569.

ON THE CYCLIDE*.

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THE Cyclide, according to the original definition, is the envelope of a variable sphere which touches three given spheres, or, more accurately, the envelope of a variable sphere belonging to one of the four series of spheres which touch three given spheres. In fact, the spheres which touch three given spheres form four series, the spheres of each series having their centres on a conic; viz. if we consider the plane through the centres of the given spheres, and in this plane the eight circles which touch the sections of the given spheres, the centres of these circles form four pairs of points, or joining the points of the same pair, we have four chords which are the transverse axes of the four conics in question.

It thus appears, that one condition imposed on the variable sphere is, that its centre shall be in a plane; and a second condition, that the centre shall be on a conic in this plane; so that the original definition may be replaced first by the following one, viz.:

The cyclide is the envelope of a variable sphere having its centre on a given plane, and touching two given spheres.

Starting herefrom, it follows that the locus of the centre will be a conic in the given plane: the transverse axis of the conic being the projection on the given plane of the line joining the centres of the given spheres; and it, moreover, follows, that if in the perpendicular plane through the transverse axis we construct a conic having for vertices the foci, and for foci the vertices, of the locus-conic, then the conic so constructed will pass through the centres of the given spheres.

* I use the term in its original sense, and not in the extended sense given to it by Darboux, and employed by Casey in his recent memoir "On Cyclides and Spheroquartics," *Phil. Trans.* 1871, pp. 582—721. With these authors the Cyclide here spoken of is a Dupin's or tetranodal Cyclide.

Two conics related in the manner just mentioned are the flat-surfaces of a system of confocal quadric surfaces; they may for convenience be termed anti-conics (fig. 1); one of them is always an ellipse and the other a hyperbola; and the property of them is that, taking any two fixed points on the two branches, or on the same branch of the

Fig. 1.

hyperbola, and considering their distances from a variable point of the ellipse: in the first case the sum, in the second case the difference, of these two distances is constant. And similarly taking any two fixed points on the ellipse, and considering their distances from a variable point of the hyperbola, then the difference, first distance *less* second distance is a constant, $+\alpha$ for one branch, $-\alpha$ for the other branch of the hyperbola.

And we thus arrive at a third, and simplified definition of the cyclide, viz. considering any two anti-conics, the cyclide is the envelope of a variable sphere having its centre on the first anti-conic, and touching a given sphere whose centre is on the second anti-conic.

And it is to be added, that the same cyclide will be the envelope of a variable sphere having its centre on the second anti-conic and touching a given sphere whose centre is on the first anti-conic, such given sphere being in fact any particular sphere of the first series of variable spheres. And, moreover, the section of the surface by the plane of either of the anti-conics is a pair of circles, the surface being thus (as will further appear) of the fourth order.

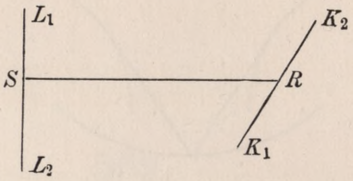
In the series of variable spheres the intersection of any two consecutive spheres is a circle, the centre of which is in the plane of the locus-anti-conic, and its plane perpendicular to that of the locus-anti-conic, this variable circle having for its diameter in the plane of the locus-anti-conic a line terminated by the two fixed circles in that plane. The cyclide is thus in two different ways the locus of a variable circle; and investigating this mode of generation, we arrive at a fourth definition as follows:—

Consider in a plane any two circles, and through either of the centres of symmetry draw a secant cutting the two circles; in the perpendicular plane through the secant, draw circles having for their diameters the chords formed by the two pairs of anti-parallel points on the secant (viz. each pair consists of two points, one on each circle, such that the tangents at the two points are not parallel to each other): the locus of the two variable circles is the cyclide.

Before going further it will be convenient to establish the definition of "skew anti-points": viz. if we have the points K_1, K_2 (fig. 2), mid-point R, and L_1, L_2, mid-point S, such that K_1K_2, RS and L_1L_2 are respectively at right angles to each other, and

$\overline{K_1R}^2+\overline{RS}^2+\overline{SL_1}^2=0$, &c.; or, what is the same thing, the distances $L_1K_1=L_1K_2=L_2K_1=L_2K_2$ are each $=0$, so that the points K_1, K_2 and L_1, L_2 are skew anti-points. Observe that the lines of the figure and the points R, S are taken to be real; but the distances $RK_1=RK_2$ and $SL_1=SL_2$ cannot be both real: it is assumed that one is real and

Fig. 2.



the other a pure imaginary, or else that they are both of them pure imaginaries. To fix the ideas we may in the figure consider the plane through K_1K_2, RS as horizontal, and that through RS, L_1L_2 as vertical.

Reverting now to the cyclide, suppose that we have (in the same plane) the two circles C, C' intersecting in K_1, K_2, and having S for a centre of symmetry, and let R be the mid-point of K_1, K_2.

The construction is:—through S draw a secant meeting the two circles in A, B and A', B' respectively, where A, A' and B, B' are parallel points, (therefore A, B' and A', B anti-parallel points), then the cyclide is the locus of the circles in the perpendicular plane on the diameters AB' and $A'B$ respectively.

The two circles have their radical axis passing through S, and not only so, but the points of intersection L_1, L_2 of the two circles are situate at a distance $SL_1=SL_2$, which is independent of the position of the secant: the points L_1, L_2 and K_1, K_2 being in fact a system of skew anti-points. And, moreover, the two circles have a centre of symmetry at the point where the plane of the two circles meets the line K_1K_2.

