx$ has a *determinate* limit as M becomes indefinitely large*, then supposing that k is indefinitely large, the integral in the last-mentioned expression for u has the value in question

$$\left(=\int_0^\infty x^{-\frac{1}{2}}\,e^{ix}\,dx\right),$$

* This is in fact the theorem $\int_0^\infty e^{ix^2}\,dx=$ a determinate value $\{=\frac{1}{2}\sqrt{(\pi)}\,e^{\frac{1}{4}i\pi}\}$, proved in the former part of the present Note.

which is independent of b, say this is

$$u = \frac{C}{\sqrt{(b)}},$$

and thence differentiating in regard to b, we find

$$\frac{du}{db} = -\frac{1}{2b}\,u,$$

the theorem in question.

But the value of $\frac{du}{db}$ *cannot be obtained by differentiating under the integral sign*, for this would give

$$\frac{du}{db} = \int_0^\infty iy^{\frac{1}{2}}\,e^{iby}\,dy,$$

and this integral is certainly indeterminate.