Consider in particular the two circles D, D' which are situate in the perpendicular plane through SR; these have the radical axis L_1L_2, and a centre of symmetry R; and if with these circles D, D' as given circles, and with R as the centre of symmetry, we obtain in a plane through K_1K_2 two circles having K_1K_2 for their radical axis, and having for a centre of symmetry the intersection of their plane with L_1L_2, the locus of these circles is the same cyclide as before; and, in particular, if their plane passes through RS, then the two circles are the before-mentioned circles C, C', having S for a centre of symmetry.

It will be noticed that, starting with the same two circles C, C' or D, D', we obtain two different cyclides according as we use in the construction one or other of the two centres of symmetry.

The cyclide is a quartic surface having the circle at infinity for a nodal line: viz. it is an anallagmatic or bicircular quartic surface; and it has besides the points

K_1, K_2, L_1, L_2, that is, a system of skew anti-points, for nodal points; these determine the cyclide save as to a single parameter. In fact, starting with the four points L_1, L_2, K_1, K_2, which give S, and therefore the plane of the circles C, C'; the circle C is then any one of the circles through K_1, K_2; and then drawing from S the two tangents to C, there is one other circle C' passing through K_1, K_2 and touching these tangents; C' is thus uniquely determined, and the construction is effected as above. Hence, with a given system of skew anti-points we have a single series of cyclides, say a series of conodal cyclides.

If in general we consider a quartic surface having a nodal conic and four nodes A, B, C, D, then it is to be observed that, taking the nodes in a proper order, we have a skew quadrilateral $ABCD$, the sides whereof AB, BC, CD, DA, lie wholly on the surface. In fact, considering the section by the plane ABC, this will be a quartic curve having the nodes A, B, C and two other nodes, the intersections of the plane with the nodal conic; the section is thus made up of a pair of lines and a conic; it follows that two of the sides of the triangle ABC, say the sides AB, BC, each meet the nodal conic, and that the section in question is made up of the lines AB, BC, and of a conic through the points A, C and the intersections of AB, BC with the nodal conic. Considering next the section by the plane through ACD, here (since AC is not a line on the surface) the lines CD, DA each meet the nodal conic, and the section is made up of the lines CD, DA and of a conic passing through the points A, C and the intersections of the lines CD, DA with the nodal conic. Thus the lines AB, BC, CD, DA each meet the nodal conic, and lie wholly on the surface; the lines AC, BD do not meet the conic or lie wholly on the surface.

A quartic surface depends upon 34 constants; it is easy to see that, if the surface has a given nodal conic, this implies 21 conditions, or say the postulation of a given nodal conic is $=21$, whence also the postulation of a nodal conic (not a given conic) is $=13$. Suppose that the surface has the given nodes A, B, C, D; the postulation hereof is $=16$; the nodal conic is then a conic meeting each of the lines AB, BC, CD, DA, viz. if the plane of the conic is assumed at pleasure, then the conic passes through 4 given points, and thus it still contains 1 arbitrary parameter; that is, in order that the nodal conic may be a given conic (satisfying the prescribed conditions) the postulation is $=4$. The whole postulation is thus $16+13+4, =33$, or the quartic surface which satisfies the condition in question (viz. which has for nodes the given points A, B, C, D, and for nodal conic a given conic meeting each of the lines AB, BC, CD, DA) contains still 1 arbitrary parameter: which agrees with the foregoing result in regard to the existence of a series of conodal cyclides.

It is to be added that, if a quartic surface has for a nodal line the circle at infinity and has four nodes, then the nodes form a system of skew anti-points and the surface is a cyclide. In fact, taking the nodes to be A, B, C, D, then each of the lines AB, BC, CD, DA meets the circle at infinity; but if the line AB meets the circle at infinity, then the distance AB is $=0$, and similarly the distances BC, CD, DA are each $=0$; that is, the nodes (A, C) and (B, D) are a system of skew anti-points.

Reverting to the cyclide, and taking (as before) the nodes to be K_1, K_2 and L_1, L_2, the line RS which joins the mid-points of K_1K_2 and L_1L_2 may be termed the axis of the cyclide, and the points where it meets the cyclide, or, what is the same thing, the circles C, C' or D, D', the vertices of the cyclide, say these are the points F, G, H, K. Supposing that the distances of these from a point on the axis are f, g, h, k, the origin may be taken so that $f+g+h+k=0$; the origin is in this case the "centre" of the cyclide. It is to be remarked, that given the vertices there are three series of cyclides; viz. we may in an arbitrary plane through the axis take for C, C' the circles standing on the diameters FG and HK respectively; and then, according as we take one or the other centre of symmetry, we have in the plane at right angles hereto for D, D' the circles on the diameters FH and GK, or else the circles on the diameters FK and GH respectively; there are thus three cases according as the two pairs of circles are the circles on the diameters

$$\begin{array}{lcl} FH,\ KG & \text{and} & FK,\ GH, \\ FK,\ GH & ,, & FG,\ HK, \\ FG,\ HK & ,, & FH,\ KG. \end{array}$$

The equation of the cyclide expressed in terms of the parameters f, g, h, k assumes a peculiarly simple form; in fact, taking the origin at the centre, so that $f+g+h+k=0$, the axis of x coinciding with the axis of the cyclide, and those of y, z parallel to the lines K_1K_2 and L_1L_2, or L_1L_2 and K_1K_2 respectively: writing also

$$fg+hk=G,$$
$$fh+kg=H,$$
$$fk+gh=K,$$

then the equation of one of the cyclides is

$$(y^2+z^2)^2+2x^2(y^2+z^2)+Gy^2+Hz^2+(x-f)(x-g)(x-h)(x-k)=0,$$

which we may at once partially verify by observing that for $z=0$ this equation becomes

$$[y^2+(x-f)(x-g)]\,[y^2+(x-h)(x-k)]=0,$$

and for $y=0$ it becomes

$$[z^2+(x-f)(x-h)]\,[z^2+(x-k)(x-g)]=0,$$

viz. the equations of the circles C, C' are

$$y^2+(x-f)(x-g)=0,\quad y^2+(x-h)(x-k)=0,$$

and those of D, D'

$$z^2+(x-f)(x-h)=0,\quad z^2+(x-k)(x-g)=0.$$

Starting from these equations of the four circles, the points K_1, K_2 are given by

$$Y^2=-(P-f)(P-g)=-(P-h)(P-k),$$

and the points L_1, L_2 by

$$Z^2 = -(Q-f)(Q-h) = -(Q-k)(Q-g).$$

Now writing for a moment

$$\beta = f+g = -h-k,$$
$$\gamma = f+h = -k-g,$$
$$\delta = f+k = -g-h,$$

we have $P = -\tfrac{1}{2}\dfrac{\gamma\delta}{\beta}$, $Q = -\tfrac{1}{2}\dfrac{\beta\delta}{\gamma}$, and thence $PQ = \tfrac{1}{4}\delta^2$. Moreover

$$\begin{aligned}2Y^2 + 2Z^2 + 2(P-Q)^2 &\\ &= -(P-f)(P-g) - (P-h)(P-k)\\ &\quad -(Q-f)(Q-h) - (Q-k)(Q-g) + 2(P-Q)^2\\ &= -(fg+hk+fh+gk) - 4PQ\\ &= \delta^2 - 4PQ\\ &= 0,\end{aligned}$$

that is,

$$Y^2 + Z^2 + (P-Q)^2 = 0,$$

which equation expresses that the four points are a system of skew anti-points.

The point $x = Q$ should be a centre of symmetry of the circles C, C'; to verify that this is so, transforming to the point in question as origin, the equations are

$$y^2 + \{x + Q - \tfrac{1}{2}(f+g)\}^2 - \tfrac{1}{4}(f-g)^2 = 0,$$
$$y^2 + \{x + Q - \tfrac{1}{2}(h+k)\}^2 - \tfrac{1}{4}(k-h)^2 = 0,$$

that is,

$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(\delta+\gamma)\right\}^2 - \tfrac{1}{4}(f-g)^2 = 0,$$
$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(\delta-\gamma)\right\}^2 - \tfrac{1}{4}(k-h)^2 = 0.$$

But $\delta+\gamma = f-g$, $\delta-\gamma = k-h$, so that these equations are

$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(f-g)\right\}^2 = \tfrac{1}{4}(f-g)^2,$$
$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(k-h)\right\}^2 = \tfrac{1}{4}(k-h)^2,$$

which are of the form

$$y^2 + (x - \alpha)^2 = c^2,$$
$$y^2 + (x - m\alpha)^2 = m^2c^2,$$

and consequently $x = Q$ is a centre of symmetry of the circles C, C'; and in like manner it would appear that $x = P$ is a centre of symmetry of the circles D, D'.

If in the last-mentioned equations of the circles C, C' we write $x=\Omega\cos\theta$, $y=\Omega\sin\theta$, and put for shortness

$$\rho\ =\alpha\cos\theta-\nabla,\quad \sigma\ =m(\alpha\cos\theta-\nabla),$$
$$\rho'=\alpha\cos\theta+\nabla,\quad \sigma'=m(\alpha\cos\theta+\nabla),$$

where $\nabla=\sqrt{(c^2-\alpha^2\sin^2\theta)}$, then the values of Ω for the first circle are ρ, σ, and those for the second circle are ρ', σ'. Hence the equations of the generating circles are

$$z^2+(r-\rho)(r-\sigma')=0,$$
$$z^2+(r+\rho')(r-\sigma)=0,$$

where r is the abscissa in the plane of the circles, measured from the point $x=Q$. Attending say to the first of these equations, to find the equation of the cyclide, we must eliminate θ from the equations

$$z^2+(r-\rho)(r-\sigma')=0,\quad x=r\cos\theta,\quad y=r\sin\theta;$$

the first equation is

$$z^2+r^2+m(\alpha^2-c^2)-r(\rho+\sigma')=0,$$

and we have

$$\rho+\sigma'=(m+1)\alpha\cos\theta-(m-1)\sqrt{(c^2-\alpha^2\sin^2\theta)},$$

and thence

$$(\rho+\sigma')r=(m+1)\alpha x-(m-1)\sqrt{\{c^2(x^2+y^2)-\alpha^2y^2\}},$$

so that we have

$$z^2+x^2+y^2+m(\alpha^2-c^2)-(m+1)\alpha x+(m-1)\sqrt{\{c^2(x^2+y^2)-\alpha^2y^2\}}=0,$$

viz. this is the equation of the cyclide in terms of the parameters α, c, m, the origin being at the point $x=Q$, the centre of symmetry of the circles C, C'.

Reverting to the former origin at the centre of the cyclide, we must write $x-Q$ for x; the equation thus is

$$\{y^2+z^2+(x-Q)^2-(m+1)\alpha(x-Q)+m(\alpha^2-c^2)\}^2-(m-1)^2[\{c^2(x-Q)^2+(c^2-\alpha^2)y^2\}]=0,$$

where

$$Q=-\tfrac{1}{2}\frac{\beta\delta}{\gamma},\quad \alpha=\tfrac{1}{2}\frac{\beta}{\gamma}(f-g),\quad c^2=\tfrac{1}{4}(f-g)^2,\quad m=\frac{k-h}{f-g},$$

whence also

$$m+1=\frac{2\delta}{f-g},\quad m-1=\frac{-2\gamma}{f-g},\quad \alpha^2-c^2=\tfrac{1}{4}(f-g)^2\frac{(f-k)(g-h)}{\gamma^2}.$$

After all reductions, the equation assumes the before-mentioned form

$$(y^2+z^2)^2+2x^2(y^2+z^2)+Gy^2+Hz^2+(x-f)(x-g)(x-h)(x-k)=0.$$

The equation may be written

$$(x^2+y^2+z^2)^2+(G+H+K)x^2+Gy^2+Hz^2-\beta\gamma\delta x+fghk=0,$$

and if we express everything in terms of β, γ, δ by the formulæ

$$\begin{aligned}
2f &= \beta+\gamma+\delta, & 2G &= \beta^2-\gamma^2-\delta^2,\\
2g &= \beta-\gamma-\delta, & 2H &= -\beta^2+\gamma^2-\delta^2,\\
2h &= -\beta+\gamma-\delta, & 2K &= -\beta^2-\gamma^2+\delta^2,\\
2k &= -\beta-\gamma+\delta, & 2(G+H+K) &= -\beta^2-\gamma^2-\delta^2;
\end{aligned}$$

then we have

$$(x^2+y^2+z^2)^2+\tfrac{1}{2}(-\beta^2-\gamma^2-\delta^2)\,x^2+\tfrac{1}{2}(\beta^2-\gamma^2-\delta^2)\,y^2+\tfrac{1}{2}(-\beta^2+\gamma^2-\delta^2)\,z^2$$
$$-\beta\gamma\delta x+\tfrac{1}{16}(\beta^4+\gamma^4+\delta^4-2\beta^2\gamma^2-2\beta^2\delta^2-2\gamma^2\delta^2)=0\,;$$

or, what is the same thing,

$$(x^2+y^2+z^2+\tfrac{1}{4}\beta^2+\tfrac{1}{4}\gamma^2-\tfrac{1}{4}\delta^2)^2-(\beta^2+\gamma^2)\,x^2-\gamma^2y^2-\beta^2z^2-\beta\gamma\delta x-\tfrac{1}{4}\beta^2\gamma^2=0.$$

An equivalent form of equation may be obtained very simply as follows: the surface

$$(x^2+y^2+z^2)^2+2Ax^2+2By^2+2Cz^2+2Kx+L=0$$

will be a cyclide if only the section by each of the planes $y=0$, $z=0$ breaks up into a pair of circles. Now for $y=0$ the equation is

$$(x^2+z^2)^2+2Ax^2+2Cz^2+2Kx+L=0,$$

that is,

$$z^4+2z^2(x^2+C)+x^4+2Ax^2+2Kx+L=0,$$

or

$$(z^2+x^2+C)^2=2(C-A)\,x^2-2Kx+C^2-L,$$

which will be a pair of circles if only

$$2(C-A)(C^2-L)=K^2\,;$$

and similarly writing $z=0$, we obtain

$$2(B-A)(B^2-L)=K^2.$$

These equations give

$$\begin{aligned}
L &= (B+C)^2-(BC+CA+AB),\\
K^2 &= -2(B-A)(C-A)(B+C),
\end{aligned}$$

so that L, K having these values the surface is a cyclide; there are two cyclides corresponding to the two different values of K, which agrees with a former result.

Reverting to the equation in terms of β, γ, δ this may be written

$$\beta^2-\gamma^2+\sqrt{\{(2\gamma x+\beta\delta)^2-4(\beta^2-\gamma^2)\,y^2\}}+\sqrt{\{(2\beta x+\gamma\delta)^2+4(\beta^2-\gamma^2)\,z^2\}}=0.$$

[Compare herewith Kummer's form

$$b^2=\sqrt{\{(ax-ek)^2+b^2y^2\}}+\sqrt{\{(ex-ak)^2-b^2z^2\}},\ \text{where}\ b^2=a^2-e^2.]$$

In fact, representing this for a moment by

$$\beta^2-\gamma^2+\sqrt{(\Theta)}+\sqrt{(\Phi)}=0,$$

we have

$$(\beta^2-\gamma^2)^2+\Theta-\Phi=-2(\beta^2-\gamma^2)\sqrt{(\Theta)},$$

or, substituting and dividing by $\beta^2-\gamma^2$, we have

$$\beta^2-\gamma^2+\delta^2-4(x^2+y^2+z^2)+2\sqrt{\{(2\gamma x+\beta\delta)^2-4(\beta^2-\gamma^2)y^2\}}=0,$$

or, similarly

$$\beta^2-\gamma^2-\delta^2+4(x^2+y^2+z^2)+2\sqrt{\{(2\beta x+\gamma\delta)^2+4(\beta^2-\gamma^2)z^2\}}=0,$$

either of which leads at once to the rational form.

The irrational equation

$$\beta^2-\gamma^2+\sqrt{\{(2\gamma x+\beta\delta)^2-4(\beta^2-\gamma^2)y^2\}}+\sqrt{\{(2\beta x+\gamma\delta)^2+4(\beta^2-\gamma^2)z^2\}}=0$$

is of the form

$$p+\sqrt{(qr)}+\sqrt{(st)}=0,$$

which belongs to a quartic surface having the nodal conic $p=0$, $qr-st=0$ (in the present case the circle at infinity), and also the four nodes ($q=0$, $r=0$, $p^2-st=0$) and ($s=0$, $t=0$, $p^2-qr=0$), viz. these are

$$x=-\tfrac{1}{2}\frac{\beta\delta}{\gamma},\quad y=0,\quad z=\pm\tfrac{1}{2}\frac{1}{\gamma}\sqrt{\{(\beta^2-\gamma^2)(\gamma^2-\delta^2)\}},$$

and

$$x=-\tfrac{1}{2}\frac{\gamma\delta}{\beta},\quad y=\pm\tfrac{1}{2}\frac{1}{\beta}\sqrt{\{(\gamma^2-\beta^2)(\beta^2-\delta^2)\}},\quad z=0,$$

and we hence again verify that the nodes form a system of skew anti-points, viz. the condition for this is

$$\delta^2\left(\frac{\beta}{\gamma}-\frac{\gamma}{\beta}\right)^2+(\beta^2-\gamma^2)\left(1-\frac{\delta^2}{\gamma^2}\right)-(\beta^2-\gamma^2)\left(1-\frac{\delta^2}{\beta^2}\right)=0,$$

that is,

$$\delta^2(\beta^2-\gamma^2)+\beta^2(\gamma^2-\delta^2)-\gamma^2(\beta^2-\delta^2)=0,$$

which is satisfied identically.

The cyclide has on the nodal conic or circle at infinity four pinch-points, viz. these are the intersections of the circle at infinity with the planes $\beta^2y^2+\gamma^2z^2=0$.

If $\beta=0$, the equation becomes

$$\tfrac{1}{2}\gamma+\sqrt{(x^2+y^2)}+\sqrt{(\tfrac{1}{4}\delta^2-z^2)}=0,$$

viz. the cyclide has in this case become a torus; there are here two nodes on the axis ($x=0$, $y=0$), and two other nodes on the circle at infinity, viz. these are the circular points at infinity of the sections perpendicular to the axes, and the pinch-points coincide in pairs with the last-mentioned two nodes; viz. each of the circular points at infinity = node + two pinch-points.

The Parabolic Cyclide.

One of the circles C, C' and one of the circles D, D' may become each of them a line; the cyclide is in this case a cubic surface. The easier way would be to treat the case independently, but it is interesting to deduce it from the general case. For this purpose, starting from the equation

$$(y^2+z^2)^2+2x^2(y^2+z^2)+Gy^2+Hz^2+(x-f)(x-g)(x-h)(x-k)=0,$$

where $f+g+h+k=0$, $G=fg+hk$, $H=fh+gk$, I write $x-\alpha$ for x, and assume $\alpha+f$, $\alpha+g$, $\alpha+h$, $\alpha+k$, equal to f', g', h', k' respectively; whence $4\alpha=f'+g'+h'+k'$; and the equation is

$$(y^2+z^2)^2+2(x-\alpha)^2(y^2+z^2)+(f'g'+h'k'-2\alpha^2)y^2+(f'h'+g'k'-2\alpha^2)z^2$$
$$+(x-f')(x-g')(x-h')(x-k')=0,$$

or, what is the same thing,

$$(y^2+z^2)^2+(2x^2-4\alpha x)(y^2+z^2)+(f'g'+h'k')y^2+(f'h'+g'k')z^2$$
$$+(x-f')(x-g')(x-h')(x-k')=0.$$

Now assuming $k'=\infty$, we have $4\alpha=k'=\infty$, or writing 4α instead of k', and attending only to the terms which contain α, we have

$$x(y^2+z^2)-h'y^2-g'z^2+(x-f')(x-g')(x-h')=0,$$

or, what is the same thing,

$$(x-f')(x-g')(x-h')+(x-h')y^2+(x-g')z^2=0,$$

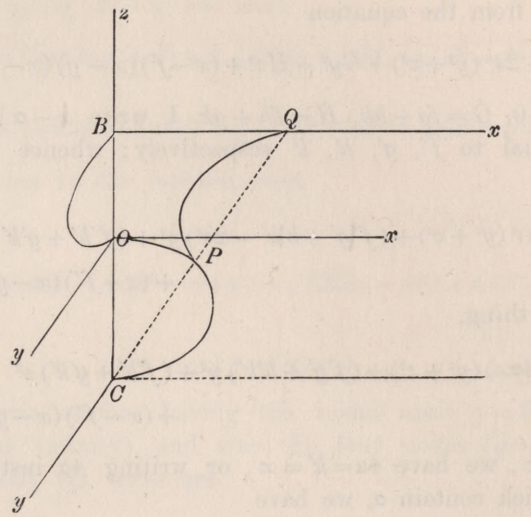
where by altering the origin we may make $f'=0$.

It is somewhat more convenient to take the axis of z (instead of that of x) as the axis of the cyclide; making this change, and writing also 0, β, γ in place of the original constants, I take the equation to be

$$z(z-\beta)(z-\gamma)+(z-\gamma)y^2+(z-\beta)x^2=0,$$

viz. this is a cubic surface having upon it the right lines $(z=\gamma,\ x=0)$, $(z=\beta,\ y=0)$; the section by a plane through either of these lines is the line itself and a circle; and in particular the circle in the plane $x=0$ is $z(z-\beta)+y^2=0$, and that in the plane $y=0$ is $z(z-\gamma)+x^2=0$. And it is easy to see how the surface is generated: if, to fix the ideas, we take β positive, γ negative, the lines and circles are as shown in fig. 3; and if we draw through Cy a plane cutting the circle CO and the line Bx in P, Q respectively, then the section is a circle on the diameter PQ; and similarly for the sections by the planes through Bx. It is easy to see that the whole surface is included between the planes $z=\beta$, $z=\gamma$; considering the sections parallel to these planes (that is, to the plane of xy) $z=\beta$, the section is the two-fold line $y=0$; $z=$ any smaller positive value, it is a hyperbola having the axis of y for its transverse axis; $z=0$, it is the pair of real lines $\gamma y^2+\beta x^2=0$; z negative and less in absolute magnitude than $-\gamma$, it is a hyperbola having the axis of x for its transverse axis; and finally $z=\gamma$, it is the two-fold line $x=0$. It is easy to see the forms of the cubic curves which are the sections by any planes $x=$ const. or $y=$ const.

Fig. 3.



The before-mentioned circles are curves of curvature of the surface; to verify this *à posteriori*, write

$$U=z(z-\beta)(z-\gamma)+(z-\gamma)y^2+(z-\beta)x^2=0$$

for the equation of the surface; and put for shortness $P=3z^2-2z(\beta+\gamma)+\beta\gamma$, $P+x^2+y^2=L$, so that $d_zU=P+x^2+y^2, =L$. The differential equation for the curves of curvature is

$$\begin{vmatrix} 2x(z-\beta) \quad , & 2y(z-\gamma) \quad , & P+x^2+y^2 \\ xdz+(z-\beta)\,dx, & ydz+(z-\gamma)\,dy, & \tfrac{1}{2}P'dz+xdx+ydy \\ dx \quad , & dy \quad , & dz \end{vmatrix}=0,$$

or, say this is

$$\begin{aligned}\Omega = dx^2\,.\,2xy\,(z-\gamma)-dy^2\,.\,2xy\,(z-\beta)+dz^2\,.\,2xy\,(\gamma-\beta)\\ +\,dzdy\,.\,x\,[-2\,(z-\beta)\,(2z-\beta)+L]\\ +\,dxdz\,.\,y\,[\;\;2\,(z-\gamma)\,(2z-\gamma)-L]\\ +\,dxdy\,.\;\;[(\gamma-\beta)\,P+(2z-\beta-\gamma)\,(y^2-x^2)]=0.\end{aligned}$$

But in virtue of the equation $U=0$, we have identically

$$\{2\,(z-\beta)\,x\,dx+2\,(z-\gamma)\,y\,dy+Ldz\}\times\left\{-\frac{z-\gamma}{z-\beta}\,y\,dx+\frac{z-\beta}{z-\gamma}\,x\,dy+\frac{xy\,(\gamma-\beta)}{(z-\beta)\,(z-\gamma)}\,dz\right\}$$
$$+\,\Omega$$
$$=(\gamma-\beta)\left\{2-\frac{L}{(z-\beta)\,(z-\gamma)}\right\}\times\{xydz^2-y(z-\gamma)\,dzdx-x\,(z-\beta)\,dzdy+(z-\beta)\,(z-\gamma)\,dxdy\}.$$

Hence in virtue of the equations $U=0$, $dU=0$ the equation $\Omega=0$ becomes

$$xy\,dz^2 - y(z-\gamma)\,dz\,dx - x(z-\beta)\,dz\,dy + (z-\beta)(z-\gamma)\,dx\,dy = 0,$$

that is,

$$\{x\,dz - (z-\gamma)\,dx\}\,\{y\,dz - (z-\beta)\,dy\} = 0,$$

whence either $x - C(z-\gamma) = 0$ or $y - C'(z-\beta) = 0$; viz. the section of the surface by a plane of either series (which section is a circle) is a curve of curvature of the surface.

The equation of the cyclide can be elegantly expressed in terms of the ellipsoidal coordinates (λ, μ, ν) of a point (x, y, z); viz. writing for shortness $\alpha = b^2 - c^2$, $\beta = c^2 - a^2$, $\gamma = a^2 - b^2$, the coordinates (λ, μ, ν) are such that

$$\begin{aligned} -\beta\gamma x^2 &= (a^2+\lambda)(a^2+\mu)(a^2+\nu),\\ -\gamma\alpha y^2 &= (b^2+\lambda)(b^2+\mu)(b^2+\nu),\\ -\alpha\beta z^2 &= (c^2+\lambda)(c^2+\mu)(c^2+\nu), \end{aligned}$$

(see Roberts, *Comptes Rendus*, t. LIII. (Dec., 1861), p. 1119), whence

$$\begin{aligned} x^2+y^2+z^2 &= a^2+b^2+c^2+\lambda+\mu+\nu,\\ (b^2+c^2)\,x^2 + (c^2+a^2)\,y^2 + (a^2+b^2)\,z^2 &= b^2c^2 + c^2a^2 + a^2b^2 - \mu\nu - \nu\lambda - \lambda\mu. \end{aligned}$$

The equation of the cyclide then is

$$\sqrt{(a^2+\lambda)} + \sqrt{(a^2+\mu)} + \sqrt{(a^2+\nu)} = \sqrt{(\delta)}.$$

In fact, starting from this equation and rationalising, we have

$$\begin{aligned} (3a^2+\lambda+\mu+\nu-\delta)^2 &= 4\,[\sqrt{\{(a^2+\mu)(a^2+\nu)\}} + \sqrt{\{(a^2+\nu)(a^2+\lambda)\}} + \sqrt{\{(a^2+\lambda)(a^2+\mu)\}}]^2\\ &= 4\,[3a^4 + 2a^2(\lambda+\mu+\nu) + \mu\nu+\nu\lambda+\lambda\mu + 2\sqrt{\{(a^2+\lambda)(a^2+\mu)(a^2+\nu)\}}\,\sqrt{(\delta)}], \end{aligned}$$

which, substituting for

$$\lambda+\mu+\nu,\ \mu\nu+\nu\lambda+\lambda\mu \text{ and } \sqrt{\{(a^2+\lambda)(a^2+\mu)(a^2+\nu)\}}$$

their values, is

$$(x^2+y^2+z^2+\gamma-\beta-\delta)^2 = 4\,\{(\gamma-\beta)\,x^2 - \beta y^2 + \gamma z^2 - \beta\gamma - 2x\sqrt{(-\beta\gamma\delta)}\},$$

or, writing $-\tfrac{1}{4}\gamma^2$, $\tfrac{1}{4}\beta^2$, $\tfrac{1}{4}\delta^2$ in place of β, γ, δ respectively, this is

$$(x^2+y^2+z^2+\tfrac{1}{4}\gamma^2+\tfrac{1}{4}\beta^2-\tfrac{1}{4}\delta^2)^2 = (\gamma^2+\beta^2)\,x^2 + \beta^2z^2 + \gamma^2y^2 + \tfrac{1}{4}\beta^2\gamma^2 + \beta\gamma\delta x,$$

which agrees with a foregoing form of the equation.

The generating spheres of the cyclide cut at right angles each of a series of spheres; viz. each of these spheres passes through one and the same circle in the plane of, and having double contact with, the conic which contains the centres of the generating spheres; the centres of the orthotomic spheres being consequently in a line meeting an axis, and at right angles to the plane of the conic in question. Or, what is the same thing, starting with a conic, and a sphere having double contact therewith, the cyclide is the envelope of a variable sphere having its centre on the conic and cutting at right angles the fixed sphere.*

* I am indebted for this mode of generation of a Cyclide to the researches of Mr Casey.

It may be remarked, that if we endeavour to generalize a former generation of the cyclide, and consider the envelope of a variable sphere having its centre on a conic, and *touching* a fixed sphere, this is in general a surface of an order exceeding 4; it becomes a surface of the fourth order, viz. a cyclide, *only* in the case where the fixed sphere has its centre on the anti-conic. But if we consider the envelope of a variable sphere having its centre on a conic and cutting at right angles a fixed sphere, this is always a quartic surface having the circle at infinity for a double line; the surface has moreover two nodes, viz. these are the anti-points of the circle which is the intersection of the sphere by the plane of the conic. If the sphere touches the conic, then there is at the point of contact a third node; and similarly, if it has double contact with the conic, then there is at each point of contact a node; viz. in this case the surface has four nodes, and it is in fact a cyclide.

There is no difficulty in the analytical proof: consider the envelope of a variable sphere having its centre on the conic $Z=0$, $-\frac{X^2}{\beta}+\frac{Y^2}{\alpha}=1$, and which cuts at right angles the sphere $(x-l)^2+(y-m)^2+(z-n)^2=k^2$.

Take the equation of the variable sphere to be

$$(x-X)^2+(y-Y)^2+z^2=c^2,$$

then the orthotomic condition is

$$(X-l)^2+(Y-m)^2+n^2=c^2+k^2,$$

or, substituting this value of c^2, the equation of the variable sphere is

$$(x-X)^2+(y-Y)^2+z^2=-k^2+(X-l)^2+(Y-m)^2+n^2,$$

all which spheres pass through the points

$$x=l,\quad y=m,\quad z=\pm\sqrt{(n^2-k^2)};$$

that is,

$$x^2+y^2+z^2+k^2-l^2-m^2-n^2-2(x-l)X-2(y-m)Y=0,$$

and considering Y, Y as variable parameters connected by the equation $-\frac{X^2}{\beta}+\frac{Y^2}{\alpha}=1$, the equation of the envelope is

$$(x^2+y^2+z^2+k^2-l^2-m^2-n^2)^2+4\beta(x-l)^2-4\alpha(y-m)^2=0,$$

viz. this is a bicircular quartic, having the two nodes $x=l$, $y=m$, $z=\pm\sqrt{(n^2-k^2)}$; these are the anti-points of the circle $(x-l)^2+(y-m)^2=k^2-n^2$, which is the intersection of the sphere $(x-l)^2+(y-m)^2+(z-n)^2=k^2$ by the plane of the conic.

The constants might be particularised so that the equation should represent a cyclide; but I treat the question in a somewhat different manner, by showing that the generating spheres of a cyclide cut at right angles each of a series of fixed spheres. Write α, β, $\gamma=b^2-c^2$, c^2-a^2, a^2-b^2; then if

$$\frac{X^2}{-\beta}+\frac{Y^2}{\alpha}=1;\quad \frac{X_1^2}{\gamma}-\frac{Z_1^2}{\alpha}=1,$$

the points $(X, Y, 0)$ and $(X_1, 0, Z_1)$ will be situate on a pair of anti-conics.

Consider the fixed sphere

$$(x - X_1)^2 + y^2 + (z - Z_1)^2 = c_1^2,$$

then if this is touched by the variable sphere

$$(x - X)^2 + (y - Y)^2 + z^2 = c^2,$$

the last-mentioned sphere will be a generating sphere of the cyclide. The condition of contact is

$$(X - X_1)^2 + Y^2 + Z_1^2 = (c + c_1)^2,$$

that is,

$$\begin{aligned}(c + c_1)^2 &= X^2 - 2XX_1 + X_1^2 + \alpha\left(1 + \frac{X^2}{\beta}\right) + \alpha\left(\frac{X_1^2}{\gamma} - 1\right)\\ &= -\frac{\gamma}{\beta}X^2 - 2XX_1 - \frac{\beta}{\gamma}X_1^2\\ &= \Omega^2,\end{aligned}$$

if for a moment

$$\Omega = X\sqrt{\left(\frac{-\gamma}{\beta}\right)} + X_1\sqrt{\left(\frac{-\beta}{\gamma}\right)},$$

that is, $c = -c_1 + \Omega$, and the equation of the variable sphere is

$$(x - X)^2 + (y - Y)^2 + z^2 = (c_1 - \Omega)^2,$$

where X, Y are variable parameters connected by

$$\frac{X^2}{-\beta} + \frac{Y^2}{\alpha} = 1.$$

Suppose that the variable sphere is orthotomic to

$$(x - X_2)^2 + y^2 + (z - Z_2)^2 = c_2^2,$$

the condition for this is

$$(X - X_2)^2 + Y^2 + Z_2^2 = c^2 + c_2^2,$$

or combining with the identical equation

$$(X - X_1)^2 + Y^2 + Z_1^2 = (c + c_1)^2,$$

we have

$$\begin{aligned}-2X(X_2 - X_1) + X_2^2 - X_1^2 + Z_2^2 - Z_1^2 &= c_2^2 - c_1^2 - 2c_1(-c_1 + \Omega)\\ &= c_2^2 + c_1^2 - 2c_1\Omega,\end{aligned}$$

or, substituting for Z_1, Ω their values, this is

$$-2X(X_2 - X_1) + X_2^2 - X_1^2 + Z_2^2 - \alpha\left(\frac{X_1^2}{\gamma} - 1\right) = c_2^2 + c_1^2 - 2c_1\left\{X\sqrt{\left(-\frac{\gamma}{\beta}\right)} + X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)}\right\},$$

viz. this will be identically true if

$$X_2 = X_1 + c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)},$$

$$X_2^2 + Z_2^2 - c_2^2 = -\frac{\beta}{\gamma}X_1^2 + 2c_1X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} - \alpha + c_1^2,$$

or, as this last equation may be written

$$Z_2^2 - c_2^2 = Z_1^2 - c_1^2\frac{\alpha}{\beta} + 2c_1X_1\left\{\sqrt{\left(-\frac{\beta}{\gamma}\right)} + \sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\}.$$

The equation of the orthotomic sphere is thus found to be

$$\left\{x - X_1 - c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\}^2 + y^2 + z^2 - 2zZ_2 + Z_1^2 - c_1^2\frac{\alpha}{\beta} + 2c_1X_1\left\{\sqrt{\left(-\frac{\beta}{\gamma}\right)} + \sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\} = 0;$$

or, what is the same thing,

$$x^2 + y^2 + z^2 - 2zZ_2 - 2x\left\{X_1 + c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\} - X_1^2\frac{\beta}{\gamma} - 2c_1X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} + c_1^2 - \alpha = 0,$$

or, as this may be written

$$\alpha\left(-\frac{x^2}{\beta} + \frac{y^2}{\alpha} - 1\right) + z^2 - 2zZ_2 - \frac{\gamma}{\beta}x^2 - 2xX_1 - \frac{\beta}{\gamma}X_1^2 - 2c_1x\sqrt{\left(-\frac{\beta}{\gamma}\right)} - 2c_1X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} + c_1^2 = 0,$$

viz. this is

$$\alpha\left(-\frac{x^2}{\beta} + \frac{y^2}{\alpha} - 1\right) + z^2 - 2zZ_2 + \left\{x\sqrt{\left(-\frac{\gamma}{\beta}\right)} + X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} - c_1\right\}^2 = 0,$$

where Z_2 is arbitrary. We have thus a series of orthotomic spheres; viz. taking any one of these, the envelope of a variable sphere having its centre on the conic $-\frac{x^2}{\beta} + \frac{y^2}{\alpha} - 1 = 0$, and cutting at right angles the orthotomic sphere, is a cyclide. The centre of the orthotomic sphere is a point at pleasure on the line

$$x = X_1 + c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)},\ y = 0;$$

and the sphere passes through the circle $z = 0$,

$$\left\{x - X_1 - c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\}^2 + y^2 + Z_1^2 - c_1^2\frac{\alpha}{\beta} + 2c_1X_1\left\{\sqrt{\left(-\frac{\beta}{\gamma}\right)} + \sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\} = 0,$$

viz. this is a circle having double contact with the conic $-\frac{x^2}{\beta} + \frac{y^2}{\alpha} = 1$; or, what is the same thing, the orthotomic sphere is a sphere having its centre on the line in question, and having double contact with the conic $-\frac{x^2}{\beta} + \frac{y^2}{\alpha} = 1$.